A fresh look at the large N limit of Matrix models and holography

Gautam Mandal,
TIFR, Mumbai

Large-N Matrix Models and Emergent Geometry
ESI, Vienna
September 4, 2023

with Ajay Mohan (in progress)

In the last few decades, spacetime and geometry have emerged in many contexts as dual descriptions to matrix valued field theories.

It is expected that a smooth continuum results if one takes a suitable large N limit where N = dimension of the matrix.

In this talk, we will revisit this issue in some examples. We will explore some subtleties of the large N limit and discuss a way to deal with them.

Examples:

AdS/CFT: Here gauge invariant operators of SU(N) Yang-Mills fields are supposed to describe bulk fields in the large N limit. To bring out possible subtleties of this limit we will describe a simpler example.

c=1 matrix model/2D string theory:

$M_{ij}(\mathrm{t}) \to \lambda_i(t), i = 1,2, \dots, N \quad \mathrm{S} = \int dt\, Tr\left(\dot{M}^2 - V(M)\right), V(M) = -M^2$

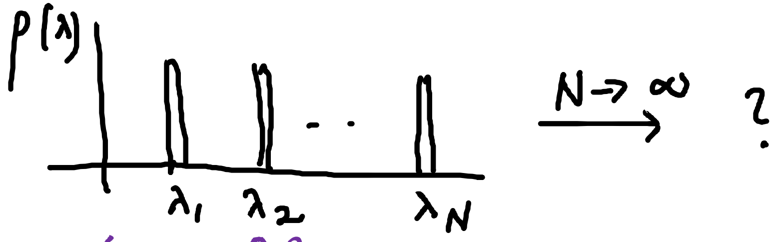
This is equivalent to N free fermions with coordinates $\lambda_i(t)$ in a potential $\sum_i V(\lambda_i)$

$\sum_i \delta\left(\lambda - \lambda_i(t)\right) = \rho(\lambda, t),$ $\quad \underbrace{\delta\rho(\lambda, t)}_{\text{matrix.}} \to \underbrace{T(x, t)}_{\text{2D string}}, \; \lambda = \sqrt{2\mu} \cosh x$ (Das-Jevicki)

Here on the RHS, T(x,t) describes a massless scalar field called the Tachyon, which is the only dynamical field of the 2D string theory. There are many quantities which agree on both sides as $N \to \infty$. (see reviews by Ginsparg, Klebanov; S-matrix=> Polchinski, GM-Sengupta-Wadia, Moore,.. Yin et al, Sen)

However, there are several reasons why such a duality cannot be exact for any finite value of N, however large. In the following many of our statements will be valid for matrix QM in general and not necessarily only the c=1 matrix model.

Problem 1: (trace identities)



$\rho(\lambda, t)$ has to have some strange properties as a function of λ .

This follows from the Cayley Hamilton identities which, at any given t, relate $Tr\ M^{N+p}, p \geq 1$ to lower traces $Tr\ M^{N-p}, p \geq 0$. Since $Tr\ M^p = \int d\lambda\, \rho(\lambda)\, \lambda^p$, this implies constraints between moments of $\rho(\lambda)$.

(These follow from the fact that it is enough to determine all N eigenvalues from the first N traces, hence higher traces cannot be independent.)

Thus, e.g. for N=2

$$Tr\ M^3 = \frac{3}{2}\ Tr\ M\ Tr\ M^2 - \frac{1}{2}(\mathrm{Tr\ M})^3$$

$$\lambda_1^3 + \lambda_2^3 = \frac{3}{2}\ (\lambda_1 + \lambda_2)(\lambda_1^2 + \lambda_2^2) - \frac{1}{2}(\lambda_1 + \lambda_2)^3$$

In terms of $\rho(\lambda, t)$ it means a constraint

$$\int d\lambda\ \rho(\lambda, t)\lambda^3 = \frac{3}{2}\ (\int d\lambda\ \rho(\lambda, t)\lambda^2)(\int d\lambda' \rho(\lambda', t)\lambda') - \frac{1}{2}(\int d\lambda\ \rho(\lambda, t)\lambda)^3$$

If we wish to write the matrix path integral in terms of a density path integral, and eventually, for c=1, the tachyon path integral, we must include these constraints and hope they go away in the $N \to \infty$ limit.

(work in progress in a c=0 context with R. Suroshe; for earlier work based on phase space path integrals and Moyal product constraint, see Dhar-GM-Wadia, discussed below)

Problem 2: (not bosons!)

