Wavefront Sets, Local Descents and Spectrum

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Nilpotent Orbits and Degenerate Whittaker Models

G: reductive algebraic group defined over a local field F of char. 0.

Write $G = G(F)$ and $\mathfrak{g}$ is the Lie algebra of G.

$\mathcal{N}_F(\mathfrak{g})$ is the set of F-rational nilpotent elements in $\mathfrak{g}(F)$.

$\mathcal{N}_F(\mathfrak{g})_\circ$ is the set of all F-rational $\mathrm{Ad}(G)$-orbits in $\mathcal{N}_F(\mathfrak{g})$.

$\mathcal{N}_F(\mathfrak{g})_\circ^{\mathrm{st}}$ is the set of F-stable orbits from $\mathcal{N}_F(\mathfrak{g})_\circ$.

Classical Theorem: The set $\mathcal{N}_F(\mathfrak{g})_\circ^{\mathrm{st}}$ is finite and is parameterized by combinatorics data (partitions or Bala-Carter data).

The classification of the set $\mathcal{N}_F(\mathfrak{g})_\circ$ is also known via various methods by many people.

Nilpotent Orbits and Degenerate Whittaker Models

For $X \in \mathcal{N}_F(\mathfrak{g})$, there is an $\mathfrak{sl}_2$-triple $\{X, \hbar, Y\}$.

Via $\mathrm{ad}(\hbar)$, $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\ \mathfrak{g}_i^\hbar$ with $\mathfrak{g}_i^\hbar := \{x \in \mathfrak{g} \colon \mathrm{ad}(\hbar)(x) = ix\}$.

Denote by $\mathfrak{u} = \oplus_{i\le -1}\ \mathfrak{g}_i^\hbar$, $\mathfrak{p} = \oplus_{i\le 0}\ \mathfrak{g}_i^\hbar$, and $\mathfrak{m} = \mathfrak{g}_0^\hbar$.

Denote by $U_X = U = \exp(\mathfrak{u})$, and similar P and M.

$P = M \ltimes U$ is a parabolic of G, depending on $\mathcal{O}_X^{\mathrm{st}}$ up to conjugate.

Take $\mathfrak{u}_{X,2} = \oplus_{i\le -2}\ \mathfrak{g}_i^\hbar$ and $U_{X,2} = \exp(\mathfrak{u}_{X,2})$.

Fix a character $\psi_0 \ne 1$ of F and an inv. nondeg. bilinear form κ.

$$\psi_X(\exp(A)) = \psi_0(\kappa(X, A)) \qquad \text{for } A \in \mathfrak{u}_{X,2}$$

defines a character of $U_{X,2}$.

Nilpotent Orbits and Degenerate Whittaker Models

If $\mathfrak{g}_{-1} \neq 0$, $\kappa_{-1}(A, B) := \kappa(\mathrm{ad}(X)A, B)$ for $A, B \in \mathfrak{g}_{-1}$ defines a symplectic form.

$\mathcal{H}_X = \mathfrak{g}_{-1} \times F$ is a Heisenberg group with a surjective morphism: $\alpha_X \colon U_X \mapsto \mathcal{H}_X$ given by

$$\alpha_X(\exp(A)\exp(Z)) = (A, \kappa(X, Z)) \qquad \text{for } A \in \mathfrak{g}_{-1}, Z \in \mathfrak{u}_{X,2}.$$

ψ_X factors through $U_{X,2}$ and defines a central character of $\mathcal{H}_X$.

The smooth oscillator representation $(\omega_{\psi_X}, V_{\psi_X})$ of $\mathcal{H}_X$ is also a representation of U_X.

$\Pi_F(G)$: the set of equiv. classes of irreducible smooth representations of $G(F)$, which is of Casselman-Wallach type if F is archimedean.

Nilpotent Orbits and Degenerate Whittaker Models

Following Mœglin-Waldspurger (1987) and Gomez-Gourevitch-Sahi (2017), for $(\pi, V_\pi) \in \Pi_F(G)$, define **degenerate Whittaker modules** of π:

$$\mathcal{J}_X(\pi) = V_\pi \widehat{\otimes} V_{\psi_X}^\vee \ / \ \overline{\{\pi \otimes \omega_{\psi_X}^\vee(u)v - v \mid u \in U_X, v \in V_\pi \widehat{\otimes} V_{\psi_X}^\vee\}}.$$

The nonvanishing of $\mathcal{J}_X(\pi)$ depends on the F-rational nilpotent orbit of X.

