Toric Stratification and Topology of Hypersurfaces

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Abstract

Toric varieties are naturally stratified into algebraic tori of various dimension. Using the theory of Danilov and Khovanskii, this can be used to describe topological data of toric hypersurfaces from combinatorics.

In these lectures, I will give a hands-on introduction to this approach and show how it can be used to efficiently address a variety of questions.

Contents

1 Introduction and Notation

In these notes I will assume that the reader has a basic familiarity with how toric varieties can be constructed from fans using homogeneous coordinates, as well as Batyrev's description of toric hypersurfaces using reflexive polytopes. We will then explain how to use the theory of [1] to compute Hodge numbers of toric hypersurfaces and understand aspects of its topology. Most of the material contained in these notes is adapted from [2, 3, 4]. The work [1] also covers complete intersections, but for the sake of simplicity, I will focus on hypersurfaces here and not comment on the more general case.

For quick reference, here is the notation I will be using

- n is used as the dimension of whatever toric variety we are interested in.
- For a fan Σ, the associated toric variety will be denoted by $\mathbb{P}_\Sigma$.
- The set of d-dimensional cones in Σ are denoted by

$$\Sigma(d) = \{\sigma_k^{[d]}\}\,, \tag{1}$$

We include a unique zero-dimensional cone (the origin) in any fan.

- I will use Δ for the **M**-lattice polytope (where the **M**onomials live) and Δ° for the **N**-lattic polytope (where the fa**N** lives).
- The ray generators of Σ are $v_i \in \mathbf{N}$. They are associated to divisors D_i and homogeneous coordinates z_i. Elements in **M** that we are interested in are typically called m.
- The inner form between elements of **M** and **N** is written as $\langle m, v \rangle$.

2 Stratification of Toric Varieties

A toric variety can be written (with some caveats for non-simplicial fans) as

$$\mathbb{P}_{\Sigma} = \frac{\mathbb{C}^k - Z}{G} \tag{2}$$

where G is the kernel of the homomorphism

$$z_i \mapsto t_k := \prod_j z_j^{\langle e_k, v_j \rangle} \tag{3}$$

with e_k a basis of **M**. One can think of the t_k as coordinates on the dense algebraic torus giving a toric variety its name:

$$(\mathbb{C}^*)^n = \{t_k | z_i \neq 0\} \,. \tag{4}$$

Any $\mathbb{Z}$-basis of **M** is fine to describe this.

One way to think about toric varieties is as a (partial) compactification of $(\mathbb{C}^*)^n$ by adding in lower-dimensional algebraic tori. One can think of these lower-dimensional tori as limits of $(\mathbb{C}^*)^n$ in which some of the t_i can go to zero. The data for how this can happen is encoded in the fan Σ. Recall that whenever Σ contains a cone σ spanned by rays generated by $\{\nu_i | i \in I_\sigma\}$, we can simultaneously set the homogeneous coordinates associated with it to zero: $\emptyset \neq \{z_i = 0 | \forall i \in I_\sigma\} \subset \mathbb{P}_\Sigma$. For a ray σ, choosing all of the e_k to sit in the dual cone $\sigma^\vee$ to σ defined by

$$\langle \sigma^\vee, \sigma \rangle \geq 0$$

we can safely set $\{z_i = 0 | \forall i \in I_\sigma\}$. Higher dimensional cones can then be found by appropriately intersecting this data.

This lands us on a $(\mathbb{C}^*)^{n-d}$ for a d-dimensional cone σ: denoting the set of d-dimensional cones by

$$\Sigma(d) = \{\sigma^{[d]}\}\,, \tag{5}$$

any cone is spanned by a number of ray generators, which in turn correspond to homogeneous coordinates z_i. To such a cone, we can associate

$$T_{\sigma^{[d]}} = \prod_j z_j^{\langle e_k^{\sigma^{[d]}}, v_j\rangle} \tag{6}$$

where the $e_k^{\sigma^{[d]}}$ span the sublattice $M^\perp_{\sigma^{[d]}}$ of M which is perpendicular to $\sigma^{[d]}$. The ray generators of a d-dimensional cone span a hyperplane of dimension d, so that the dimension of $M^\perp_{\sigma^{[d]}}$ is $n-d$.

We can hence stratify any toric variety according to the data of our fan as

$$\mathbb{P}_\Sigma = \coprod_d \coprod_{\sigma^{[d]}\in\Sigma(d)} T_{\sigma^{[d]}} \quad = \quad \coprod_d \coprod_{\sigma^{[d]}\in\Sigma(d)} (\mathbb{C}^*)^{n-d}\,. \tag{7}$$

The definition of a fan then makes sure that all of these fit together in a consistent way. Note that this also immediately shows that $(\mathbb{C}^*)^n$ is open and dense and naturally acts on $\mathbb{P}_\Sigma$.

Let us discuss some examples. The fan corresponding to $\mathbb{P}^1$ lives in $\mathbb{R}$ and is composed of three cones: the origin, a ray generated by 1 and a ray generated by -1. We recover the standard presentation from (2) as

$$\mathbb{P}^1 = \frac{(z_1,z_2) - \{(0,0)\}}{(z_1,z_2)\sim(\lambda z_1,\lambda z_2)}\,. \tag{8}$$

The open dense $\mathbb{C}^*$ can be described by the coordinate z_1/z_2, or equivalently by z_2/z_1, and the strata corresponding to the two one-dimensional cones are simply given by $z_1=0$ and $z_2=0$, respectively. We hence recover the description of $\mathbb{P}$ as the Riemann sphere, i.e. adding the point at infinity to $\mathbb{C}$.

The fan corresponding to $\mathbb{P}^2$ is shown in figure 1. Again, (2) gives the standard presentation as

$$\mathbb{P}^2 = \frac{(z_1,z_2,z_3) - \{(0,0,0)\}}{(z_1,z_2,z_3)\sim(\lambda z_1,\lambda z_2,\lambda z_3)}\,. \tag{9}$$

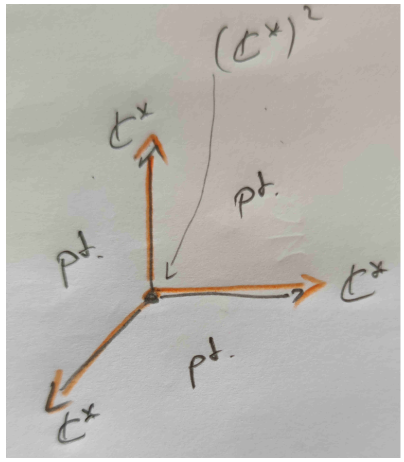

Figure 1: The fan of $\mathbb{P}^2$ contains a zero-dimensional cone, three rays (one-dimensional cones) and three two-dimensional cones.