$\rho(\lambda, t)$ and its conjugate field, do not have the bosonic Heisenberg algebra, again related to finite N. Note that $\rho(\lambda,t) = \psi^+(\lambda,t)\psi(\lambda,t)$

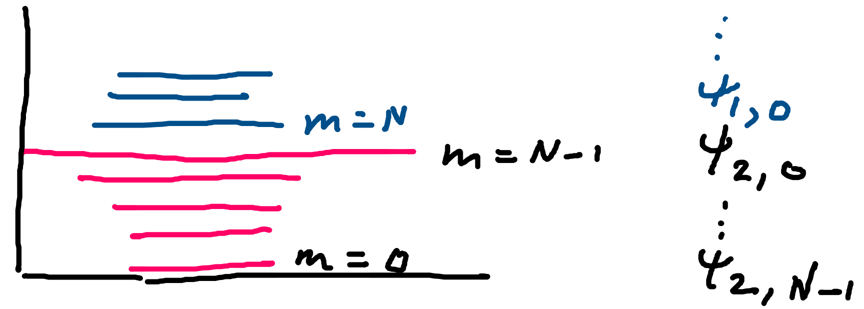
Recall the standard bosonization of relativistic fermions in 1D:

$$\psi(z) = (\psi_1(z) + i\psi_2(z))/\sqrt{2} \quad \rightarrow \text{right}$$

$$\bar{\psi}(\bar{z}) = (\psi_1(\bar{z}) + i\psi_2(\bar{z}))/\sqrt{2} \quad \rightarrow \text{left}$$

$$\partial_z \phi = \psi^+(z)\psi(z) = i\psi_1(z)\psi_2(z)$$

modes: $a_n = \sum_m \psi^+_m \psi_{n-m}$



$$[a_n, a^+_p] = n\delta_{n,p} \quad \longrightarrow \text{requires infinite range of } m$$

For finite N number of fermions, say in a box, the two species of fermions corresponds to particles and holes; the latter have a finite number of modes, namely N. Hence, the Heisenberg algebra does not hold. In fact, the a_n do not have a closed algebra!
This is the Tomonaga problem.

$$[a_m, a_{-m}] = \sum - [\psi_{1,n}\psi_{2,p}, \psi_{1,n'}\psi_{2,p'}] \qquad n+p=m, \quad n'+p'=-m$$

$m=1$

$\vdots$

$\psi_{2,-3/2}\psi_{2,3/2} - \psi_{1,-5/2}\psi_{1,5/2}$ $\qquad n = -n' = 5/2 \qquad p = -p' = -3/2$

$\psi_{2,-\frac{1}{2}}\psi_{2,\frac{1}{2}} - \psi_{1,-3/2}\psi_{1,\frac{3}{2}}$ $\qquad n = -n' = 3/2 \qquad p = -p' = -1/2$

$1 - \psi_{2,-\frac{1}{2}}\psi_{2,\frac{1}{2}} - \psi_{1,-\frac{1}{2}}\psi_{1,\frac{1}{2}}$ $\qquad n = -n' = \frac{1}{2} \qquad p = -p' = \frac{1}{2}$

$\psi_{1,-\frac{1}{2}}\psi_{1,\frac{1}{2}} - \psi_{2,-3/2}\psi_{2,3/2}$ $\qquad n = -n' = -\frac{1}{2} \qquad p = -p' = 3/2$

$\psi_{1,-\frac{3}{2}}\psi_{1,\frac{3}{2}} - \psi_{2,-5/2}\psi_{2,5/2}$ $\qquad n = -n' = -3/2 \qquad p = -p' = 5/2$

$\vdots$

For finite N number of fermions, a closed operator algebra of fermion bilinears is the W-infinity algebra:

$$W_{mn} = \psi^\dagger_m \psi_n , \quad [W_{mn}, W_{pq}] = \delta_{np} W_{mq} - \delta_{qm} W_{pn}$$

Alternative bases of the algebra are

$$\Phi_{xy} = \psi_x \psi^\dagger_y$$

$$u(x,p) = \int d\eta \; \psi^\dagger\left(x+\frac{\eta}{2}\right) \psi\left(x-\frac{\eta}{2}\right) e^{ip\eta}$$

<-- Wigner distribution (phase space density operator)

Bosonization can be done in terms of phase space density operator. This operator is constrained, however, following from

$$\int dy \; \Phi_{xy} \Phi_{yz} = \psi_x \underbrace{\int \psi^\dagger(y) \psi(y)}_{N} \psi^\dagger_z = N \Phi_{xz}$$

Dhar-GM-Wadia, Das-Dhar-GM-Wadia,.... Kulkarni-GM-Morita (1992-2019)

Problem 3: (problem with particle interpretation)

Correlators of single trace operators $Tr\ M^l \equiv O_l$ do not have a good large N limit even when $\frac{l}{N} \to 0$ in the large N limit:

$$\frac{\langle O_{l_1} O_{l_2} O_{l_3} \rangle}{|O_{l_1}|\,|O_{l_2}|\,|O_{l_3}|} \propto \frac{\sqrt{l_1 l_2\, l_3}}{N}, \qquad |O_l| \equiv \sqrt{O_l O_l}$$

For $l_i = O(1)$, there is a good large N limit, namely $=0$, which, in fact shows the orthogonality of "1-particle" states (single trace) to "2-particle" states (double traces), leading to a Fock space interpretation of single-trace states.
For $l_i > O(N^{2/3})$, the correlator diverges, ruling out a particle interpretation of single trace states.

