The surprising occurrence of Toric Geometry at the interface of String Theory, computations and elliptic fibrations.

Antonella Grassi

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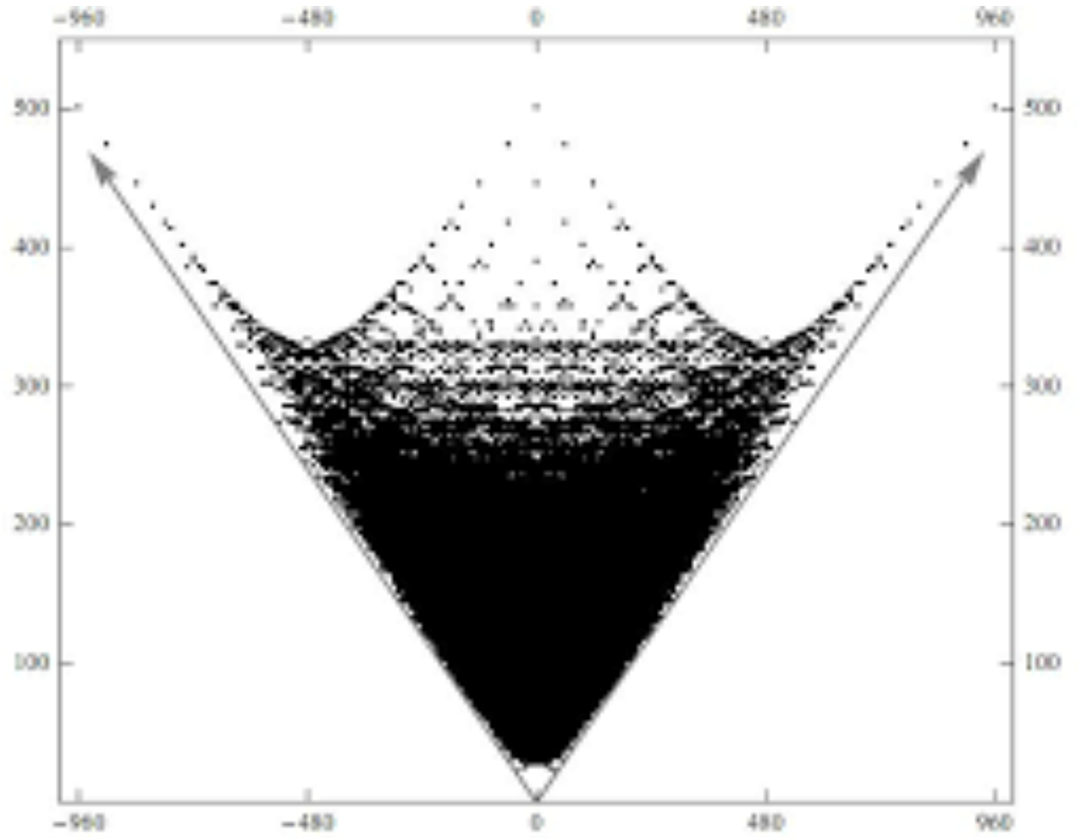