Define the **algebraic wavefront set** of π by

$$\mathrm{WF}_{\mathrm{wm}}(\pi) := \{\mathcal{O} \in \mathcal{N}_F(\mathfrak{g})_\circ \mid \mathcal{J}_X(\pi) \neq 0, \text{ for some } X \in \mathcal{O}\}.$$

Question: How to understand and compute this invariant of π?

Wavefront Sets of Representations

R. Howe (1981) introduced the notion of **wavefront sets** for representations of Lie groups through the singular support of the distribution characters.

It is an important geometric invariant that describes the size and other properties of the representations.

Since then, variants of wavefront sets were considered by various people from different perspectives.

I refer to D. Vogan's Takagi Lecture 2016

The size of infinite-dimensional representations

for more detailed discussion of the significance of wavefront sets and their relatives.

Wavefront Sets of Representations

R. Howe (1974) and Harish-Chandra (1978): When F is p-adic, the distribution character Θ_π has an expansion near the identity:

$$\Theta_\pi = \sum_{\mathcal{O} \in \mathcal{N}_F(\mathfrak{g})_\circ} c_{\mathcal{O}} \cdot \widehat{\mu}_{\mathcal{O}}, \quad c_{\mathcal{O}} \in \mathbb{C},$$

where $\widehat{\mu}_{\mathcal{O}}$ is the Fourier transform of the measure $\mu_{\mathcal{O}}$ on $\mathcal{O}$.

Define the **analytic wavefront set** of $\pi \in \Pi_F(G)$:

$$\mathrm{WF}_{\mathrm{tr}}(\pi) := \{\mathcal{O} \in \mathcal{N}_F(\mathfrak{g})_\circ \mid c_{\mathcal{O}} \neq 0\}.$$

Mœglin-Waldspurger (1987) show:

$$\mathrm{WF}_{\mathrm{wm}}(\pi)^{\max} = \mathrm{WF}_{\mathrm{tr}}(\pi)^{\max}$$

Wavefront Sets of Representations

By Barbasch-Vogan (1980), one may ask for the same equality

$$\mathrm{WF}_{\mathrm{wm}}(\pi)^{\max} = \mathrm{WF}_{\mathrm{tr}}(\pi)^{\max}$$

for $\pi \in \Pi_F(G)$ when F is archimedean.

However, in the archimedean case, this equality is not fully understood, except several special cases considered in D. Vogan (1978), H. Matumoto (1987, 1990, 1992), H. Yamashita (2001), D. Gourevitch-S. Sahi (2015).

In particular, R. Gomez-D. Gourevitch-S. Sahi (2017) show the equality holds when G is a complex reductive group, and it is true when $G = \mathrm{GL}_n$ and $F = \mathbb{R}$.

N. Li (2022) proves the equality for some irreducible constituents of certain degenerate principal series representations of $\mathrm{Sp}_{2n}(\mathbb{R})$.

Wavefront Sets of Representations

In 1996, C. Mœglin proves that the orbits in $\mathrm{WF}_{\mathrm{wm}}(\pi)^{\max}$ are special in the sense of Lusztig for any $\pi \in \Pi_F(G)_{\mathrm{temp}}$ when F is p-adic and G is classical.

By using the argument of raising nilpotent orbits, D. Jiang-B. Liu-G. Savin (2016) generalizes the results of Mœglin. The method is applicable to the automorphic case.

Some partial results analogous to the results above in the archimedean case was obtained in A. Joseph (1980), D. Barbasch-D. Vogan (1982, 1983), B. Harris (2012).

Gomez-Gourevitch-Sahi (2021) prove the most general result so far: the orbits in $\mathrm{WF}_{\mathrm{wm}}(\pi)^{\max}$ are quasi-admissible for general reductive groups and all local fields of char. zero.

For classical groups, *quasi-admissible* is equivalent to *special*.

Wavefront Sets of Representations

Mœglin-Waldspurger (1987) posed the following *deep conjecture:*

The nilpotent orbits in $\mathrm{WF}_{\mathrm{wm}}(\pi)^{\max}$ are associated to one single F-stable orbit.