Problem 4: (entanglement entropy)

Entanglement entropy of the matrix model
(equivalently, of the fermion field theory $\psi(\lambda, t)$)

By using standard methods (see Das-Hampton-Liu), the entanglement entropy of a subregion $A \equiv [\lambda_1, \lambda_2]$ with respect to the complement set, in the fermi ground state (filled Fermi sea) is given by

$$S_A = \frac{1}{3}\ln[(\lambda_2 - \lambda_1)P_F(\lambda_0)] + constant\,, \text{ where } P_F(\lambda) = \sqrt{(2(E_F - V(\lambda))}\,, \quad \lambda_0 = \frac{\lambda_1+\lambda_2}{2}$$

For fermions in a box of length L, $P_F(\lambda) \sim \frac{N}{L}$

$$S_A = \frac{1}{3}\ln\left(\frac{\lambda_2-\lambda_1}{\epsilon}\right), \qquad \text{where } \epsilon = \frac{L}{N}$$

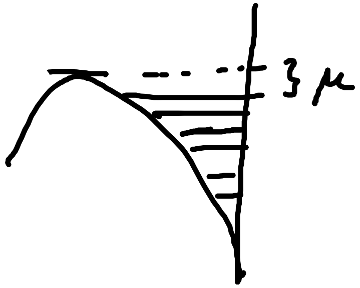
This agrees with the Calabrese-Cardy formula for a boson with uv cut-off $\Lambda = \frac{N}{L}$

It appears that the fermion number N is transformed to a UV cut-off for the "bosons" ! **What bosons are these?**

Problem 4a

c=1: the above formula becomes $S_A = \frac{1}{3}\ln\left(\frac{x_2-x_1}{g_s(x_0)}\right)$, Das 1995, Hartnoll-Mazenc 2015

where $g_s(x) = \frac{1}{\mu N}\frac{1}{sh^2(x)}$, $\quad x = cosh^{-1}\left(\frac{\lambda}{\sqrt{2\mu N}}\right)$



$\mu N =$ fixed in $N \to \infty$ limit (together with $\mu \to 0$) $= \frac{1}{g_s}$

Reinstating l_s $\quad S_A = \frac{1}{3}\, ln\left(\frac{x_2-x_1}{g_s\, l_s}\, sh^2(x_0)\right) = \frac{1}{3}\, ln\left(\frac{x_2-x_1}{\frac{l_s}{N}}\mu\, sh^2(x_0)\right)$

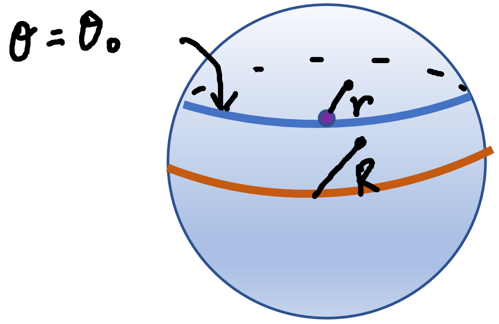
$\Downarrow \qquad\qquad\qquad \Downarrow$

$\varepsilon \sim g_s\, l_s \qquad\qquad \varepsilon \sim \frac{l_s}{N}$ (for fixed μ)

This is surprising from 2D string viewpoint, one expects $\epsilon \sim l_s$ (we will come back to this)

Solution

We will first tackle problems 1-4 and address 4a subsequently.
The hint of the solution comes from a situation similar to problem 3 which appears in the physics of giant gravitons.



S^2

θ, ϕ

Giant gravitons

½ BPS geometry has S3 x (ρ, t) x S3 x (θ, ϕ)

Giant gravitons are D3 branes, wrapping the second S3, and moving in the other coordinates.

SUSY demands that $\rho = 0, \theta = \theta_0, \phi = t$

The size of the giant graviton is given by $\frac{r^2}{R^2} = \frac{l}{N}$ L=1,2,..., N. There are gravitons with the same quantum numbers, but with increasing l $(> \sqrt(g_s N))$, the graviton shrinks below string length. Hence the correct representation is in terms of giant gravitons. (Susskind-Toumbas)

Boundary theory = 1/2 BPS sector of N=4 SYM on S^3 x time.

The half BPS sector is defined in terms of the charges

$(E,S_1,S_2;\ R_1,R_2,R_3) = (E,0,0;\ 0,0,J)$, with E=J,

~~$A_\mu(t,\Omega)$~~, ~~$\psi_\alpha(t,\Omega)$~~, ~~Φ_1~~, ~~Φ_2~~, ~~Φ_3~~, ~~Φ_4~~, Φ_5, Φ_6

The action, projected from the original SYM action to this sector, is

$\int dt\, \mathrm{Tr}(\,|D_t Z|^2 - |Z|^2\,) = \sum \int dt(\,|\partial_t z_i|^2 - |z_i|^2\,)$, $\quad Z = \Phi_5 + i\Phi_6$

fermions in 2D (Landau)

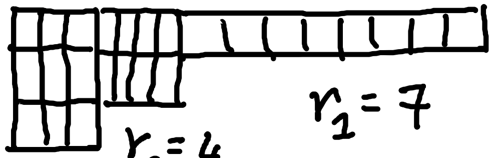
in the last expression we have gone from Z to the eigenvalues z_i by a gauge choice.