From the fan, we can directly read off the stratification data (7), which becomes

$$\mathbb{P}^2 = (\mathbb{C}^*)^2 \coprod_{i=1..3} \mathbb{C}^* \coprod_{k=1..3} pt. \tag{10}$$

A direct consequence of this is that we can work out the topological Euler characteristic $\chi(\mathbb{P}_\Sigma)$ with ease using its additivity:

$$\begin{aligned} \chi(\mathbb{P}_\Sigma) &= \chi(\coprod_{d} \coprod_{\sigma_k^{[d]} \in \Sigma(d)} (\mathbb{C}^*)^{n-d}) \\ &= \sum_{d} \sum_{\sigma_k^{[d]} \in \Sigma(d)} \chi\left((\mathbb{C}^*)^{n-d}\right) \\ &= \sum_{\sigma_k^{[n]} \in \Sigma(n)} \chi\left((\mathbb{C}^*)^{0}\right) = |\Sigma(n)| \end{aligned} \tag{11}$$

where we have used that the topological Euler characteristic is additive, vanishes for an algebraic torus of dimension > 0 vanishes and is 1 for a point.

Finally, note that we can also obtain a stratification of toric subvarieties

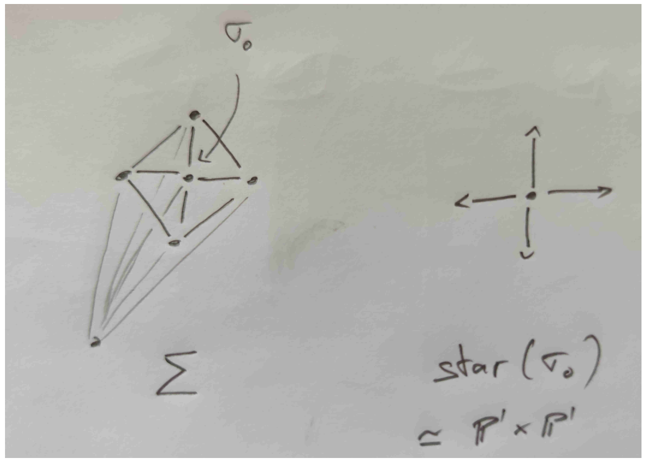

Figure 2: The fan Σ of 'local $\mathbb{P}^1 \times \mathbb{P}^1$' i.e. the total space of $-K_{\mathbb{P}^1 \times \mathbb{P}^1}$ and the star fan associated with the central compact ray. We can find the stratification of $\mathbb{P}^1 \times \mathbb{P}^1$ from either Σ or $star(\sigma_0)$

this way. For any cone σ there is an associated toric subvariety

$$X_{star(\sigma)} = X_\Sigma \cap \{z_i = 0 \forall i \in I_\sigma\} \tag{12}$$

with $star(\sigma)$ the star fan of sigma. We have shown an example in Figure 2.

3 Polytopes and Toric Hypersurfaces

We now want to talk about toric hypersurfaces defined by the vanishing of a polynomial $P \in \Gamma(\mathcal{O}(D))$ for some divisor class D. It turns out that this is where polytopes make a pleasant appearance.

For any polytope Δ the **M**-lattice, there is an associated normal fan $\Sigma_n(\Delta)$ giving rise to a toric variety $\mathbb{P}_{\Sigma_n(\Delta)}$ along with a divisor D_Δ.

The normal fan of a polytope is defined as follows: to every face $\Theta^{[k]}$ of a polytope Δ, we may associate a cone

$$\check{\sigma}_N(\Theta^{[k]}) = \bigcup_{r \geq 0} r \cdot (p_\Delta - p_{\Theta^{[k]}}) \tag{13}$$

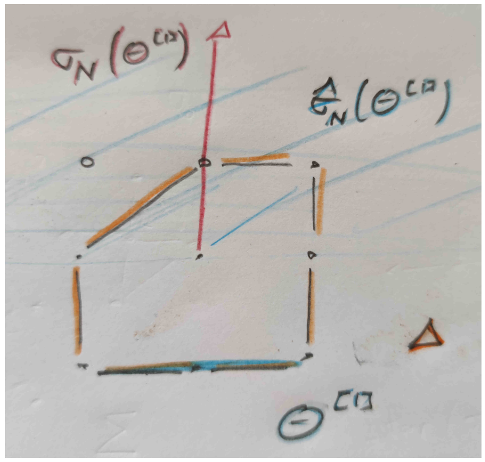

Figure 3: The construction of the cone dual to the face $\Theta^{[1]}$ of a convex polytope Δ. Here $\check{\sigma}_N(\Theta^{[1]})$ is the upper halfplane and its dual $\sigma_N(\Theta^{[1]})$ is just the ray going upwards.

where p_Δ is an arbitrary point lying inside Δ and $p_\Theta^{[k]}$ is an arbitrary point lying inside $\Theta^{[k]}$. The dual cones $\sigma_N(\Theta^{[k]})$, defined by

$$\langle \check{\sigma}_N, \sigma_N \rangle \geq 0\,, \tag{14}$$

form a complete fan which is called the *normal fan* $\Sigma_N(\Delta)$ of Δ. Here, k-dimensional faces $\Theta^{[k]}$ of Δ are associated with $4-k$-dimensional cones $\sigma_N(\Theta^{[k]})$ (for four-dimensional polytopes).