It is also expected to be true when π is automorphic.

It is known only for some special cases.

When $G = \mathrm{GL}_n$, it is known by Mœglin-Waldspurger (1987) when F is p-adic and by Gomez-Gourevitch-Sahi (2017) when F is archimedean.

For the automorphic case, some special cases were known through the work of J. Shalika (1974), I. Piatetski-Shapiro (1979), D. Jiang-B. Liu (2013), B.-Liu-B. Xu (2021) for $G = \mathrm{GL}_n$; and D. Ginzburg-D. Soudry (2022) for some residual spectrum of Sp_{2n}.

Computing Wavefront Sets

Motivated by the understanding of the unitary dual via the orbit method, it is important to compute $\mathrm{WF}_{\mathrm{tr}}(\pi)^{\max}$ for $\pi \in \Pi_F(G)$.

Adams-Vogan (2021) provides an (atlas) algorithm to compute the wavefront sets or associated varieties for $\pi \in \Pi_{\mathbb{R}}(G)$ by using the Kazhdan-Lusztig theory, the Knapp-Zuckerman theory, the equivariant K-theory, and the work of W. Schmid-K. Vilonen.

If F is p-adic and π unipotent representations in the sense of Arthur (1984), Waldspurger (2018,2019,2020) determines the wavefront sets for SO_{2n+1} and Ciubotaru, Mason-Brown, and Okada (2021) determines that for general reductive groups, but for Iwahori-spherical π, using the Lusztig classification of unipotent representations (1995, 2002).

D. Jiang-L. Zhang (2018) aimed to understand (and computing) $\mathrm{WF}_{\mathrm{wm}}(\pi)^{\max}$ by using local descent (I will explain below).

Arithmetic Wavefront Sets

D. Jiang, D. Liu, and L. Zhang (2022) introduce the **arithmetic wavefront set** $\mathrm{WF}_{\mathrm{ari}}(\pi)$ for $\pi \in \Pi_F(G)$, for classical groups G over any local field of char. zero (assuming that π has a generic local L-parameter at this moment).

The definition involves the local Langlands parameter of π, the relevant local root numbers (the Gan-Gross-Prasad conjecture) and the combinatorics of relevant F-rational nilpotent orbits.

Wavefront Set (WFS) Conjecture:

$$\mathrm{WF}_{\mathrm{ari}}(\pi)^{\max} = \mathrm{WF}_{\mathrm{wm}}(\pi)^{\max} = \mathrm{WF}_{\mathrm{tr}}(\pi)^{\max}.$$

The **WFS Conjecture** is helpful for us to understand either $\mathrm{WF}_{\mathrm{wm}}(\pi)^{\max}$ or $\mathrm{WF}_{\mathrm{tr}}(\pi)^{\max}$ because the arithmetic wavefront set $\mathrm{WF}_{\mathrm{ari}}(\pi)^{\max}$ can be computed in terms of the arithmetic datum (ϕ, χ).

Arithmetic Wavefront Sets

If F is p-adic, Mœglin-Waldspurger (1987) show

$$\mathrm{WF}_{\mathrm{wm}}(\pi)^{\max} = \mathrm{WF}_{\mathrm{tr}}(\pi)^{\max}$$

It is enough to show either

$$\mathrm{WF}_{\mathrm{ari}}(\pi)^{\max} = \mathrm{WF}_{\mathrm{wm}}(\pi)^{\max}$$

or

$$\mathrm{WF}_{\mathrm{ari}}(\pi)^{\max} = \mathrm{WF}_{\mathrm{tr}}(\pi)^{\max}.$$

The **WFS Conjecture** can be verified for some lower rank groups, like SO_7.

It can also be verified for some families of tempered unipotent representations π of SO_{2n+1}, based on the work of Waldspurger (2018, 2019, 2020).

Arithmetic Wavefront Sets

For F-quasisplit classical groups, we show that

$$\mathrm{WF}_{\mathrm{ari}}(\pi)^{\max} = \mathrm{WF}_{\mathrm{wm}}(\pi)^{\max}$$

when π is generic. This implies the **WFS Conjecture** for GL_n.