Just like in the bulk, the description in terms of gravitons is replaced by D branes (giant gravitons), in the boundary theory, the description in terms of single trace operators like Tr Z^l are replaced by Schur polynomials. The reason for this replacement was precisely problem 3. In particular, for $l \sim N$, the single trace and double trace operators were not orthogonal, they went as

$\langle\, O_l^\dagger\, O_{l_1}\, O_{l_2}\,\rangle \sim \sqrt{l\, l_1 l_2}/N \sim \sqrt{N}$ which blows up in the large N limit!

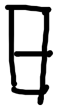
Schur polynomials:

r_1 giant gr. at $\ell=1$
r_2 " " " $\ell=2$
r_3 " " " $\ell=3$

For a representation $R =$ 

$\chi_R(Z) = \mathrm{Tr}_R(Z^{\otimes n})$ $\quad n = \dim R$

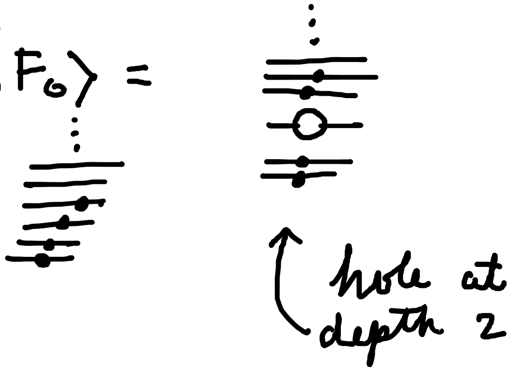
$Z|v_i\rangle = z_i|v_i\rangle$

E.g. $R =$  $\chi_R(Z) = \mathrm{Tr}_R(Z\otimes Z) = \sum_{i<j} z_i z_j = \frac{1}{2}\left(\left(\sum_i z_i\right)^2 - \sum z_i^2\right) = \frac{(\mathrm{Tr}\,Z)^2 - \mathrm{Tr}\,Z^2}{2}$

$R =$  $\chi_R(Z) = \mathrm{Tr}_R(Z\otimes Z) = \sum_{i<j} z_i z_j + \sum_i z_i^2 = \frac{(\mathrm{Tr}\,Z)^2 + \mathrm{Tr}\,Z^2}{2}$

$\therefore$ Schur polynomials are polynomials of single trace operators.

How do they act on the fermi sea?

$\chi_{\square\!\square}(Z)\,|F_0\rangle =$ 

Proof: $\sum_{i<j} z_i z_j \det\begin{bmatrix} 1 & 1 & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 \\ z_1^2 & z_2^2 & z_3^2 & z_4^2 \\ z_1^3 & z_2^3 & z_3^3 & z_4^3 \end{bmatrix} = \det\begin{bmatrix} 1 & 1 & 1 & 1 \\ z_1 & z_2 & z_3 & z_4 \\ z_1^3 & z_2^3 & z_3^3 & z_4^3 \\ z_1^4 & z_2^4 & z_3^4 & z_4^4 \end{bmatrix}$

Nomura, Jevicki, Balasubramaniam et al, Corley-Javicki-Ramgoolam

$\chi_{\underbrace{\text{Young diagram}}_{l}}(Z)$ = single giant graviton of ang. mom. l

$$\frac{\langle F_0 | \chi_l^\dagger \, \chi_{l_1} \chi_{l_2} | F_0 \rangle}{\| \chi_l \| \, \| \chi_{l_1} \| \, \| \chi_{l_2} \|} = \sqrt{\frac{N!}{(N-l)!} \frac{(N-l_1)!}{N!} \frac{(N-l_2)!}{N!}} \qquad l = l_1 + l_2 \quad \text{— (1)}$$

e.g. $l = N, \quad l_1 = \frac{N}{2} + L, \quad l_2 = \frac{N}{2} - L$

$$= \sqrt{\frac{(\frac{N}{2}+L)! \, (\frac{N}{2}-L)!}{N!}}$$

decreases $\sim e^{-N}$ for $L = 0, 1, \ldots$

$\sim \frac{1}{\sqrt{N}}$ for $L = \frac{N}{2} - 1$

$\longrightarrow 0$ in the entire range $L = 0, 1, \ldots, \frac{N}{2} - 1$

More generally,

$\chi_l | F_0 \rangle \perp \chi_{l_1} \chi_{l_2} | F_0 \rangle$ in the large N limit

Consistent with their interpretation as 1-"particle", 2-"particle" states.

$\longrightarrow$ (1) Solves problem 3.

In the above, the results continue to be valid if $Z \to M$ (a hermitian matrix)

Problem (1) is also solved.

