Exact Volumes of Semi-Algebraic Convex Bodies

MAX-PLANCK-INSTITUT FÜR MATHEMATIK IN DEN NATURWISSENSCHAFTEN

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Overview

We compute volumes of semi-algebraic convex bodies defined by finitely many concave polynomials with arbitrary precision. This is motivated by geometric statistics, where intersections of convex bodies arise as maximum likelihood estimator (MLE) sets [1].

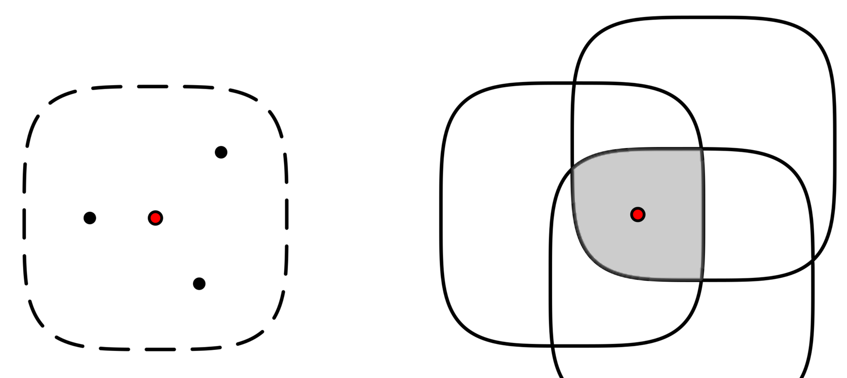

Figure: On the left: An ℓ_4-ball with unknown center θ in $\mathbb{R}^2$ and $p_1, \dots, p_k$ samples. On the right, K is translated to each of the p_i. The MLE set is the shaded region on the right.

Consider a convex body $K + \theta$ with unknown center θ and random samples $X_1, \dots, X_k$ from $K + \theta$. The MLE set is then the bounded convex set

$$C = \cap_{i=1}^{k} (K + X_i).$$

One can think of the MLE set as an analog of the confidence interval in higher dimensions, i.e.

$$\mathrm{vol}(C) = \mathrm{uncertainty}(\mu \mid X_1, \dots, X_k).$$

We present an algorithm for the computation of the volume of these MLE sets based on the approach of [2]. The setting of concave polynomial, however, allows for various simplifications. We provide a full implementation of the algorithm in SageMath available on Github and Zenodo [3].

Volumes as Periods of Rational Functions

Classically, the volume of semi-algebraic sets C is computed by *counting points* in C, for example with a Monte Carlo approach. This amounts to integrate the *volume form* over C. When $C = \{f(x) > 0\}$ for f a smooth polynomial, one can also represent the volume as a period of a rational function:

$$\int_C 1 dx_1 \wedge \ldots \wedge dx_n \quad = \quad \mathrm{Vol}(C) \quad = \quad \frac{1}{2\pi i} \int_\Gamma \frac{x_1 \partial_1 \bullet f(x)}{f(x)} dx_1 \wedge \ldots \wedge dx_n.$$

Whereas in the classical view C is an *open* and *real* integration contour in $\mathbb{R}^n$, the expression as a period of a rational function is an integral over a *closed* and *complex* contour Γ in $\mathbb{C}^n$. This contour is the *Leray coboundary* for $\{x \in \mathbb{R}^n \mid f(x) = 0\}$.

For $f(t, x)$, such that $I(t) = \int_{\Gamma(t)} A(t, x) dx$, D-module theory naturally provides non-trivial differential equations for such parametric periods: Any element of the *integration ideal*

$$D_t \cap (\mathrm{Ann}_{D_{t,\mathbf{x}}}(A(t,x)) + \partial_{x_1} D_{t,\mathbf{x}} + \ldots + \partial_{x_n} D_{t,\mathbf{x}}), \quad \text{where} \quad D_{t,\mathbf{x}} = \mathbb{Q}[t, \mathbf{x}]\langle \partial_t, \partial_{x_1}, \ldots, \partial_{x_n} \rangle,$$

annihilates $I(t)$, provided that the contour $\Gamma(t)$ is locally constant in a suitable sense. Integration ideals formalize *integration-by-parts* identities and are computed by a process called *creative telescoping*.