When G is F-split SO_{2n+1}, we can verify

$$\mathrm{WF}_{\mathrm{ari}}(\pi)^{\max} = \mathrm{WF}_{\mathrm{wm}}(\pi)^{\max}$$

if π has Bessel model of special type, but not generic. Those representations are associated to the sub-regular nilpotent orbits.

In general, it is our intention to understand this conjecture by using the theory of local descents as developed in [JZ2018].

Arithmetic Wavefront Sets

Local Langlands Conjecture: Any $\pi \in \Pi_F(G)$ can be written as

$$\pi = \pi(\phi, \chi)$$

with L-parameters ϕ and irreducible representations χ of $\mathcal{S}_\phi$.

More precise formulation: A. Borel (1979) and D. Vogan (1993).

The **LLC** is known for many cases: Langlands (1989, $F = \mathbb{R}$); L. Fargues and P. Scholze (2021, F p-adic), ...

For G classical, and ϕ generic, [JLZ22] defines $\mathcal{Y}(\phi, \chi)$, a set of L-Young tableaux associated to the arithmetic data (ϕ, χ).

Then [JLZ22] defines the **arithmetic wavefront set** $\mathrm{WF}_{\mathrm{ari}}(\pi)$ to a subset of F-rational nilpotent orbits in $\mathcal{N}_F(\mathfrak{g})_\circ$ that correspond canonically with those admissible L-Young tableaux in $\mathcal{Y}(\phi, \chi)$, whose underlying ordered partitions are decreasing.

Arithmetic Wavefront Sets

Arithmetic Wavefront Set (AWFS) Conjecture (JLZ22): Let G be a classical group over F and $\pi = \pi(\phi, \chi)$ with ϕ a generic.

(1) The given arithmetic data (ϕ, χ) of π determines a unique special partition $\underline{p}(\phi, \chi)$, such that $\mathrm{WF}_{\mathrm{ari}}(\pi)^{\max} \subset \mathcal{O}^{\mathrm{st}}_{\underline{p}(\phi,\chi)}$.

(2) The subset $\mathrm{WF}_{\mathrm{ari}}(\pi)^{\max} \subset \mathcal{N}_F(\mathfrak{g})_\circ$ can be completely determined by inductive process that defines the set $\mathcal{Y}(\phi, \chi)$.

(3) The subset $\mathrm{WF}_{\mathrm{ari}}(\pi)^{\mathrm{st}} \subset \mathcal{N}_F(\mathfrak{g})^{\mathrm{st}}_\circ$ is given by the closure $\overline{\{\underline{p}(\phi, \chi)\}}$ of the single special partition $\underline{p}(\phi, \chi)$.

Theorem (Jiang-Liu-Zhang (2022)): The following hold.

1. The arithmetic wavefront set $\mathrm{WF}_{\mathrm{ari}}(\pi)$ is an invariant of π.
2. If $F = \mathbb{R}$, the **AWFS** conjecture holds.

The **AWFS** conjecture for p-adic fields F has been checked for some cases, but the general case is in progress.

It is possible to extend the above discussion to general local Arthur parameters based on our calculation on GL_n.

$\mathrm{WF}_{\mathrm{wm}}(\pi)$ and Local Descents: $G = \mathrm{SO}_{2n+1}(F)$, p-adic F

G is a split classical group, say, $G = \mathrm{SO}_{2n+1}(F)$.

$\mathrm{WF}_{\mathrm{wm}}(\pi)$ consists of orthogonal partitions $\underline{p} = [p_1 p_2 \cdots p_r]$ of $2n+1$, with $p_1 \geq p_2 \geq \cdots \geq p_r > 0$.

Question: For a given $\pi \in \Pi_F(G)$, what is the largest part p_1 such that the partition

$$\underline{p} = [p_1 \cdot 1^*]$$

occurs in $\mathrm{WF}_{\mathrm{wm}}(\pi)$?

For $G = \mathrm{SO}_{2n+1}(F)$, such a partition must be of the form

$$\underline{p}_\ell := [(2\ell+1)1^{2(n-\ell)}]$$

with $\ell = 0, 1, \cdots, n$.