A polytope hence determines a (possibly singular) toric variety $\mathbb{P}_{\Sigma_N(\Delta)}$. However, it is not hard to see that shrinking or enlarging the polytope gives us the same fan, so there is more data here, which turns out to be that of a divisor on $\mathbb{P}_{\Sigma_N(\Delta)}$. This can be seen from the existence of a convex support function Ψ_Δ, linear on each cone of Σ_N. Each cone of maximal dimension of $\Sigma_N(\Delta)$ is associated with a vertex m_α of Δ. Ψ_Δ can then be described by setting:

$$\Psi_\Delta|_{\sigma_N(m_\alpha)}(p) = \langle m_i, p \rangle \tag{15}$$

for each point p in $\sigma_N(m_\alpha)$. This also determines Ψ_Δ for all cones of lower dimension. The associated divisor is then

$$D_\Delta = \sum_{\nu_i \in \Sigma_N(1)} a_j D_j \tag{16}$$

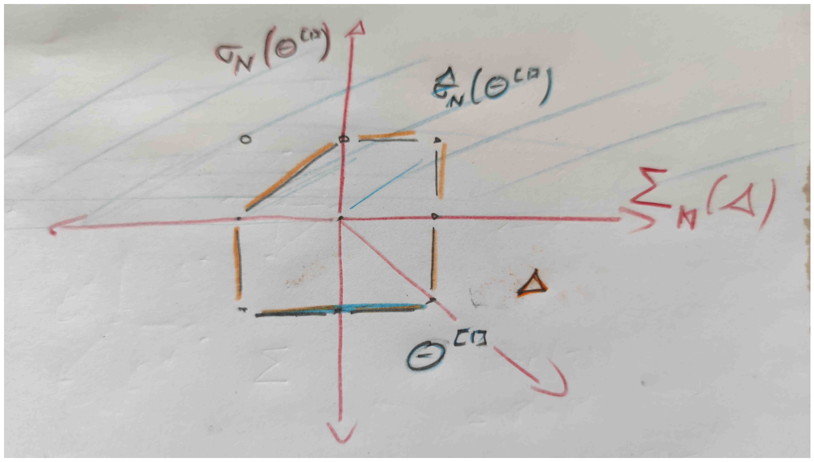

Figure 4: The normal fan $\Sigma_N(\Delta)$ of Δ.

can then be determined from

$$\Psi_\Delta|_{\sigma_N(m_\alpha)}(\nu_j) = -a_j \;\; \forall \nu_j \in \sigma_N(m_\alpha), \tag{17}$$

and convexity means that

$$\Psi_\Delta|_{\sigma_N(m_\alpha)}(\nu_j) > -a_j \;\; \forall \nu_j \notin \sigma_N(m_\alpha) \tag{18}$$

In this construction, points m of Δ are associated with holomorphic sections μ_m of $\mathcal{O}(D_\Delta)$ given by

$$\mu_m = \prod z_i^{\langle m,\nu_i\rangle + a_i} \tag{19}$$

and Δ becomes the Newton polyhedron of a generic hypersurface defined by the zero locus of a section of the line bundle $\mathcal{O}(D_\Delta)$. We can write such a hypersurface as

$$X_\Delta := \{0 = \sum_{m\in\Delta} c_m \mu_m\} \subset \mathbb{P}_{\Sigma_N(\Delta)} \tag{20}$$

As $\mathbb{P}_{\Sigma_N(\Delta)}$ is typically singular, such a hypersurface is typically singular as well.

Choosing a bigger/smaller Δ with the same normal fan hence corresponds to choosing a different D_Δ.

This construction is particularly simple in the case of reflexive pairs of polytopes. Here Δ is a lattice polytope such that it polar dual Δ° defined by

$$\langle \Delta, \Delta^\circ \rangle \geq -1 \tag{21}$$

is another lattice polytope. This implies that the normal fan $\Sigma_N(\Delta)$ is equal to the face fan (fan over the faces) $\Sigma_f(\Delta^\circ)$ of its polar dual and there is a one-to-one correspondence

$$\Theta^{[k]} \leftrightarrow \Theta^{\circ[n-k-1]} . \tag{22}$$

between faces of Δ and Δ°. As the facets Θ^{n-1} (faces of highest dimension) of Δ° are given by

$$\langle m_i, \Theta^{n-1} \rangle = -1 . \tag{23}$$

where m_i are the vertices of Δ, this determines $a_i = 1$ for all i, so that $D_\Delta = -K_{\mathbb{P}_{\Sigma_n(\Delta)}}$. A generic linear combination of the μ_m hence has vanishing first Chern class by adjunction and hence defines a Calabi-Yau hypersurface.

As $\mathbb{P}_{\Sigma_N(\Delta)}$ is typically singular, such a hypersurface is typically singular as well. We can find a maximal projective crepant partial desingularization (MPCP) by refining $\Sigma \to \Sigma_N(\Delta)$ using $\nu_i \subset \Delta^\circ$ as ray generators. The defining equation of a Calabi-Yau hypersurface then becomes

$$X_{(\Delta,\Delta^\circ,\Sigma)} := \{0 = \sum_m c_m \prod z_i^{\langle m,\nu_i \rangle + 1}\} . \tag{24}$$

While it is clear from adjunction that the above has vanishing first Chern class, this can also be seen from (20) by working our the proper transform of the blowups associated with fan refinements associated with $\nu_i \in \Delta^\circ$.

Note that the refinement $\Sigma \to \Sigma_N(\Delta)$ is not unique, an the choices correspond to different triangulations of Δ° (subject to being 'regular $\sim$ projective).

Batyrev has shown that this can always be done such that it results in a smooth hypersurface $X_{(\Delta,\Delta^\circ,\Sigma)}$ for $n \leq 4$.

4 Topology of Hypersurfaces

We now move on to the main part of these notes, which is a summary of parts of [1] and [5].

4.1 Stratification for (refinements of) Normal Fans

The stratification (7) induces a stratification of any hypersurface $X \in \mathbb{P}_\Sigma$ (20):

$$X_{\sigma^{[k]}} = X \cap T_{\sigma^{[k]}} \tag{25}$$

so that

$$X = \coprod_k \coprod_{\sigma^{[k]} \in \Sigma(k)} X_{\sigma^{[k]}} \tag{26}$$

For the construction of (families of) hypersurfaces from polytopes, the fan Σ is the normal fan $\Sigma_N(\Delta)$ of Δ, the cones of which are in one-to-one correspondence with faces $\Theta^{[k]}$ of Δ:

$$\Theta^{[k]} \leftrightarrow \sigma_N^{[n-k]} \tag{27}$$

Note that we count Δ as a face of itself, which corresponds to the 0-dimensional cone in $\Sigma_N(\Delta)$. We can hence define

$$X_{\Theta^{[k]}} = X \cap T_{\Theta^{[k]}} \tag{28}$$

and write (26) as

$$X = \coprod_k \coprod_{\Theta^{[k]} \in \Delta} X_{\Theta^{[k]}} \tag{29}$$

For a non-degenerate hypersurface the dimension of these strata is $k-1$, which is implied by the Bertini theorem for a sufficiently generic choice of the defining polynomials of X.