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Volumes as Solutions of Differential Equations

Volumes of a convex semi-algebraic set $C = \bigcap_i \{f_i > 0\}$ can be expressed as solutions to *univariate linear differential* equations in the following two ways. It relies on realizing C as one of the regions of $\{\prod f_i > 0\}$, therefore placing it in a family of smooth varieties defined by $\prod f_i - t$.

(A) - Deformation:

$$\mathrm{Vol}(C) = \lim_{0 \leftarrow t} \mathrm{Vol}(C_t)$$

(B) - Integration:

$$\mathrm{Vol}(C_\epsilon) = \int_{a(\epsilon)}^{b(\epsilon)} \mathrm{Vol}(C_\epsilon \cap \{x_1 = s\}) ds$$

The volume of the semi-algebraic set (or region thereof) is then realized as the limit respectively the integral of a holonomic function. And the linear differential operator P that annihilates it only depends on the defining polynomials $f_1, \dots, f_k$. If P has order$(P) = r$, then that holonomic function can be solved to *arbitrary precision* by *recursively* providing either r volumes of *deformations* respectively *lower-dimensional* slices.

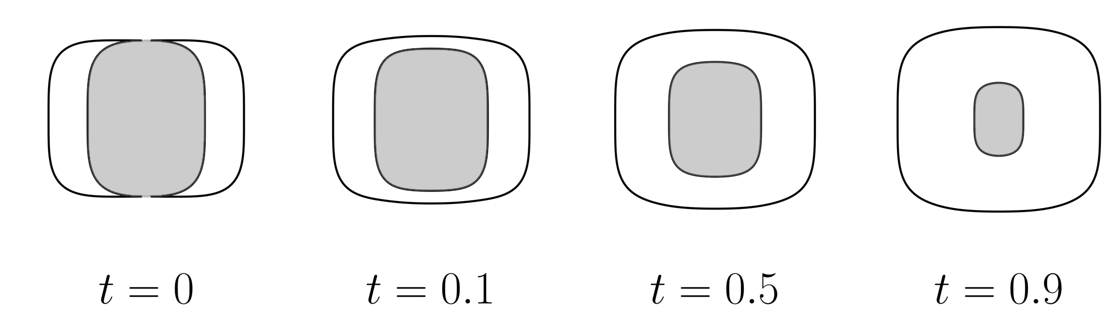

Figure: The intersection of two ℓ_4-balls in $\mathbb{R}^2$ defined by $f_1 = 1 - x^4 - y^4$ and $f_2 = 1 - (x - 1/2)^4 - y^4$ is deformed into the family of smoothly bounded sets $C_t \subset \{\prod f_i - t > 0\}$. $Vol(C_t)$ is annihilated for t small by a linear differential operator P_t of order $\mathrm{ord}(P_t) = 6$ and degree $\deg(P_t) = 18$.

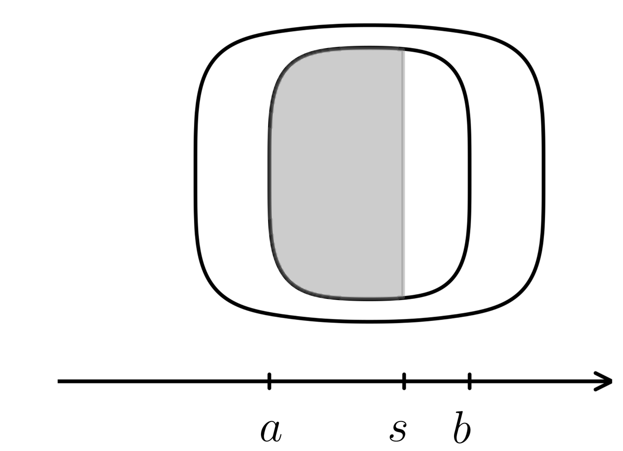

Figure: For f_1, f_2 as on the left and for generic small t_0, the volume of the one-dimensional slices $\mathrm{Vol}(C_{t_0} \cap \{x = s\})$ is annihilated by a linear differential operator P_s of order $\mathrm{ord}(P_s) = 2$ and degree $\deg(P_s) = 23$.

In the above example, the volume of the intersection is then determined by computing a total of 7 Picard–Fuchs operators and 12 one-dimensional slice volumes. Using high-precision solvers for linear ODEs one obtains to arbitrary precision Vol(C) $= 2.708344826299720090001844936980643720528690610575117510163356615319\ldots$

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