$\mathrm{WF}_{\mathrm{wm}}(\pi)$ and Local Descents: $G = \mathrm{SO}_{2n+1}(F)$, p-adic F

When $\underline{p}_\ell$ corresponds to $\mathcal{O}_\ell$, for $X \in \mathcal{O}_\ell$, one may choose

$$U_X = U_{\ell,X} = \{u = \begin{pmatrix} z & x & w \\ 0 & \mathrm{I}_{2(n-\ell)+1} & x' \\ 0 & 0 & z' \end{pmatrix}\}, \quad \mathfrak{g}_{-1} = 0,$$

with z a upper-triangular unipotent in GL_ℓ. The character $\psi_X = \psi_{\ell,X}$ can be chosen to be

$$\psi_{\ell,X}(n) = \psi(z_{12} + \cdots + z_{\ell-1,\ell})\psi(< x_\ell, \alpha_X >)$$

with x_ℓ the last low of x, which is a vector in $F^{2(n-\ell)+1}$, and α_X is an anisotropic vector $F^{2(n-\ell)+1}$, corresponding $X \in \mathcal{O}_\ell$

$M_{\ell,X} = \mathrm{Stab}_{\mathrm{SO}_{2(n-\ell)+1}}(\alpha_X)^\circ$ is an F-rational form of $\mathrm{SO}_{2(n-\ell)}$.

For $\pi \in \Pi_F(G)$, $\mathcal{J}_X(V_\pi)$ is a representation of $M_{\ell,X}$.

$\mathrm{WF}_{\mathrm{wm}}(\pi)$ and Local Descents: $G = \mathrm{SO}_{2n+1}(F)$, p-adic F

Multiplicity One Theorem: For any $\sigma \in \Pi_F(M_{\ell,X})$, we have

$$\mathrm{Hom}_{M_{\ell,X}}(\mathcal{J}_X(V_\pi)\widehat{\otimes}\sigma^\vee, 1) \leq 1.$$

When F is p-adic, it is proved by A. Aizenbud, D. Gourevitch, S. Rallis, and G. Schiffmann (2010) for $\ell = 0$.

For general ℓ, it is proved by W. T.Gan, B. Gross, and D. Prasad (2012), using Frobenius reciprocity law and the case $\ell = 0$.

When F is archimedean, it is proved by B. Sun and C. Zhu (2012) for $\ell = 0$.

For general ℓ, it is proved by Jiang-Sun-Zhu (2010) by using the case $\ell = 0$ and explicit construction of certain intertwining operators.

Local Descents: $G = \mathrm{SO}_{2n+1}(F)$, p-adic F

$$
\begin{array}{ccccc}
 & & & \mathcal{J}_{X_0}(V_\pi) & \mathrm{SO}_{2n} \\
 & & \vdots & \vdots & \vdots \\
 & & & \mathcal{J}_{X_{\ell-1}}(V_\pi)) & \mathrm{SO}_{2n-2\ell+2} \\
 & & \nearrow & & \\
\mathrm{SO}_{2n+1} & \pi & \rightarrow & \mathcal{J}_{X_\ell}(V_\pi)) & \mathrm{SO}_{2n-2\ell} \\
 & & \searrow & & \\
 & & & \mathcal{J}_{X_{\ell+1}}(V_\pi)) & \mathrm{SO}_{2n-2\ell-2} \\
 & & \vdots & \vdots & \vdots \\
 & & & \mathcal{J}_{X_n}(V_\pi) & \mathrm{SO}_0
\end{array}
$$

Local Descents: $G = \mathrm{SO}_{2n+1}(F)$, p-adic F

For $\pi \in \Pi_F(G)$, there is an ℓ_0 s.t. $\mathcal{J}_{X_0}(V_\pi)) \neq 0$ for some $X_0 \in \mathcal{O}_{\ell_0}$, but $\mathcal{J}_X(V_\pi)) = 0$ for any $\ell_0 < \ell \leq n$; any $X \in \mathcal{O}_\ell$.

This integer $\ell_0 = \ell_0(\pi)$ is called the **first occurrence index** of π.

Write $\mathcal{D}_{\ell_0, X_0}(V_\pi) = \mathcal{J}_{X_0}(V_\pi)$ and call it the **local descent** of π.

Jiang-Zhang (2018): If $\pi \in \Pi_F(G)$ is tempered, then $\ell_0 = \ell_0(\pi)$ can be determined by the endoscopic classification data of π as given by J. Arthur 2013.