Let us now discuss the effect of the resolution of X_Δ. We are interested in a refinement Σ of the fan $\Sigma_N(\Delta)$ in order to resolve the hypersurface X_Δ. Fortunately, this process results in only minor modifications of the stratification (26) above. In a slight abuse of notation, we will use X_Δ also for the resolved hypersurface. Under a refinement $\pi : \Sigma \to \Sigma_N(\Delta)$, the stratification associated with faces of Δ becomes:

$$X = X_\Delta \coprod_{\Theta^{[n-1]}} X_{\Theta^{[n-1]}} \coprod_{k \geq 2} \coprod_{\Theta^{[n-k]}} E_{\Theta^{[n-k]}} \times X_{\Theta^{[n-k]}} \, . \tag{30}$$

Here $E_{\Theta^{[n-k]}}$ is the exceptional set of the refinement of the cone in $\Sigma_N(\Delta)$ associated with the face $\Theta^{[n-k]}$. As $\Theta^{[n-1]}$ corresponds to rays which cannot be refined, there is no such contribution there.

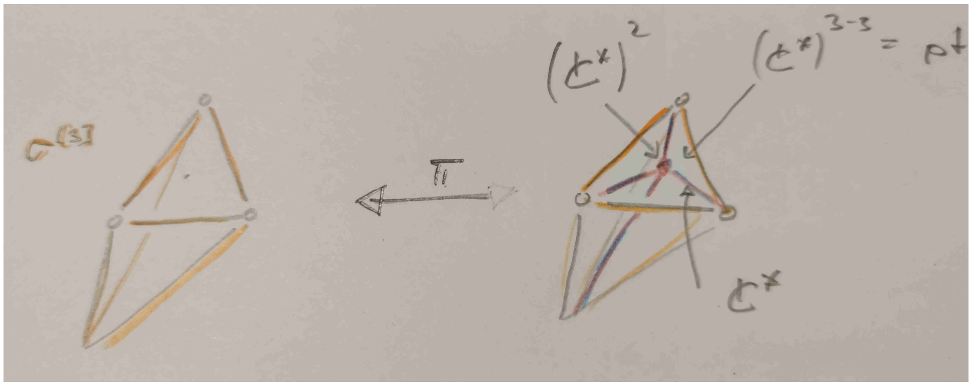

Figure 5: A refinement of a cone and the associated strata of the exceptional locus. Note that while the dimension of the strata of the cones of the original fan depend on the dimension of the fan $\Sigma_N(\Delta)$, the strata of the exceptional locus of the blowup only depend on the codimension of the new cones appearing in the refinement. In this case, the exceptional locus is a $\mathbb{P}^2$.

We can write

$$E_{\Theta^{[n-k]}} = \coprod_{\sigma^{[l]} \subset \sigma^{[k]}} (\mathbb{C}^*)^{k-l}\,. \tag{31}$$

For every l-dimensional cone in $\pi^{-1}(\sigma^{[k]})$, where $\sigma^{[k]} \in \Sigma_N(\Delta)$ is the cone in the normal fan associated with the face $\Theta^{[n-k]}$, there is a corresponding stratum $(\mathbb{C}^*)^{k-l}$ in $E_{\Theta^{[n-k]}}$. See Fig 5 for an example.

4.2 Hodge Deligne Numbers

The idea of [1] is to work out the topology of X from the topology of the $X_{\sigma^{[d]}} = X_{\Theta^{[n-d]}}$ which can in turn be computed combinatorially. This is made possible by introducing the Hodge-Deligne numbers.

The strata appearing in the stratification (30) naturally carry a mixed Hodge structure. In very simple terms, this means that the Hodge-Deligne numbers

$$h^{p,q}(H^k(X,\mathbb{C})) \tag{32}$$

can be nonzero even when $p+q \neq k$, but still obey $h^{p,q} = h^{q,p}$. See e.g. [6, 7] for a proper introduction.

These data can be packed into the numbers (which we will also call Hodge-Deligne numbers in the following)

$$e^{p,q}(X) = \sum_k (-1)^k h^{p,q}(H^k(X,\mathbb{C}))\,, \tag{33}$$

which are convenient for a number of reasons. First of all, in case these is a pure Hodge structure (e.g. when X is smooth and compact), they agree (up to a sign) with the usual Hodge numbers. Secondly, they behave in the same way as the topological Euler characteristic under unions and products of spaces

$$e^{p,q}(X_1 \amalg X_2) = e^{p,q}(X_1) + e^{p,q}(X_2) \tag{34}$$

$$e^{p,q}(X_1 \times X_2) = \sum_{\substack{p_1+p_2=p\\ q_1+q_2=q}} e^{p_1,q_1}(X_1) \cdot e^{p_2,q_2}(X_2)\,. \tag{35}$$

Hence knowing the numbers $e^{p,q}(X_{\Theta^{[n-d]}})$ (and of $(\mathbb{C}^*)^n$) is sufficient to compute the Hodge numbers of a toric hypersurface. This information has been supplied (in the form of an algorithm) in the work of [1]. Before reviewing their algorithm, let us first discuss toric varieties themselves to illustrate the method. We have that

$$e^{p,q}((\mathbb{C}^*)^n) = \delta_{p,q}(-1)^{n+p}\binom{n}{p}\,. \tag{36}$$

For a smooth toric variety V, we hence see that $h^{p,0} = 0$ for $p > 0$. A direct consequence of this formula combined with the stratification of a toric variety read off from its fan is the standard formula [8] for the nontrivial Hodge numbers of a smooth n-dimensional toric variety. Letting $|\Sigma(l)|$ denote the number of l-dimensional cones in Σ we can write

$$h^{p,p}(\mathbb{P}_\Sigma) = \sum_{k=0}^{n} (-1)^{p+k}\binom{k}{p}|\Sigma(n-k)|\,. \tag{37}$$

4.3 Hodge Deligne Numbers of Hypersurface Strata

For the smooth hypersurface strata $X_{\Theta^{[k]}}$, the following relations, shown in [1], allow to compute the Hodge-Deligne numbers. First we have that for

$p > 0$

$$e^{p,0}(X_{\Theta^{[k]}}) = (-1)^{k-1} \sum_{\Theta^{[p+1]} \leq \Theta^{[k]}} \ell^*(\Theta^{[p+1]}) \tag{38}$$

where $\ell^*(\Theta^{[p+1]})$ counts the number of lattice points in the relative interior of $\Theta^{[p+1]}$ and the sum runs over all faces $\Theta^{[p+1]}$ of dimension $p+1$ contained in the face $\Theta^{[k]}$. Note that this sum only has one term for $p+1 = k$ which corresponds to the face $\Theta^{[k]}$ itself.