−960
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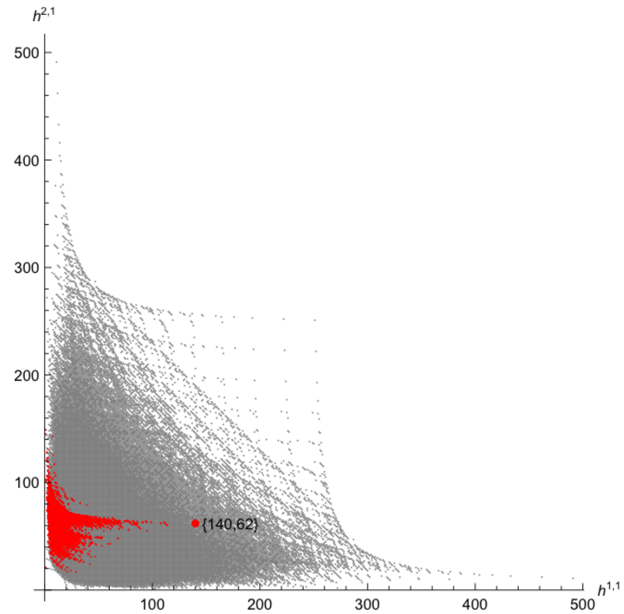
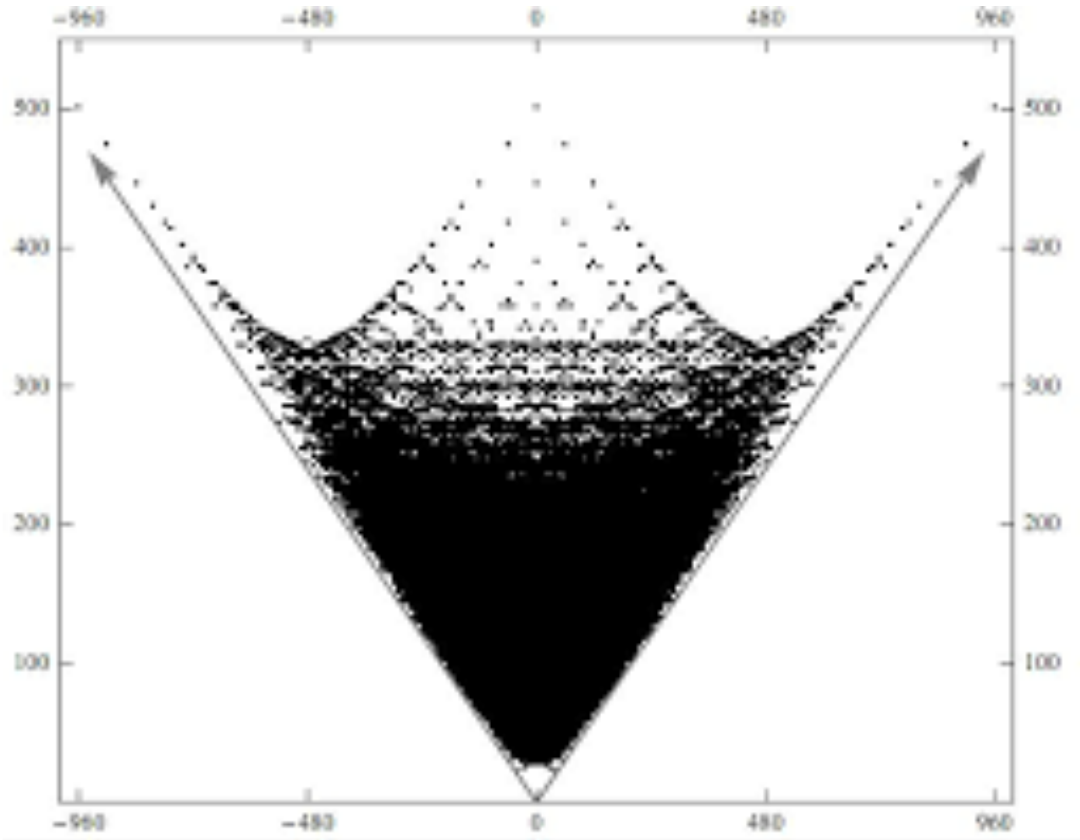
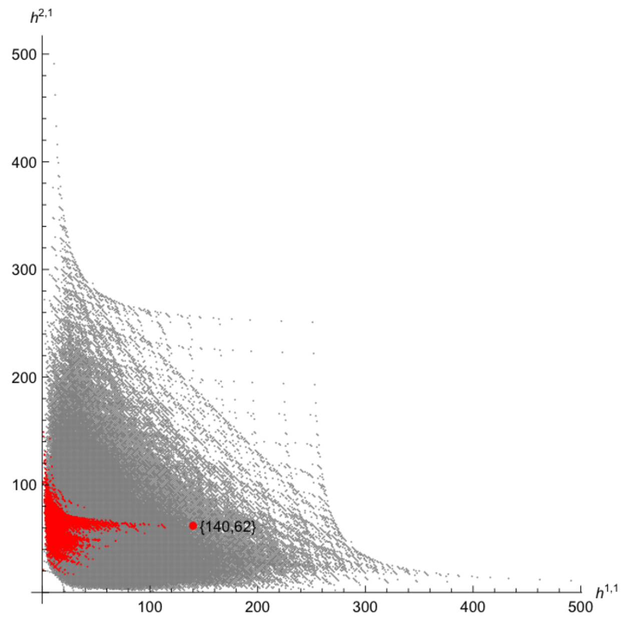