Conjecture (Jiang-Zhang (2018)): For any $\pi \in \Pi_F(G)$ with generic L-parameter, if p_1 is the largest part in a partition $\underline{p} = [p_1 p_2 \cdots p_r] \in \mathrm{WF}_{\mathrm{wm}}(\pi)^{\max}$, with $p_1 \geq p_2 \geq \cdots p_r > 0$, then

$$p_1 = 2\ell_0 + 1,$$

with $\ell_0 = \ell_0(\pi)$.

Local Descents: $G = \mathrm{SO}_{2n+1}(F)$, p-adic F

Theorem (Jiang-Zhang (2018)): Let G be SO_{2n+1} or SO_{2n}. If $\pi \in \Pi_F(G)$ has a generic L-parameter, then the following hold

(1) The **local descent** $\mathcal{D}_{\ell_0,x_0}(V_\pi)$ is a multiplicity free direct sum of irred. square-integrable representations of M_{ℓ_0,x_0}, with all its irred. summands belonging to different Bernstein components.

(2) **(Spectrum)** The local arithmetic data (ϕ', χ') for each irred. summands in the **local descent** $\mathcal{D}_{\ell_0,x_0}(V_\pi)$ can be completely determined by the descent of local arithmetic data (ϕ, χ) associated to π via the local Langlands correspondence.

The descent of local arithmetic data (ϕ, χ) is a computation based on the local Gan-Gross-Prasad conjecture for classical groups, which is now a theorem of Waldspurger, Mœglin, Beuzart-Plessis, H. Atobe, Gan-Ichino, H. Xue, Z. Luo and C. Chen. (Some special cases were done by H. He (2017); Kobayashi-Speh (2015,2018)).

Local Descents of Non-tempered Case and Functoriality

Interesting examples of local descents with non-tempered $\pi \in \Pi_F(G)$ yield local **Langlands functorial transfers**.

$G_{2n} = \mathrm{SO}_{4n}$ with Levi $M_{2n} = \mathrm{GL}_{2n}$ and maximal parabolic $P = M_{2n}N$ over p-adic F.

$\tau \in \Pi_F(\mathrm{GL}_{2n})$ supercuspidal. Consider the normalized induction

$$\mathrm{Ind}_P^{\mathrm{SO}_{4n}}(\tau|\det|^s).$$

Assume it is reducible at $s = \frac{1}{2}$, with the Langlands quotient $\mathcal{L}(\frac{1}{2}, \tau)$, which is of the **Speh type**.

According to the endoscopic classification of Arthur, $\mathcal{L}(\frac{1}{2}, \tau)$ has the local Arthur parameter $(\tau, 2)$ with sign equal 1.

The wavefront set $\mathrm{WF}_{\mathrm{wm}}(\mathcal{L}(\frac{1}{2}, \tau))^{\mathrm{st}} = \overline{\{[(2n)^2]\}}$.

Local Descents of Non-tempered Case and Functoriality

[Jiang-Soudry 2003; Jiang-Nien-Qin 2010] prove

$\ell_0 = \ell_0(\mathcal{L}(\frac{1}{2}, \tau)) = n - 1.$

There is a $X_{n-1} \in \mathcal{O}_{n-1}$ such that the local descent $\mathcal{D}_{n-1, X_{n-1}}(\mathcal{L}(\frac{1}{2}, \tau))$ is irreducible generic supercuspdal representation of F-split $\mathrm{SO}_{2n+1}(F)$.

The complex dual group of $\mathrm{SO}_{2n+1}(F)$ is $\mathrm{Sp}_{2n}(\mathbb{C})$, which has the natural embedding

$$\mathrm{Sp}_{2n}(\mathbb{C}) \to \mathrm{GL}_{2n}(\mathbb{C}).$$

The local Langlands functorial transfer of the local descent $\mathcal{D}_{n-1, X_{n-1}}(\mathcal{L}(\frac{1}{2}, \tau))$ from SO_{2n+1} to GL_{2n} is the given τ.