The remaining Hodge-Deligne numbers satisfy the 'sum rule'

$$(-1)^{k-1} \sum_q e^{p,q}(X_{\Theta^{[k]}}) = (-1)^p \binom{k}{p+1} + \varphi_{k-p}(\Theta^{[k]}) \,, \tag{39}$$

where the function $\varphi_n(\Theta^{[k]})$ is defined by

$$\varphi_n(\Theta^{[k]}) := \sum_{j=1}^{n} (-1)^{n+j} \binom{k+1}{n-j} \ell^*(j\Theta) \,. \tag{40}$$

Here $j\Delta$ denotes the polytope that is found by scaling all of the vertices of Δ by j, and $j\Theta$ is the image of Θ under this scaling.

For a face of dimension $k \geq 4$ there is the simple formula

$$e^{k-2,1}(X_\Theta^{[k]}) = (-1)^{k-1} \left(\varphi_2(\Theta^{[k]}) - \sum_{\Theta^{[k-1]} \leq \Theta^{[k]}} \varphi_1(\Theta^{[k-1]}) \right) . \tag{41}$$

Finally, for higher Hodge numbers, $p+q \geq n$, one has

$$e^{p,q}(X_{\Theta^{[n]}}) = \delta_{p,q} (-1)^{n+p+1} \binom{n}{p+1} \,. \tag{42}$$

By subsequent application of these formulae, one can derive combinatorial formulae for the Hodge-Deligne numbers of strata $X_{\Theta^{[k]}}$ for arbitrarily high k. It is convenient to use $\ell^n(\Theta)$ to denote the number of points on the n-skeleton of Θ. Let us derive the Hodge-Deligne numbers $e^{p,q}$ of strata $X_{\Theta^{[k]}}$ for $k \leq 4$.

First,

$$e^{0,0}(X_{\Theta^{[k]}}) = (-1)^{k-1} \left(\ell^1(\Theta^{[k]}) - 1 \right) \,, \tag{43}$$

can be shown by combining the 'sum rule' for $p = 0$ with the relations (38) (note that these require $p > 0$), so that $e^{0,0}(X_{\Theta^{[1]}}) = \ell^*(\Theta^{[1]}) + 1$, i.e. the stratum $X_{\Theta^{[1]}}$ consists of $\ell^*(\Theta^{[1]}) + 1$ points. Hence

$$e^{p,q}(X_{\Theta^{[1]}}) = \begin{array}{|cc} 0 & 0 \\ \ell^*(\Theta^{[1]}) + 1 & 0 \\ \hline \end{array} \tag{44}$$

For $X_{\Theta^{[2]}}$, we immediately find $e^{1,1}(X_{\Theta^{[2]}}) = 1$. Furthermore, $e^{1,0}(X_{\Theta^{[2]}}) = -\ell^*(\Theta^{[2]})$, and we can write

$$e^{p,q}(X_{\Theta^{[2]}}) = \begin{array}{|cc} -\ell^*(\Theta^{[2]}) & 1 \\ 1 - \ell^1(\Theta^{[2]}) & -\ell^*(\Theta^{[2]}) \\ \hline \end{array} \tag{45}$$

Similarly, we find for $X_{\Theta^{[3]}}$ that

$$\begin{array}{|ccc} \ell^*(\Theta^{[3]}) & 0 & 1 \\ \ell^2(\Theta^{[3]}) - \ell^1(\Theta^{[3]}) & -3 + \ell^*(2\Theta^{[3]}) - 4\ell^*(\Theta^{[3]}) - \ell^2(\Theta^{[3]}) + \ell^1(\Theta^{[3]}) & 0 \\ \ell^1(\Theta^{[3]}) - 1 & \ell^2(\Theta^{[3]}) - \ell^1(\Theta^{[3]}) & \ell^*(\Theta^{[3]}) \\ \hline \end{array} \tag{46}$$

and for $X_{\Theta^{[4]}}$

$$\begin{aligned} e^{0,0}(X_{\Theta^{[4]}}) &= 1 - \ell^1(\Theta^{[4]}) \\ e^{1,0}(X_{\Theta^{[4]}}) &= -\ell^2(\Theta^{[4]}) + \ell^1(\Theta^{[4]}) \\ e^{2,0}(X_{\Theta^{[4]}}) &= -\ell^3(\Theta^{[4]}) + \ell^2(\Theta^{[4]}) \\ e^{2,1}(X_{\Theta^{[4]}}) &= \ell^3(\Theta^{[4]}) - \ell^2(\Theta^{[4]}) - \varphi_2(\Theta^{[4]}) \\ e^{2,2}(X_{\Theta^{[4]}}) &= -4\,. \end{aligned} \tag{47}$$

Let us describe the example of a degree k hypersurface in $\mathbb{P}^2$. The stratification of $\mathbb{P}^2$ has already been discussed in (10), and its Newton polytope Δ_k has vertices $(n, 0), (0, n), (-n, -n)$.