$h^{2,1}$
500
400
300
200
100
{140,62}
$h^{1,1}$
100
200
300
400
500

From : Kreuzer- Skarke



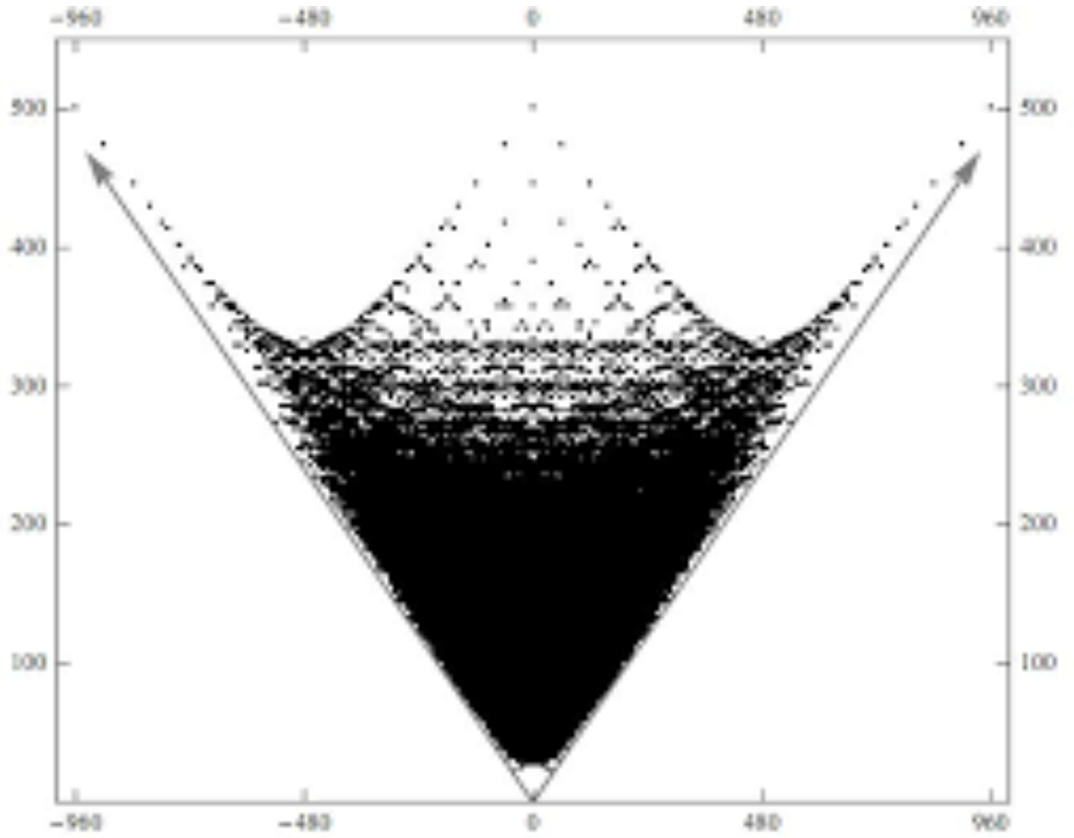


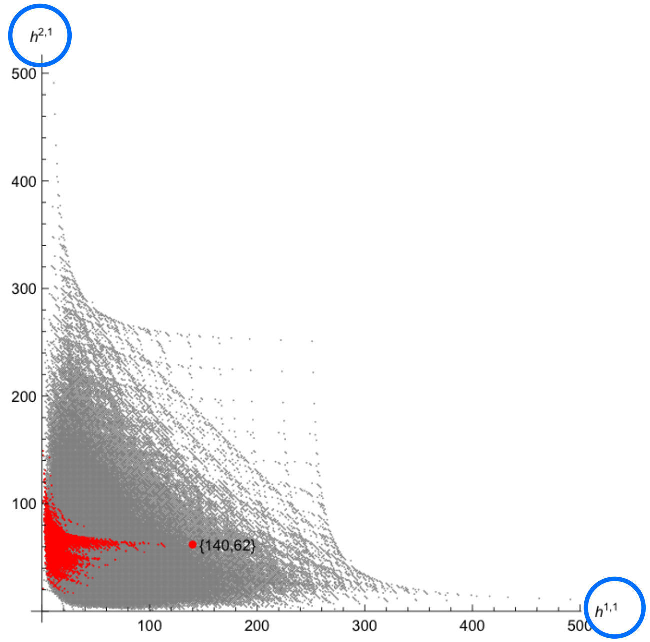
$h^{1,1}(X) - h^{2,1}(X)$

$h^{1,1}(X) + h^{2,1}(X)$

- X smooth Calabi-Yau threefold
  $X \subset X_\Sigma$ toric.

Jejjala- Taylor- Turner 2022

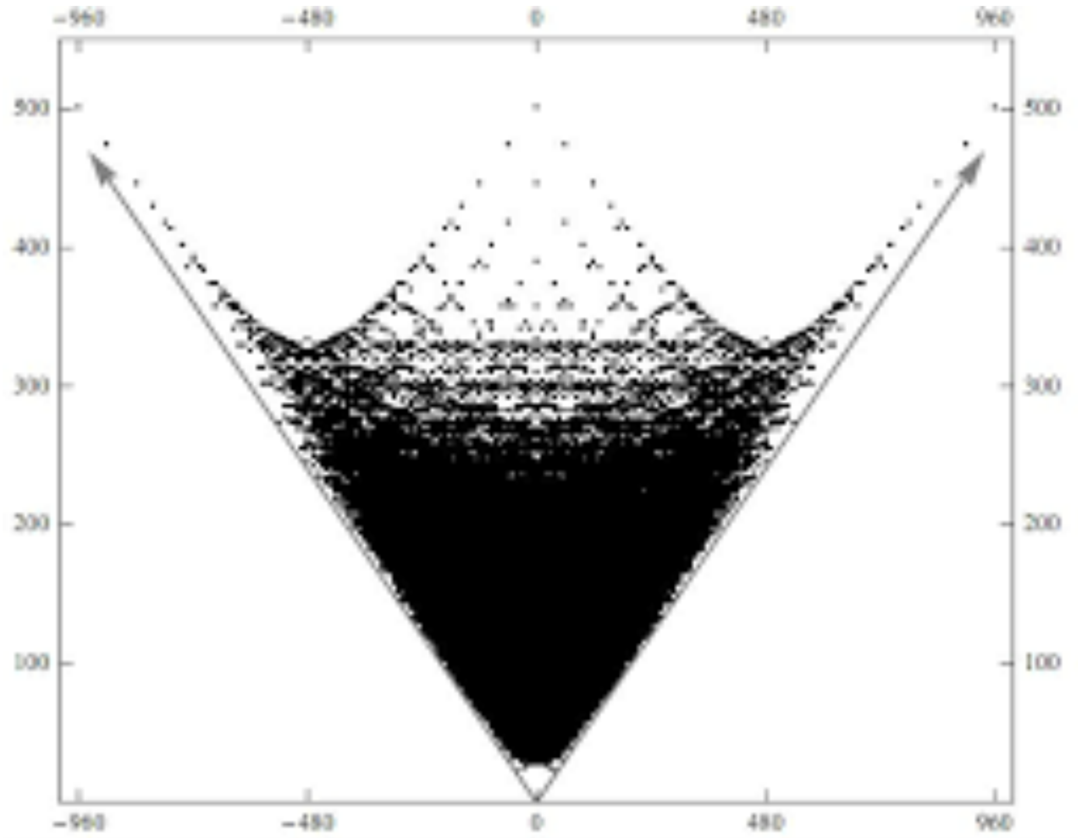


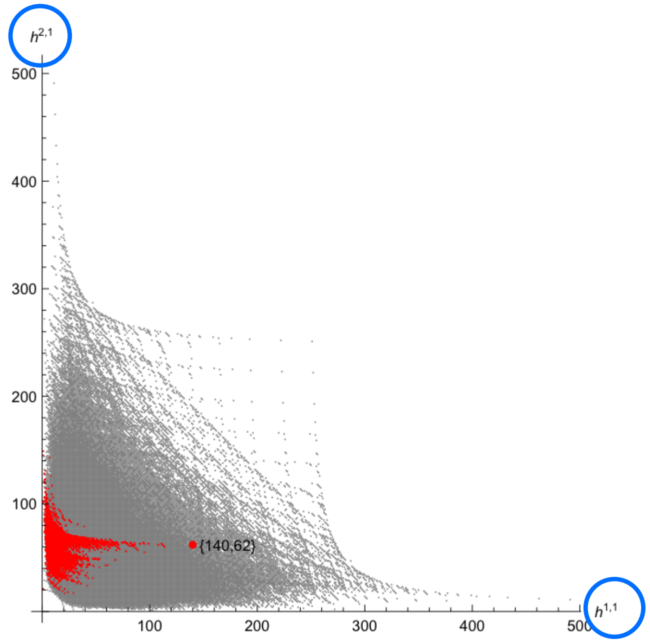


· 4738 million formulas

• 29223 without obvious elliptic fibrations

Jejjala- Taylor- Turner 2022





· 4738 million families

· 29223 without obvious elliptic fibrations

● 99.3% CICY have obvious elliptic fibration

Anderson- Gray- Gao- Lee

$\leadsto$ Boundedness of Calabi-Yau ??