The Schur operators χ_l exist only for $l=1,2,\ldots,N$; they involve only $\mathrm{Tr}\, Z^l$ for $l=1,2,..,N$ and are all independent. In particular, they are not constrained by any trace identities.

We will realize the independence of the χ_l by constructing their action in terms of independent Heisenberg oscillators $a_1, a_2, \ldots, a_N$.

We will find below that they also solve problems (2) and (4)!

More generally, a composite giant graviton operator $\chi(r_1, r_2, r_3, \ldots, r_N)$ applied to the fermi sea changes it to a new state with the filling $(f_1, f_2, \ldots, f_N)$

where $f_1 = r_N$, $f_2 = r_{N-1} + r_N + 1$, $f_3 = r_{N-2} + r_{N-1} + r_N + 2, \ldots$

$r_N = f_1$, $r_{N-1} = f_2 - f_1 - 1$, $r_{N-i} = f_{i+1} - f_i - 1$, $i = 1, 2, \ldots N-1$ — (*)

clearly $f_1 = 0, f_2 = 1, f_3 = 2, \ldots, f_N = N-1 \Rightarrow r_N = 0, r_{N-1} = 0, \ldots r_1 = 0$ (no giant graviton)

$|F_0\rangle \longleftrightarrow |0\rangle$

It is possible to invent $a_1, a_2, \ldots a_N$ such that

$$[a_i, a_j^\dagger] = \delta_{ij} \qquad i, j = 1, 2, \ldots N$$

→ solves problem (2)

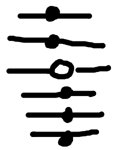
and the state $\underbrace{\chi(r_1, r_2, \ldots r_N)|F_0\rangle}$ is represented by $\frac{(a_1^\dagger)^{r_1}}{\sqrt{r_1!}} \frac{(a_2^\dagger)^{r_2}}{\sqrt{r_2!}} \cdots \frac{(a_N^\dagger)^{r_N}}{\sqrt{r_N!}} |0\rangle$

$$||| \qquad |f_1, f_2, \ldots f_N\rangle \equiv |r_1, r_2, \ldots r_N\rangle \qquad \text{— (*)}$$

Note the special case $\chi_\ell = \chi(r_1 = 0, r_2 = 0, \ldots r_\ell = 1, r_{\ell+1} = 0, \ldots)$

$\Rightarrow f_1 = 0, f_2 = 1, \ldots, f_{N-\ell+1} = N-\ell+1, f_{N-\ell+2} = N-\ell+2, \ldots$

$f_{N-\ell} = N-\ell-1$



← hole at depth ℓ

$$a_k^\dagger\,|f_1,\ldots,f_N\rangle = \sqrt{f_{N-k+1}-f_{N-k}}\,|f_1,\ldots,f_{N-k},f_{N-k+1}+1,\ldots,f_N+1\rangle, \quad k=1,\ldots,N-1$$

$$a_N^\dagger\,|f_1,\ldots,f_N\rangle = \sqrt{f_1+1}\,|f_1+1,\ldots,f_N+1\rangle. \tag{2.15}$$

Thus, $a_k^\dagger$ moves each of the top k fermions, counting down from the topmost filled level, up by one step. Similarly, the action of a_k is to move each of the top k fermions down by one step:

$$a_k\,|f_1,\ldots,f_N\rangle = \sqrt{f_{N-k+1}-f_{N-k}-1}\,|f_1,\ldots,f_{N-k},f_{N-k+1}-1,\ldots,f_N-1\rangle, \quad k=1,\ldots,N-1$$

$$a_N\,|f_1,\ldots,f_N\rangle = \sqrt{f_1}\,|f_1-1,\ldots,f_N-1\rangle. \tag{2.16}$$

The N bosonic oscillators a_k, a_k^+ as explicit operators in the Fermion Hilbert space.

$$a_k^\dagger \equiv \sum_{m_k>m_{k-1}>\cdots>m_0} \sqrt{m_1-m_0}\,(\psi_{m_0}^\dagger\psi_{m_0})(\psi_{m_1+1}^\dagger\psi_{m_1})\cdots(\psi_{m_k+1}^\dagger\psi_{m_k}) \times \delta\Big(\sum_{m=m_0+1}^{m_1-1}\psi_m^\dagger\psi_m\Big)\,\delta\Big(\sum_{m=m_1+1}^{m_2-1}\psi_m^\dagger\psi_m\Big)\cdots\delta\Big(\sum_{m=m_{k-1}+1}^{m_k-1}\psi_m^\dagger\psi_m\Big) \times \delta\Big(\sum_{m=m_k+1}^{\infty}\psi_m^\dagger\psi_m\Big), \quad k=1,2,\ldots,(N-1) \tag{2.17}$$

and

$$a_N^\dagger \equiv \sum_{m_N>m_{N-1}>\cdots>m_1} \sqrt{m_1+1}\,(\psi_{m_1+1}^\dagger\psi_{m_1})\cdots(\psi_{m_N+1}^\dagger\psi_{m_N}) \times \delta\Big(\sum_{m=m_1+1}^{m_2-1}\psi_m^\dagger\psi_m\Big)\cdots\delta\Big(\sum_{m=m_{N-1}+1}^{m_N-1}\psi_m^\dagger\psi_m\Big) \times \delta\Big(\sum_{m=m_N+1}^{\infty}\psi_m^\dagger\psi_m\Big). \tag{2.18}$$

The bosonic oscillators a_k, a_k^+ written in terms of Fermion bilinears.