The open dense torus $(\mathbb{C}^*)^2$ in $\mathbb{P}^2$ gives rise to an open dense subset of C_k, i.e. a Riemann surface with a number of points excised. Its Hodge-Deligne numbers are given by

$$\begin{aligned} e^{p,q}(X_{\Delta_k}) &= \begin{array}{|cc} -\ell^*(\Delta_k) & 1 \\ 1 - \ell^1(\Delta_k) & -\ell^*(\Delta_k) \\ \hline \end{array} \\ &= \begin{array}{|cc} -(k-1)(k-2)/2 & 1 \\ 1 - 3k & -(k-1)(k-2)/2 \\ \hline \end{array} \end{aligned} \tag{48}$$

The stratum associated with each of the three 1-faces $\Theta^{[1]}$ of Δ_k consists of

$$1 + \ell^*(\Theta^{[1]}) = k \tag{49}$$

points. We hence find the well-known result $h^{0,0} = 1$ and $h^{1,0} = (k-1)(k-2)/2$ (which is more easily derived using adjunction).

5 Examples

5.1 Hodge Numbers of CY 3-fold Hypersurfaces

To compute the Hodge numbers of Calabi-Yau threefold hypersurfaces, let us first note that they enjoy a stratification

$$\begin{aligned} X_{(\Delta,\Delta^\circ,\Sigma)} = X_\Delta \amalg X_{\Theta^{[3]}} \amalg X_{\Theta^{[2]}} \times [& \underbrace{\sum \mathbb{C}^*}_{\text{points on } \Theta^{\circ[1]}} + \underbrace{\sum pt}_{\text{1-simplices on } \Theta^{\circ[1]}}] \\ \amalg X_{\Theta^{[1]}} \times [& \underbrace{\sum (\mathbb{C}^*)^2}_{\text{points on } \Theta^{\circ[2]}} + \underbrace{\sum \mathbb{C}^*}_{\text{1-simplices on } \Theta^{\circ[2]}} + \underbrace{\sum pt}_{\text{2-simplices on } \Theta^{\circ[2]}}] \end{aligned} \tag{50}$$

after we have resolved the variety $\mathbb{P}_{\Sigma_n(\Delta)}$ using all lattice points on Δ°. Here we have used that the location of cones of Σ within the normal fan $\Sigma_n(\Delta)$ is identical to the location of the corresponding simplex within faces of Δ°. Note that there is no term associated with strata interior to facets of Δ° which are dual to vertices $\Theta^{[0]}$ of Δ. Recall that the polar duality between Δ and Δ° implies a one-to-one correspondence

$$\Theta^{[k]} \leftrightarrow \Theta^{\circ[3-k]} \,. \tag{51}$$

The subvarieties $X_{\Theta^{[k]}}$ have dimension $k-1$: the face $\Theta^{[k]}$ is dual to a cone of dimension $n-k$ which defines a stratum $(\mathbb{C}^*)^k$ and the hypersurface equation reduces this by one. Vertices of Δ $k=0$ correspond to subvarieties of dimension -1, i.e. these are empty. Hence the interior of maximal dimensional cones of $\Sigma_N(\Delta)$ does not contribute to the hypersurface X at all. These cones are nothing but the cones over the facets of Δ°, which means that we can ignore anything which happens in facets of Δ°. This can also be argued from a simple intersection argument.

In the following, we will abbreviate $X_{(\Delta,\Delta^\circ,\Sigma)} = X$ in order not to clutter the notation to badly.

We may now simply compute $h^{1,1}(X) = h^{2,2}(X) = e^{2,2}(X)$ by summing the numbers $e^{2,2}$ of the individual strata. In the expressions below, $\Theta^{\circ[k]}$ of Δ° is considered to be the dual to the face $\Theta^{[4-k-1]}$ being summed over.

$$
\begin{aligned}
h^{1,1}(X) = e^{2,2}(X_\Delta) + \sum_{\Theta^{[3]}<\Delta} e^{2,2}(\Theta^{[3]}) + \sum_{\Theta^{[2]}<\Delta} e^{1,1}(\Theta^{[2]})\ell^*(\Theta^{\circ[1]}) \\
+ \sum_{\Theta^{[1]}<\Delta} e^{0,0}(\Theta^{[1]}) \cdot \ell^*(\Theta^{\circ[2]}) \cdot e^{2,2}((\mathbb{C}^*)^2) \\
= -4 + \sum_{\Theta^{[3]}<\Delta} 1 + \sum_{\Theta^{[2]}<\Delta} \ell^*(\Theta^{\circ[1]}) + \sum_{\Theta^{[1]}<\Delta} (\ell^*(\Theta^{[1]})+1)\ell^*(\Theta^{\circ[2]}) \\
= \ell(\Delta) - 5 - \sum_{\Theta^{\circ[3]}<\Delta^\circ} \ell^*(\Theta^{\circ[3]}) + \sum_{\Theta^{\circ[2]}<\Delta^\circ} \ell^*(\Theta^{[1]})\ell^*(\Theta^{\circ[2]})\,.
\end{aligned}
\tag{52}
$$

In the last line we have used that we may associate a unique vertex to each face $\Theta^{[3]}$ and that we can decompose all points on Δ° according to the face containing them in their relative interior.

The above can also be interpreted as toric divisors meeting the hypersurface modulo linear relations, plus a correction term for reducible divisors (more on thosein a bit).

Similarly, we can compute $h^{2,1}(X) = -e^{2,1}(X)$. As $e^{2,1}(X_{\Theta^{[3]}}) = 0$, we find

$$
e^{2,1}(X) = e^{2,1}(X_\Delta) + \sum_{\Theta^{[2]}<\Delta} e^{1,0}(X_\Theta^{[2]}) \cdot \ell^*(\Theta^{\circ[1]}) \cdot e^{1,1}(\mathbb{C}^*) \tag{53}
$$

$$
= \ell^3(\Delta) - \ell^2(\Delta) - \varphi_2(\Delta) - \sum_{\Theta^{[2]}<\Delta} \ell^*(\Theta^{[2]})\ell^*(\Theta^{\circ[1]}) \tag{54}
$$

$$
= \sum_{\Theta^{[3]}<\Delta} \ell^*(\Theta^{[3]}) + 5 - \ell(\Delta) - \sum_{\Theta^{[2]}<\Delta} \ell^*(\Theta^{[2]})\ell^*(\Theta^{\circ[1]})\,. \tag{55}
$$

Here, we have used that because $\ell^*(2\Delta) = \ell(\Delta)$ for a reflexive polytope and

$\ell^*(\Delta) = 1$, the function $\varphi_2(\Delta)$ is simply given by

$$\varphi_2(\Delta) = \sum_{j=1}^{2} (-1)^j \binom{5}{2-j} \ell^*(j\Delta) \tag{56}$$

$$= -5\ell^*(\Delta) + \ell^*(2\Delta) \tag{57}$$

$$= -5 + \ell(\Delta)\,. \tag{58}$$

Hence we end up with the final result

$$h^{2,1}(X) = \ell(\Delta) - 5 - \sum_{\Theta^{[3]}<\Delta} \ell^*(\Theta^{[3]}) + \sum_{\Theta^{[2]}<\Delta} \ell^*(\Theta^{[2]})\ell^*(\Theta^{\circ[1]})\,. \tag{59}$$

This is interpreted as monomials defining the hypersurface equation modulo automorphisms of the toric ambient space, plus a correction term for non-polynomial deformations.