Boundedness for elliptically fibered Calabi-Yau ??

Elliptic fibrations $\leftrightarrow$ F-Theory

- Elliptic curve $\longleftarrow$ complex torus.

Classical

Physics

$\downarrow$

- ~~Ellipses~~ ?

$\longleftarrow$ Elliptic integrals.

(Abel : date 1826)

Take ① $\longrightarrow$ Take ②

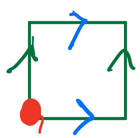
E : Plane curve

- $y^2 = x^3 + ax + b$

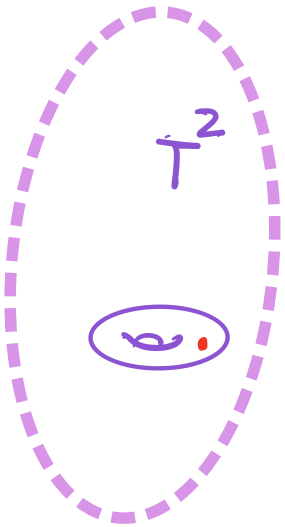
$a, b \in \mathbb{C}$

(Flex point)

E : Quotient , $K = \mathbb{C}$



$\mathbb{C} / \mathbb{Z} \oplus (\tau)\mathbb{Z}$



Less classical : Physics of string theory.

(G. Veneziano '60s)

Model(s) of the Universe "$\simeq$" $\mathbb{R}^{3,1} \times X$

compact "manifold" (possibly singular),
constrained

Question : [Last century] : Properties of X ?

• How many X ?

• "Toy model X" : E, T^2, elliptic curve.

§. Beyond the Toy model : T^2, $E/\mathbb{C}$

$$\begin{array}{ccccccc} X & & \supseteq & & E(\mathbb{C}) & & \\ \downarrow & & & & \downarrow & & \\ B & \underset{\neq}{\supset} & U & \ni & p & & \end{array}$$

".X constrained by symmetries"

dense open in the Zariski topology

- $y^2 = x^3 + a(t)x + b(t)$ $a, b \in K = \mathbb{C}(B)$

" t on B "   Function Field of B.

Question : [This century] :

"spectrum" $\overset{?}{\rightsquigarrow}$ "invariants" of X, bounds??

A Bit more about the constraints on X:

Calabi, Yau [50s – 70s]

$X/\mathbb{C}$ is an algebraic projective variety.

- $K_X \cong \mathcal{O}_X$ $(\leftrightarrow c_1(X)=0)$
- $h^i(\mathcal{O}_X)\ (= h^{i,0}) = 0 \qquad 0 < i < \dim X$

X often assumed to be smooth in the physics literature.

X is Calabi–Yau

- dimensions of interest:
  - $\dim X = 3, 4$
  - $\dim X = 2$: K3 surface

a metrically *convex* Boolean metric space over a Boolean algebra that contains the original one as a sub-algebra. The congruence is, in fact, an isomorphism, and since metric convexity of B is easily seen to be equivalent to lack of atoms in the underlying algebra, it follows that every Boolean algebra may be embedded isomorphically in an atom-free Boolean algebra. Assuming from now on that B is atom-free (metrically convex) and lattice-complete, concepts involving continuity, based on the introduction of the Birkhoff-Kantorovich order topology (sequential or directed set) are defined. It is seen that B is also metrically complete. Arcs are defined as homeomorphs of maximal chains. Since every motion (congruence of B with itself) is a homeomorphism and every congruence between any two subsets of B is extendible to a motion, segments are arcs. One seeks to determine what metric and topological properties of arcs and segments in ordinary metric spaces are valid in the very different environment provided by a Boolean metric space. Homeomorphisms between maximal chains with the same end-elements, connectedness properties of chains, characterizations of segments among arcs (by having a length equal to the distance of the end-elements, for example) are considered. A theory of continuous curves is also begun.

UNIVERSITY OF MISSOURI,
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BEMERKUNGEN ZUR HOMOTOPIETHEORIE

EWALD BURGER

Kleinere Bemerkungen über den Zusammenhang zwischen Homologie- und Homotopieinvarianten der Abbildungen eines n-dimensionalen Polyeders K in einen Raum, der in den Dimensionen $< n$ asphärisch ist, unter Benutzung der von Eilenberg-MacLane (Ann. of Math. 43 (1942)) gegebenen Beschreibung der Kohomologiegruppen von K durch die ganzzahligen Homologiegruppen.

FRANKFURT A. M.,
BRÜDER GRIMM STR. 57

ICM- 1954

THE SPACE OF KÄHLER METRICS

EUGENIO CALABI

Let M^n be a closed, n-dimensional complex manifold. We assume that M^n admits at least one Kähler metric $g_{\alpha\beta^*}$; its associated closed exterior form $\omega = \sqrt{-1}\, g_{\alpha\beta^*}\, dz^\alpha \wedge dz^{\beta^*}$ determines a real cohomology class, called the principal class of the metric. Consider the space Ω of all infinitely differentiable

Kähler metrics in M^n with the same principal class; the topology of Ω is defined by the L^2 topology of the tensorial components of metrics in Ω in compact subregions of co-ordinate domains. If $R_{\alpha\beta^*}$ is the Ricci tensor of any metric in Ω, then the Ricci form $\sqrt{-1}\, R_{\alpha\beta^*}\, dz^\alpha \wedge dz^{\beta^*}$ is closed and its cohomology class is $2\pi C^{(1)}$ ($C^{(r)} = r$th Chern class).