Local Descents of Non-tempered Case and Functoriality

The parabolic induction, the local descent and the local functorial transfer form the following commutative diagram:

$$\begin{array}{ccccc} & & & \mathcal{L}(\frac{1}{2},\tau) & \mathrm{SO}_{4n} \\ & & \swarrow & \uparrow & \\ \mathrm{SO}_{2n+1} & \sigma = \mathcal{D}_{n-1,\chi_{n-1}}(\mathcal{L}(\frac{1}{2},\tau)) & \rightarrow & \tau & \mathrm{GL}_{2n} \end{array}$$

$p_1 = 2n - 1$ is the largest part of partitions of type

$$\underline{p} = [p_1 \cdot 1^*] \in \mathrm{WF}_{\mathrm{wm}}(\mathcal{L}(\frac{1}{2},\tau))^{\mathrm{st}}.$$

Those examples of the local Langlands functoriality via the local descent became special cases of the local Gan-Gross-Prasad conjecture (2020) for non-tempered L-parameters.

Local Descents of Non-tempered Case and Functoriality

Parabolic induction, **local descent**, **functoriality**, and **local Howe correspondence** yield the following commutative diagram:

$$
\begin{array}{ccccc}
\mathrm{SO}_{4n} & & & & \mathrm{Sp}_{4n} \\
 & \mathcal{L}^{\mathrm{SO}}_{\frac{1}{2},\tau} & \Longleftrightarrow & \mathcal{L}^{\mathrm{Sp}}_{\frac{1}{2},\tau} & \\
 & & \nwarrow \qquad \nearrow & & \\
 & \downarrow & \tau \in \Pi_F(\mathrm{GL}_{2n})_{sc} & \downarrow & \\
 & & \nearrow \qquad \nwarrow & & \\
 & \mathcal{D}(\mathcal{L}^{\mathrm{SO}}_{\frac{1}{2},\tau}) & \Longleftrightarrow & \widetilde{\mathcal{D}}(\mathcal{L}^{\mathrm{Sp}}_{\frac{1}{2},\tau}) & \\
\mathrm{SO}_{2n+1} & & & & \widetilde{\mathrm{Sp}}_{2n}
\end{array}
$$

Local Descents and Submodule Problem

$G = \mathrm{SO}_{2n+1}$ (same formulation for other classical groups).

Given $\pi \in \Pi_F(G)$, for $1 \leq \ell \leq n$, the twisted Jacquet module $\mathcal{J}_{X_\ell}(\pi)$ is a smooth representation of $M_{\ell, X_\ell}(F)$.

Theorem (Jiang-Zhang (2018)): When F is p-adic and $\pi \in \Pi_F(G)$, if $\mathcal{J}_{X_\ell}(\pi) \neq 0$, then there exists $\sigma \in \Pi_F(M_{\ell, X_\ell})$, s.t.

$$\pi \hookrightarrow \mathrm{Ind}_{R_{\ell, X_\ell}}^{G}(\sigma \otimes \psi_{X_\ell})$$

as a *submodule*, where $R_{\ell, X_\ell} := M_{\ell, X_\ell} \ltimes U_{X_\ell}$.

The idea is to prove: each Bernstein component of $\mathcal{J}_{X_\ell}(\pi)$ is of finite type.

Question: How to prove this type of **Submodule Theorem** when $F = \mathbb{R}$ and $\pi \in \Pi_F(G)$ is tempered and Casselman-Wallach?

Local Descents and Spectrum Problem

From the above discussion of local descents, it suggests that the local descent (or the twisted Jacquet module)

$$\mathcal{D}_{X_{\ell_0}}(\pi) = \mathcal{J}_{X_{\ell_0}}(\pi)$$

has a nicer spectral structure at the first occurrence index $\ell_0(\pi)$.

Jiang-Zhang (2018): If F is p-adic and π has a generic L-parameter, $\mathcal{D}_{X_{\ell_0}}(\pi)$ is a direct sum of discrete series.

Jiang-Soudry (2003); Jiang-Nien-Qin (2010): If F is p-adic, the local descent $\mathcal{D}_{X_{n-1}}(\mathcal{L}(\frac{1}{2}, \tau))$ is an irreducible generic supercuspdal representation.

Question: What can one say about the spectral decomposition of the local descent $\mathcal{D}_{X_{\ell_0}}(\pi)$ at the first occurrence index $\ell_0 = \ell_0(\pi)$ when F is a local field and π is of Arthur type? **(Thanks!)**