5.2 Topology of Toric Divisors of CY 3-folds

In this section we derive combinatorial formulas for the Hodge numbers $h^{0,i}$ for toric divisors of Calabi-Yau threefolds. The Hodge numbers $h^{1,1}$ of toric divisors depend on the triangulation data and are captured by slightly more complicated expression, which can however also easily be obtained from the methods used, see [4] for details.

5.2.1 Vertices

A divisor Y_i for which ν_i is a vertex, dual to $\Theta^{[3]}$, has a stratification

$$Y_i = X_{\Theta^{[3]}} \amalg_{\Theta^{[2]}<\Theta^{[3]}} X_{\Theta^{[2]}} \times (pt) \amalg_{\Theta^{[1]}<\Theta^{[3]}} X_{\Theta^{[1]}} \times \left(\sum \mathbb{C}^* \amalg \sum pt\right)\,. \tag{60}$$

Here the pt multiplying $X_{\Theta^{[2]}}$ originates from the unique 1-simplex on $\Theta^{\circ[1]}$ dual to $\Theta^{[2]}$ ($d = 2, l = 2$ in (31)). The $\mathbb{C}^*$s multiplying $X_{\Theta^{[1]}}$ originate from 1-simplices and the pts from 2-simplices on each face $\Theta^{\circ[2]}$ dual to $\Theta^{[1]}$.

With the stratification (60) at hand, we can start computing the Hodge numbers. As $X_{\Theta^{[3]}}$ is an irreducible open complex surface, also the divisor Y_i

is irreducible. For $h^{1,0}(Y_i)$, only the first two strata contribute and we find

$$\begin{aligned}
h^{1,0}(Y_i) &= -e^{1,0}(Y_i) \\
&= -\left(e^{1,0}(X_{\Theta^{[3]}}) + \sum_{\Theta^{[2]}<\Theta^{[3]}} e^{1,0}(X_{\Theta^{[2]}})\right) \\
&= -\left(\sum_{\Theta^{[2]}<\Theta^{[3]}} \ell^*(\Theta^{[2]}) - \sum_{\Theta^{[2]}<\Theta^{[3]}} \ell^*(\Theta^{[2]})\right) \quad = 0
\end{aligned} \tag{61}$$

For $h^{2,0}$, only the first stratum in (60) contributes and we find

$$h^{2,0}(Y_i) = e^{2,0}(X_{\Theta^{[3]}}) = \ell^*(X_{\Theta^{[3]}})\,. \tag{62}$$

We can hence think of such divisors as 'generic' hypersurfaces which have $h^{1,0} = 0$.

5.2.2 Divisors Interior to 1D Faces of Δ°

Divisors originating from points ν_i interior to edges $\Theta^{\circ[1]}$ of Δ° dual to two-dimensional faces $\Theta^{[2]}$ of Δ have a stratification

$$Y_i = X_{\Theta^{[2]}} \times (\mathbb{C}^* + 2pts)\,\amalg_{\Theta^{[1]}<\Theta^{[2]}}\, X_{\Theta^{[1]}}\left(\sum \mathbb{C}^* + \sum pt\right)\,. \tag{63}$$

Here, the $\mathbb{C}^*$ multiplying $X_{\Theta^{[2]}}$ is due to the point itself ($k = 2, l = 1$), whereas the 2 points correspond to the two 1-simplices on $\Theta^{\circ[1]}$ containing ν_i ($k = 2, l = 2$). Each $\mathbb{C}^*$ multiplying $X_{\Theta^{[1]}}$ corresponds to a 1-simplex containing ν_i which is interior to the dual $\Theta^{\circ[2]}$ and each pt corresponds to a 2-simplex containing ν_i which is interior to the dual $\Theta^{\circ[2]}$.

Again, $h^{0,0}(Y_i) = 1$ as Y_i is irreducible. The computation for $h^{1,0}$ now becomes

$$\begin{aligned}
h^{1,0}(Y_i) &= -e^{1,0}(X_{\Theta^{[2]}} \times (\mathbb{C}^* \amalg 2pts)) \\
&= \ell^*(\Theta^{[2]}) \cdot (e^{0,0}(\mathbb{C}^*) + 2e^{0,0}(pt)) \\
&= \ell^*(\Theta^{[2]})\,.
\end{aligned} \tag{64}$$

We have $h^{2,0}(Y_i) = e^{2,0}(Y_i) = 0$ as no stratum contributes. Already for the highest stratum $X_{\Theta^{[2]}}$, we have to count interior points to 3-dimensional faces of $\Theta^{[2]}$, of which there are none.

As is already apparent from the stratification, divisors associated with lattice points interior to edges of Δ° result from a blow up over a Riemann surface. We can think of them as ruled ($\mathbb{P}^1$ fibred) surfaces, with reducible fibres in general, over a Riemann surface of genus determined by the dual 2D face of Δ as $g = \ell^*(\Theta^{[2]})$.

5.2.3 Divisors Interior to 2D faces of Δ°

Divisors originating from points ν_i interior to 2-dimensional faces $\Theta^{\circ[2]}$ of Δ° dual to 1-dimensional faces $\Theta^{[1]}$ of Δ have a stratification

$$Y_i = X_{\Theta^{[1]}} \times \left((\mathbb{C}^*)^2 + \sum \mathbb{C}^* + \sum pt \right) , \tag{65}$$

where the $(\mathbb{C}^*)^2$, $\mathbb{C}^*$ and pt originate from 0, 1 and 2 simplices interior to $\Theta^{[2]}$ containing ν_i. As $X_{\Theta^{[1]}}$ is a 0-dimensional stratum made up of $\ell^*(\Theta^{[1]}) + 1$ points, such divisors are reducible with $\ell^*(\Theta^{[1]})+1$ components, each of which is toric (the corresponding variety is determined by the fan $star(\nu_i)$. Hence $h^{1,0}(Y_i) = h^{2,0}(Y_i) = 0$.