Theorem 1. Given in M^n any real, closed, infinitely differentiable exterior form Σ of type (1, 1) and cohomologous to $2\pi C^{(1)}$, there exists exactly one Kähler metric in Ω whose Ricci form equals Σ.

The proof proceeds by joining the Ricci form of one metric in Ω with Σ by a differentiably parametrized arc in the same linear space of forms (for example by linear interpolation) and constructing a corresponding parametrized path in Ω, then by proving that it is unique, and that its terminal point is independent of the path.

One can introduce in Ω a generalized Riemannian structure by defining the metric form $ds^2 = \left(\frac{\partial\omega(t)}{\partial t}, \frac{\partial\omega(t)}{\partial t}\right) dt^2$, where $\omega(t)$ represents a differentiable path in Ω. One shows that between any two points in Ω there exists exactly one geodesic, that the topology induced by the geodesic distance coincides with the chosen one, and that Ω with this metric is isometric with a geodesically convex subset of a Hilbert sphere.

If $g_{\alpha\beta^*}$ belongs to Ω and Σ is its Ricci form, then (Σ, Σ) is stationary with respect to first order variations in Ω, if and only if the contravariant components of the gradient of the scalar curvature generate a group of complex analytic transformations of M^n. If M^n has the property that each one-parameter complex transformation group of M^n (if any exist) leaves no point fixed, or if it is the Gaussian sphere, then this implies that the scalar curvature is constant, that Σ is harmonic, and that (Σ, Σ) is minimized.

Theorem 2. If M^n has the restrictive property described above, then there exists a metric in Ω which minimizes (Σ, Σ), unique but for analytic transformations of M^n, and characterized by the property that the scalar curvature is constant.

LOUISIANA STATE UNIV.
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ON TWO DIMENSIONAL ASPHERICAL COMPLEXES

WILFRED HALLIDAY COCKCROFT

Let L be a connected two dimensional C. W. complex. Let K denote the complex $L \cup \{e_i^2\}$, where $\{e_i^2\}$, $i = 1, 2, \ldots$, is a set of 2-cells which are adjoined

Unless otherwise specified, the varieties are assumed to be: projective, algebraic, normal, irreducible, defined over $\mathbb{C}$

Def: • X is <u>elliptic</u> if:

$$\begin{tikzcd} X \arrow[d, "\pi"'] & \supseteq & E(\mathbb{C}) \arrow[d] \\ B & \supsetneq \; \mathcal{U} \; \ni & p \end{tikzcd}$$

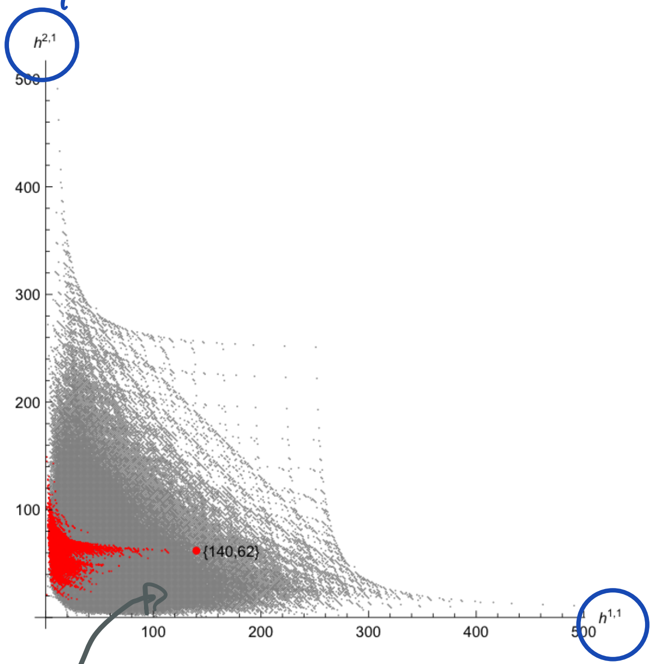
, $\mathcal{U}$ dense open

• $\Sigma \left(= \Sigma_{X/B}\right) \overset{\text{def}}{=} \{ q \in B \setminus \mathcal{U} \}$ the discriminant.

(X, B, π) , (B, Σ) $\rightsquigarrow$ F-theory.

$\Sigma \leftrightarrow$ 7 branes.

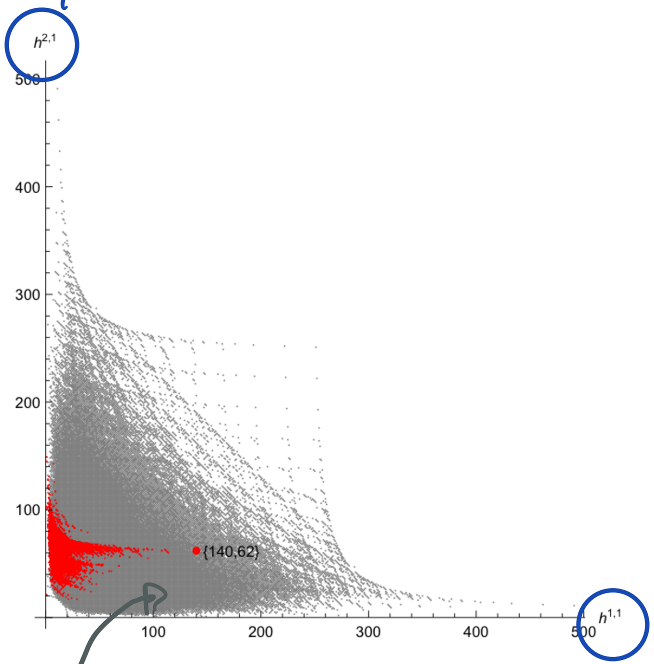
$h^{2,1} = b^3/2 - 1$, b_3 : Betti number, topological.