Here,

$$\delta(\hat{O}) \equiv \int_0^{2\pi}\frac{d\theta}{2\pi}\,\exp(i\theta\hat{O}),$$

$$
\begin{aligned}
\psi_n^\dagger\,\psi_n =\; & \sum_{k=1}^{N} \delta\Big(\sum_{i=k}^{N} a_i^\dagger a_i - n + N - k\Big) \\
\psi_{n+1}^\dagger\,\psi_n =\; & \sigma_1^\dagger\,\delta\Big(\sum_{i=1}^{N} a_i^\dagger a_i - n + N - 1\Big) \\
& + \sum_{k=1}^{N-1} \sigma_k\,\sigma_{k+1}^\dagger\,\theta_+(a_k^\dagger a_k - 1)\,\delta\Big(\sum_{i=k+1}^{N} a_i^\dagger a_i - n + N - k - 1\Big) \\
\psi_{n+2}^\dagger\,\psi_n =\; & \sigma_1^{\dagger 2}\,\delta\Big(\sum_{i=1}^{N} a_i^\dagger a_i - n + N - 1\Big) \\
& + \sum_{k=1}^{N-1} \sigma_k^2\,{\sigma_{k+1}^\dagger}^2\,\theta_+(a_k^\dagger a_k - 2)\,\delta\Big(\sum_{i=k+1}^{N} a_i^\dagger a_i - n + N - k - 1\Big) \\
& - \sum_{k=2}^{N-1} \sigma_{k-1}\,\sigma_{k+1}^\dagger\,\theta_+(a_{k-1}^\dagger a_{k-1} - 1)\,\delta(a_k^\dagger a_k)\,\delta\Big(\sum_{i=k+1}^{N} a_i^\dagger a_i - n + N - k - 1\Big) \\
& - \sigma_2^\dagger\,\delta(a_1^\dagger a_1)\,\delta\Big(\sum_{i=1}^{N} a_i^\dagger a_i - n + N - 2\Big) \qquad (2.21)
\end{aligned}
$$

The Fermion bilinears written in terms of the bosonic oscillators a_k, $a_k^\dagger$

Exact bosonization of N non-relativistic fermions in 1D

A.Dhar, GM, N.Suryanarayana, M. Smedback

The N-fermion states are given by (linear combinations of)

$$|f_1, \cdots, f_N\rangle = \psi^\dagger_{f_1}\psi^\dagger_{f_2}\cdots\psi^\dagger_{f_N}|0\rangle_F, \qquad (2)$$

where f_m are arbitrary integers satisfying $0 \le f_1 < f_2 < \cdots < f_N$, and $|0\rangle_F$ is the usual Fock vacuum annihilated by ψ_m, $m = 0, 1, \cdots, \infty$.