As is apparent from the stratification, such divisors are in fact $\ell^*(\Theta^{[1]}) + 1$ copies of a toric variety. The same is true for lower-dimensional strata contained in two-dimensional faces of Δ°.

5.2.4 Comments and Generalizations

Let us finish the discussion of threefolds with some comments and hints at generalizations.

First, note that we may use a similar logic to talk about curves which are the intersections of divisors, which allows us to immediately read off their topology. E.g. vertices meet adjacent vertices or adjacent points on edges in Riemann surfaces of genus $g = \ell^*(\Theta^{[2]})$, and points interior to 2D faces in $\ell^*(\Theta^{[1]}) + 1$ copies of $\mathbb{P}^1$.

Anything associated to a $2D$ face $\Theta^{\circ[2]}$ of Δ° always has a factor $X_{\Theta^{[1]}}$ where $\Theta^{[1]}$ is the dual face of $\Theta^{\circ[2]}$. As $X_{\Theta^{[1]}}$ is just a copy of $1+\ell^*(\Theta^{[1]})$ points, there is a 'multiplicity' attached to every 2D face. We can hence thing of 2D faces as being 'puffed up' into $1 + \ell^*(\Theta^{[1]})$ equivalent layers.

The relations we have shown in fact generalize to arbitrary dimension, where for an n-dimensional polytope divisors contained in k-dimensional faces received a non-trivial contribution to $h^{p,0}$ only for $p = n - k - 2$, see [4] for a proof.

In particular, divisors interior to faces of dimension $n - 2$ are potentially reducible and toric, as such faces are always dual to edges of Δ°. This can also be argued from the hypersurface directly by studying the defining equation when setting the associated homogeneous coordinate to zero.

5.3 K3 surfaces

5.3.1 Lattice Polarized Families and Mirror Symmetry

In this section, we'll discuss how to find divisors and their intersection on two-dimensional toric hypersurfaces (i.e. $n = 3$), which are families K3 surfaces. Such an analysis was first done in [9], and we'll rederive it using stratification.

For a three-dimensional reflexive polytope (of which there are 4319), a generic hypersurface is a K3 surface. These are hyper-Kähler surfaces and their moduli space of Ricci-flat metrics is given by a Grassmanian of three-planes in $\mathbb{R}^{3,19}$. One can think of the three-plane as spanned by the Kähler form J and the (real and imaginary parts of the) holomorphic two-form Ω, but may freely choose which one to call which.

For hypersurfaces in toric varieties, however, Kählerity is inherited from the ambient space, and we end up with a family of lattice-polarized K3 surfaces, i.e. there is a lattice N that is fixed by the ambient space and Ω must be such that

$$\Omega \perp N \qquad \Omega \cdot \Omega = 0 \qquad \Omega \cdot \bar{\Omega} = 1 \tag{66}$$

up to rescaling.

Let us try to describe N for a K3 hypersurface in a toric variety. Here, every lattice point ν_i on the one-skeleton of Δ° has a non-zero intersection with the hypersurface X. Let us denote the divisors of the ambient toric variety by D_i and their restriction to X by $Y_i := D_i \cap X$.

We can describe the generators of N as follows:

- For a vertex ν, we get an irreducible divisor Y of genus $g = \ell^*(\Theta^{[2]})$ where $\Theta^{[2]}$ is the dual 2D face on Δ. This implies that $Y^2 = -2 + 2g$.
- For a point interior to an edge $\Theta^{\circ[1]}$ of Δ°, we get $\ell^*(\Theta^{[1]}) + 1$ disjoint $\mathbb{P}^1$s, so that $Y = \sum_\alpha Y_\alpha$. with $Y_\alpha^2 = -2$.

As no strata interior to facets meet X, intersections can only happen along edges of Δ°. Note that this makes the relevant part of the triangulation unique as 1D faces have a unique triangulation.

- Pairs of divisors from vertices which have a one-simplex connecting them have intersection $\ell^*(\Theta^{[1]}) + 1$ where $\Theta^{[1]}$ is the dual of the face connecting them.
- For ν_1 such that $Y_1 = \sum_\alpha Y_{1,\alpha}$ is connected to a vertex ν_2, we have $Y_2 \cdot Y_{1,\alpha} = 1$.
- For neighbooring Y_1, Y_2 inside an edge we have $Y_{2,\beta} \cdot Y_{1,\alpha} = \delta_{\alpha\beta}$.

After modding out by the linear relations, these intersections determine a sublattice $N' \subset N$. The reason we cannot conclude that they give N immediately is that there may be other divisors not given as a (component of a) restriction of toric divisors. However, we can use mirror symmetry to argue this is not the case. For the setup just discussed, there is another lattice N_{tor} which is spanned by the image of the restriction map from the ambient topriv variety, i.e. for $Y_1 = \sum_\alpha Y_{1,\alpha}$ only $Y_i \in N_{tor}$. As a consequence of Batyrev's version of mirror symmetry (with a physics proof implied by [10]) X_{Δ,Δ° and $X_{\Delta^\circ,\Delta}$ should give mirror families, which for lattice polarized families implies that [11]

$$N(\Delta, \Delta^\circ) \oplus N_{tor}(\Delta^\circ, \Delta) \oplus U \tag{67}$$

has a primitive embedding into $H^2(X, \mathbb{Z}) \simeq \Lambda^{3,19}$. Here, U is a copy of the hyperbolic lattice. By working through the 4319 reflexive polytopes this can be explicitly checked to be the case for the N' described above [12, 3], so that $N' = N$ (such a relation could not hold if it was a sublattice).

For a K3 surface, shrinking cycles with the topology of $\mathbb{P}^1$ results in an ADE singularity with the Cartan given by (minus) the inner form. One can hence 'read off' potential singularities from the polytope and it follows from the above description that the Dynkin diagram of the ADE group must be visible in the polytope, see [9] for examples of this.

5.3.2 K3 Fibrations

5.3.3 Tops

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