$h^{11} = b_2$: Betti number topological.

elliptic CY 3folds in Toric ambient space.

$h^{2,1} = b^3/2 - 1$, b_3 : Betti number, topological.



$h^{11} = b_2$: Betti number topological.

elliptic CY 3folds in Toric ambient space.

<u>Theorem</u> (Filipazzi - Hacon - Svaldi 2022)

There are only finitely many topological types of elliptic Calabi-Yau threefolds.

, ∃ Th: by Gross '94 (later)

Theorem (Filipazzi-Hacon-Svaldi 2022)

There are only finitely many topological types of elliptic Calabi-Yau threefolds.

, ∃ Th by Gross '94 (later)

bir-bounded + CAVEAT

Theorem (Engel-Filipazzi-Greer-Mauri-Svaldi, 2025)

X elliptic Calabi-Yau are birationally bounded, in any dimension.

Next!

Corollary: X elliptic, Calabi Yau $\xrightarrow{b_2}$ $h^{1,1}(X)$ is bounded

Corollary: X elliptic Calabi-Yau 3fold $\xrightarrow{b_3}$ $h^{2,1}$ is bounded.

$\boxed{\text{Corollary: } X \text{ elliptic, Calabi Yau} \longrightarrow h^{1,1}(X) \text{ is bounded}}$

$\boxed{\text{Corollary: } X \text{ elliptic Calabi-Yau 3fold} \longrightarrow h^{2,1} \text{ is bounded.}}$

Why is this interesting? $\rightsquigarrow$ Spectrum of Theory

H, V, T : multiplets.

Question: Are there explicit bounds?

Answer: There are, some ; Math idea of proofs

Answer: There are, some bounds proven

Math idea of proof

Toric variety

Birational Geometry.

CAVEAT : $\exists$ elliptic fibrations

Kollár '15

Bakker - Dutta - Filipazzi - Lazić - Li - Svaldi - Wang, Zhao '25

$\pi: X \to B$ (elliptic) s.t. X smooth, $mK_X \sim \mathcal{O}_X$, $h^2(X, \mathcal{O}_X) = 0$

$\rightsquigarrow$ Any small deformation carries the fibration.

[Reid conjecture] ,

§ Birational geometry:

Def: $X \overset{\text{bir}}{\sim} Y \quad \Leftrightarrow \quad \exists\ \mathcal{U}_1, \mathcal{U}_2$ open, $\mathcal{U}_i \neq \emptyset : X \supseteq \mathcal{U}_1 \overset{\text{isom.}}{\cong} \mathcal{U}_2 \subseteq Y$ (Zariski open)

$(X \dashleftarrow\dashrightarrow Y)$

Example: X, Y toric $\overset{\dim X = \dim Y}{\Longrightarrow}$ $X \overset{\text{bir}}{\sim} Y$

$$\begin{array}{ccc} X & \overset{\text{bir}}{\sim} & Y \\ \cup & & \cup \\ (\mathbb{C}^*)^{\dim X} & \cong & (\mathbb{C}^*)^{\dim Y} \end{array}$$

- $\dim_{\mathbb{C}} X = 1$, X, Y non singular: $X \overset{\text{bir}}{\sim} Y \leftrightarrow X \cong Y$

Example: $\dim_{\mathbb{C}} X = 1$, X toric non singular $\Rightarrow X = \mathbb{P}^1$

- $\dim_{\mathbb{C}} X \geq 2$ X, Y toric, $X \overset{\text{bir.}}{\sim} Y \Leftrightarrow \Sigma_X \dashleftarrow\dashrightarrow \Sigma_Y$

adding / removing (d-) faces.