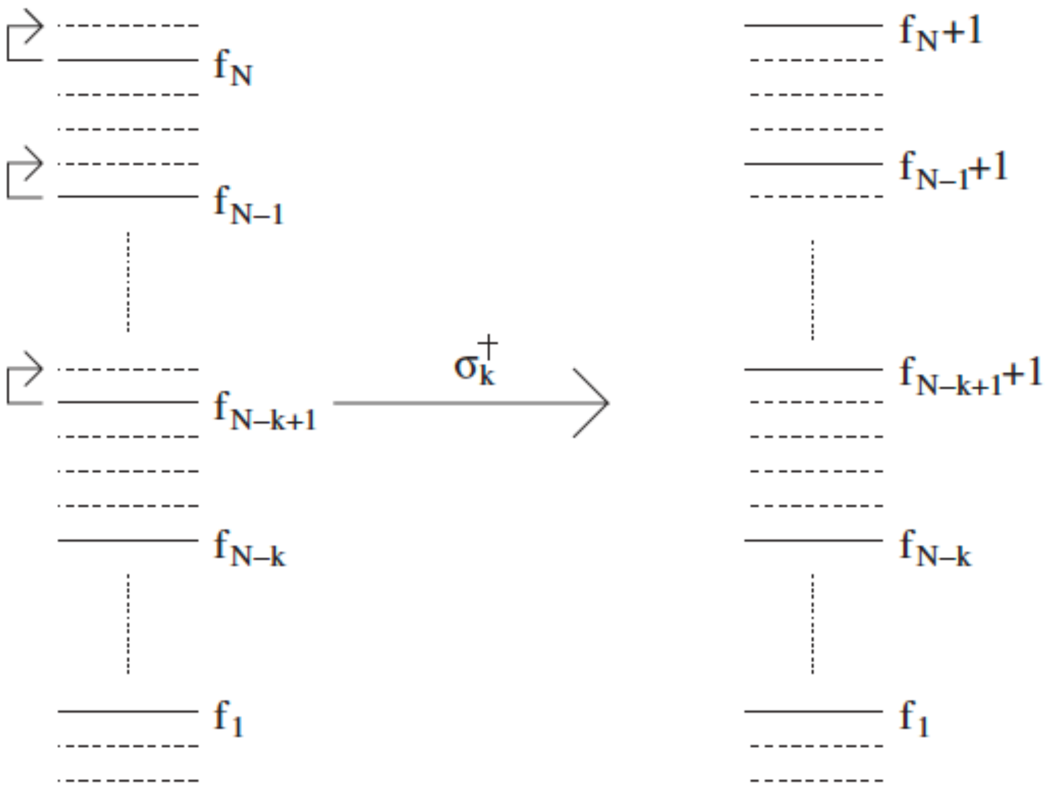


FIG. 1. The action of $\sigma_k^\dagger$.

$$\sigma_k = \frac{1}{\sqrt{a_k^\dagger a_k + 1}} a_k,$$

$$\sigma_k^\dagger = a_k^\dagger \frac{1}{\sqrt{a_k^\dagger a_k + 1}},$$

Note: only N bosonic Oscillators!!

$$[a_k, a_l^\dagger] = \delta_{kl}, \qquad k, l = 1, \cdots, N.$$

$$|r_1, \cdots, r_N\rangle = \frac{(a_1^\dagger)^{r_1}\cdots(a_N^\dagger)^{r_N}}{\sqrt{r_1!\cdots r_N!}}|0\rangle.$$

$|f_1, f_2, \ldots, f_N\rangle \leftrightarrow |r_1, r_2, \ldots, r_N\rangle$ with

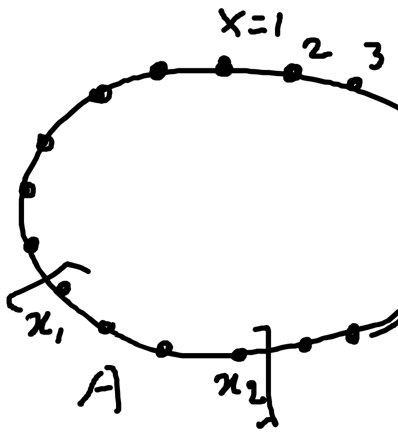
$$r_k = f_{N-k+1} - f_{N-k} - 1, \qquad k = 1, 2, \cdots N-1,$$

$$r_N = f_1.$$

$|F_0\rangle \leftrightarrow |0\rangle, \quad f_N^\dagger f_{N-1}|F_0\rangle \leftrightarrow a_1^\dagger|0\rangle$

Geometry of these oscillators:

"Construct" real space = circle of length L :



$$x_j = j\varepsilon \qquad \varepsilon = \frac{L}{N}$$
$$= \frac{jL}{N}$$

$$\phi(x_j) \equiv \phi_j = \sum_{m=1}^{N} \frac{a_m}{\sqrt{2\omega_m}} e^{i\frac{2\pi m}{L}x_j} + c.c.$$

$$= \sum_{m=1}^{N} \frac{a_m}{\sqrt{2\omega_m}} e^{i2\pi\frac{mj}{N}} + c.c.$$

$$\omega_m = \frac{4}{\varepsilon^2}\sin^2\frac{\pi m}{N}$$

$\Pi(x_j)$ determined by demanding:

$$[\phi(x_j), \Pi(x_k)] = i\delta_{jk}$$

$$\Pi(x_j) = \frac{1}{i}\sum_m \frac{1}{\sqrt{2}}\sqrt{\omega_m}\left(a_m e^{i2\pi jm/N} - a_m^\dagger e^{-i\frac{2\pi jm}{N}}\right)$$

The ground state: $\quad a_m|0\rangle = 0$

Entanglement entropy (Holzhey-Wilczek, Calabrese-Cardy, Casini-Huterta, Herzog,...

$$S_A = \frac{1}{3}\ln\left(\frac{x_2 - x_1}{\varepsilon}\right) \qquad \varepsilon = \frac{L}{N}$$

Solves problem (4) for fermions in a box:

Note that a finite number N of oscillators modes force us to have a finite number (N) of lattice points.

Alternatively, a finite number of oscillators can be represented by a finite number of energy levels of a single particle problem. Semiclassically, these correspond to phase space orbits. E.g. $\omega_m \sim m$, $m=1,2,\ldots, N$ implies a fuzzy phase space torus. In fact, in the giant graviton problem, the two-dimensional plane of the fermion is non-commutative, in the AdS geometry it maps to a fuzzy sphere.

Dynamics:

By exploiting the maps (*), (*) one can map the problem of N free fermions in a single particle Hamiltonian spectrum $\mathcal{E}(m)$, to the bosonic Hamiltonian $H=\sum_{i=1}^{N} \mathcal{E}(\sum_{k=1}^{i} a_k^{+} a_k)$

Zero potential: N free fermions in a box have a single particle Hamiltonian spectrum $\mathcal{E}(m) = 4\pi^2 \frac{m^2}{L^2}$. This maps to a quartic bosonic Hamiltonian $H=\sum_{i=1}^{N} \frac{4\pi^2}{L^2} (\sum_{k=1}^{i} a_k^{+} a_k)^2$. For small excitations, $a_k^{+} a_k$ are non-zero only for small k, hence the Hamiltonian becomes effectively quadratic.

This solves the Tomonaga problem. (A. Dhar, GM, Nemani S)

This Hamiltonian can be expressed in terms of the lattice variables ϕ_x, π_y introduced above. (complicated).

Non-zero potential: How does one find a bosonic theory which reproduces the fermionic entanglement entropy

$$S_A = \frac{1}{3}\ln\left[(x_2 - x_1)\,P_F(x_0)\right] + \text{constant}$$

where $P_F(x) = \sqrt{2(E_F - V(x))}$

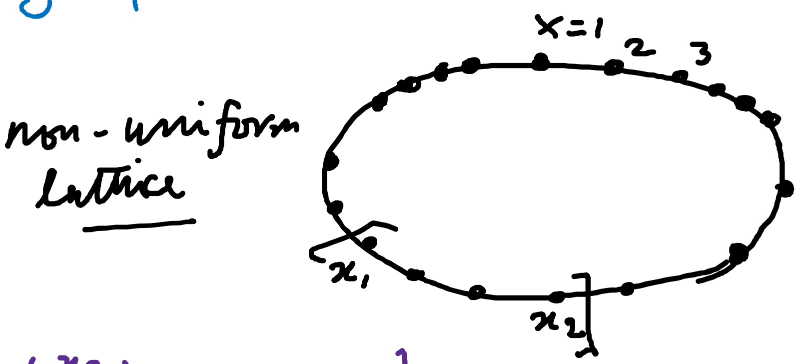
and $x_0 = \frac{x_1 + x_2}{2}$

This can be achieved by putting the bosonic theory in a circle with a metric

$$ds^2 = -dt^2 + \gamma^2(x)\,dx^2 = -dt^2 + d\tilde{x}^2 \qquad \tilde{x} = \int^{x} \gamma(x')\,dx'$$

In terms of the $\tilde{x}$ coordinate, the entanglement entropy iof the bosionic theory, as we found earlier, is

$$S_A = \frac{1}{3}\ln\frac{\tilde{x}_2 - \tilde{x}_1}{\varepsilon} \qquad \varepsilon = L/N$$



Using the relation between x and $\tilde{x}$, we find

$$S_A = \frac{1}{3}\ln\left(\sqrt{\frac{\gamma(x_1)}{\varepsilon}\frac{\gamma(x_2)}{\varepsilon}}\,(x_2 - x_1)\right) \simeq \frac{1}{3}\ln\left(\frac{\gamma\left(\frac{x_1+x_2}{2}\right)}{\varepsilon}(x_2 - x_1)\right)$$

Choosing $\gamma(x) = P_F(x)$, we find the correct bosonic theory

→ this gives a complete solution of problem 4.

Back to Problem 4a:

In terms of the above bosonization, using matric $\gamma(x) = sh^2 x$, and $\varepsilon = l_s/\mu N = g_s\, l_s$, we reproduce the c=1 EE

$$S_A = \frac{1}{3}\, ln\left(\frac{x_2 - x_1}{g_s\, l_s}\, sh^2(x_0)\right) = \frac{1}{3}\, ln\left(\frac{x_2 - x_1}{\frac{l_s}{N}}\mu\, sh^2(x_0)\right)$$

Note $\epsilon \sim g_s\, l_s$ (double scaling)

⇒ The boson in question is not the tachyon (the 2D string field), presumably it's a D0 brane, which has the characteristic length scale $g_s\, l_s$ (more confirmation needed)

Note also that $ln\left(\frac{1}{g_s}\, sh(x_1)\, sh(x_2)\right) = \frac{1}{2}\left(\Phi(x_1) + \Phi(x_2)\right)$ where $\Phi(x) = 2\,x$ is the value of the Dilaton at large x (weak coupling region). While this is suggestive, the correct classical contributions to hol. EE (RT) would involve Exp[-2 $\Phi(x)$], (=area of a point). *

* Discussion with Juan Maldacena

Conclusion:

We showed problems with the standard bosonization of matrix models, or of non-relativistic fermions for any N, however large. These follow from (i) trace identity constraints, (ii) failure to satisfy Heisenberg commutation relations, (iii) failure of particle interpretation of the bosonic theory, and (iv) N-dependent entanglement entropy.

We show that all these problems can be solved by using an exact bosonization of N non-interacting non-relativistic fermions. The real space bosons are constructed on a lattice circle; alternatively they can be understood in terms of fuzzy phase spaces.

The EE of c=1 matrix model can be explained in terms of such a bosonic theory, which has apparent differences from the 2D string.

It is important to study how these observations apply to more general instances of holography.