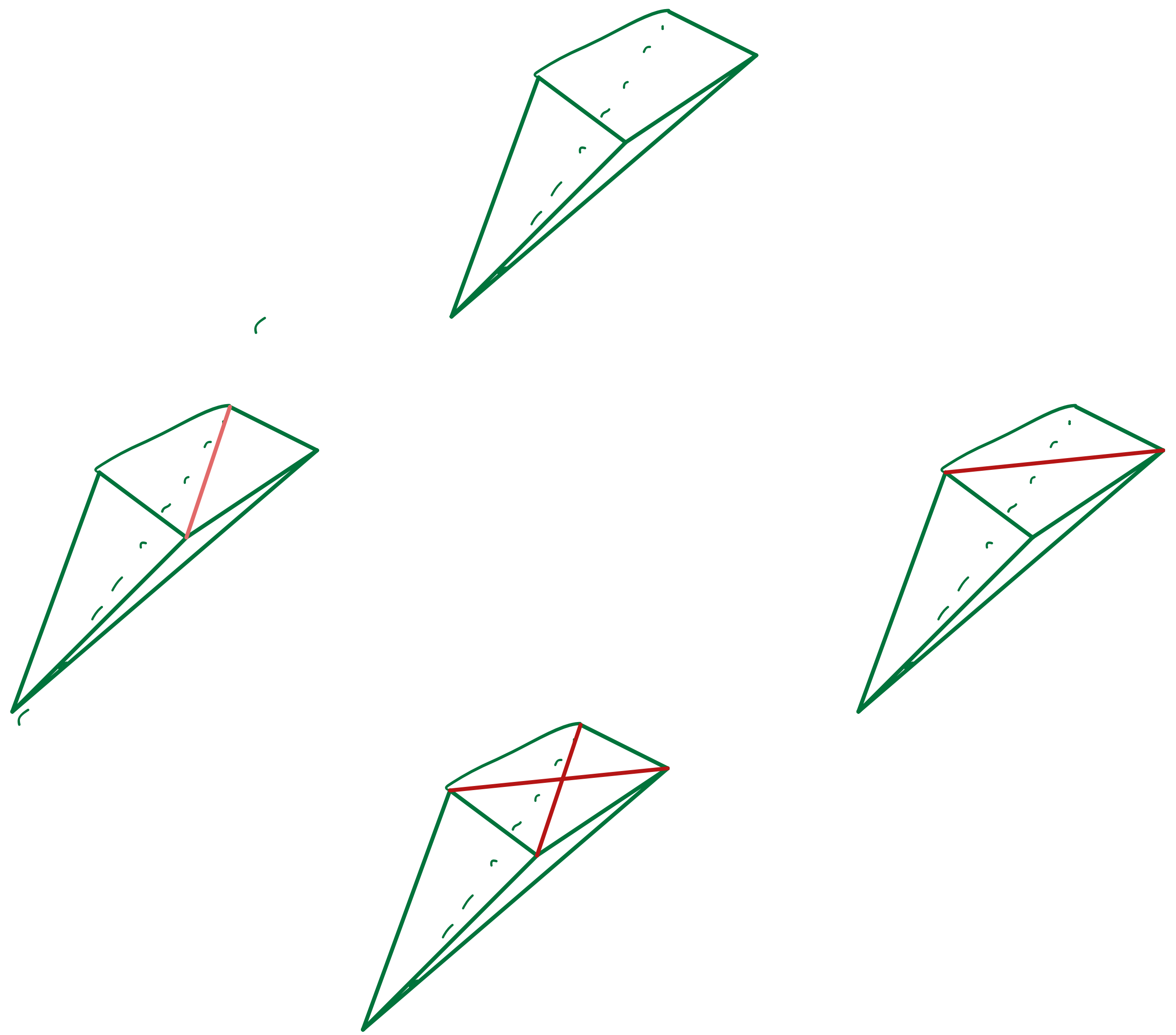
- X Toric, proj : $\mapsto$ $X \overset{\text{bir}}{\sim} \mathbb{P}^{\dim X}$

- <u>Def</u>: $X \overset{\text{bir}}{\sim} \mathbb{P}^{\dim X}$ is <u>RATIONAL</u>

$\Updownarrow$

($\exists$ a rational parametrization of X)

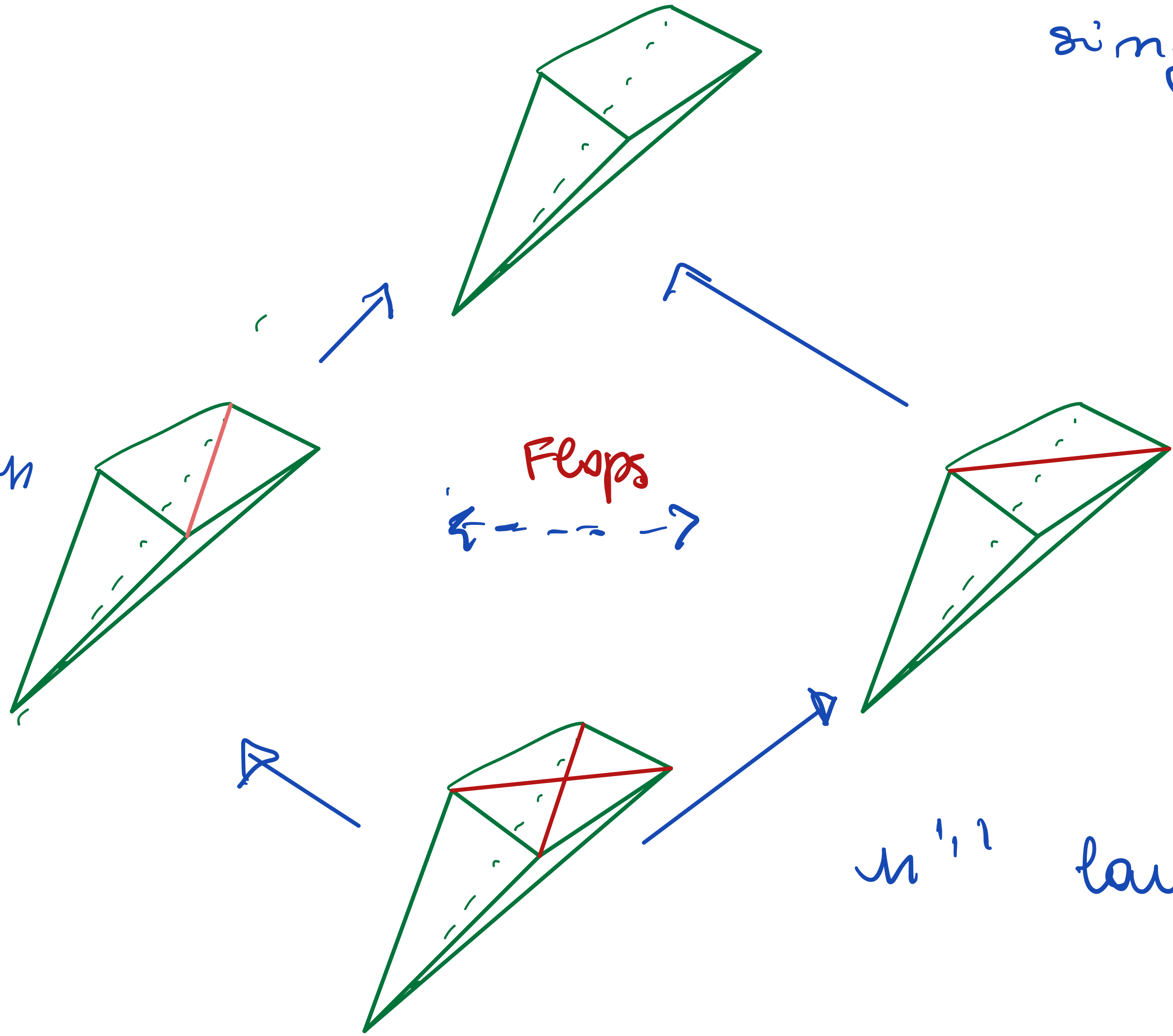
Example: X°_Σ, local form Σ°



<u>Example:</u> X°, local form Σ°

singular

smooth



smooth

$h^{1,1}$ same

(b_2 same)

$h^{1,1}$ larger, smooth.

All the arrows / dash arrows are birational

Minimal model program:

Given X find Y such that

- $X \overset{\text{bir}}{\sim} Y$
- Y is "simpler" than X

"better"

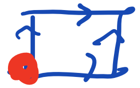
M.M.P - took off from:

- $\dim_{\mathbb{C}} X \geq 3$ X_Σ toric, "min. model program."

Next:

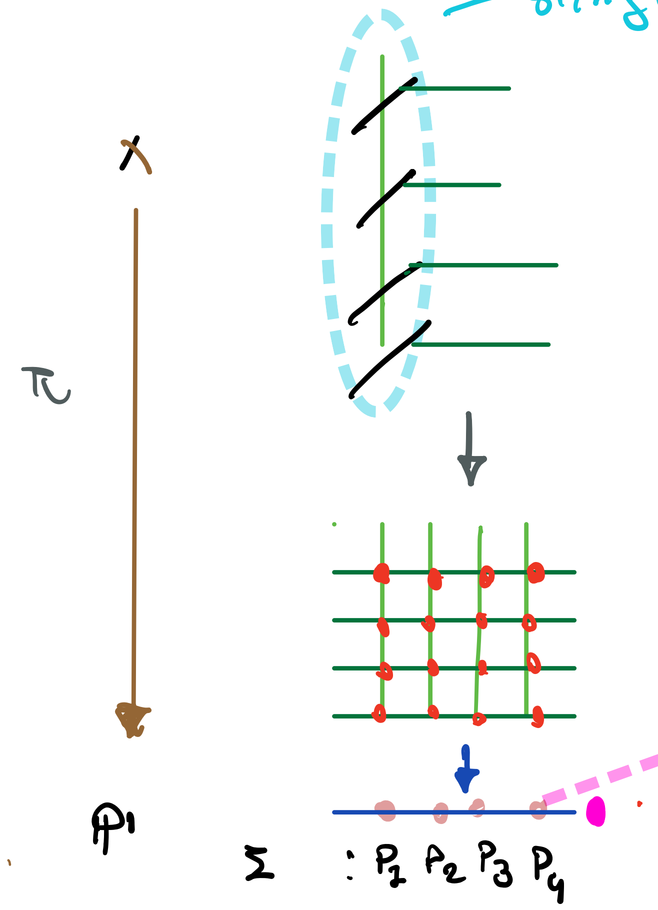
Birationality; Elliptic fibration, Toric Varietie

Example.

X, K3, Kummer surface, $E \simeq T^2$ 

$$\begin{array}{l} X \\ \downarrow \\ S = E\times E/(\pm 1) \quad \text{singular at 16 points} = \{P_1, \dots, P_{16}\} \\ \downarrow \pi_1 \qquad A_1 \text{ singularities: locally: } s^2+t^2+u^2=0 \\ B = \mathbb{P}^1 = E/(\pm 1) \end{array}$$

singular fiber (I_0^*)



$$K_X \simeq \pi^*(K_{\mathbb{P}^1} + \Delta)$$

$$\Delta = \sum_{i=1}^{4} \frac{1}{2} P_i$$

$$\frac{1}{2} = \frac{1}{12}\chi_{top}(\pi^{-1}(P_i))$$

Question: given $\pi : X \longrightarrow B$ elliptic,

does $K_X = \pi^*(K_B + \underbrace{\Delta}_{})$? $\longleftrightarrow$ singular fibers?

Kodaira: $\dim_{\mathbb{C}} X = 2$, $\exists\ Y$ smooth "minimal" (relatively minimal).

$$\begin{tikzcd} X \arrow[r, dashed, "\text{bir}"] \arrow[d, "\pi"'] & Y \arrow[dl, "\bar{\pi}"] \\ B & \end{tikzcd}$$

$K_Y = \bar{\pi}^*(K_B + \Delta)$

$\dim B = 1$

- If $\dim_{\mathbb{C}} X \geq 3$, NO, even with Y mildly ("terminal") singular.

Theorem [Particular case] [AG- 92;95]

$\pi : Y \to B_Y$, $/\mathbb{C}$ elliptic, dim $Y = 3$

$$\exists \quad \begin{tikzcd} Y \arrow[r, dashed, "\psi"] \arrow[d, "\pi_Y"'] & X \arrow[d, "\pi"] \\ B_Y \arrow[r, "\Phi"] & B \end{tikzcd}$$

Φ, ψ birational

(i) $K_X \equiv \pi^*(K_B + \Delta)$, Δ supported on Σ / B

$\Rightarrow$ (ii) X $\mathbb{Q}$-factorial, terminal, minimal

singularities of B are $\mathbb{C}^2/\Gamma$. $\Gamma \subset GL(2,\mathbb{C})$

- If, in addition, $K_B + \Delta \equiv 0$, no multiple fibers, $\Delta \not\equiv 0$. X Calabi-Yau,

$\to$ • B smooth:

• $\exists B'$

$X \to B \xrightarrow{\text{bir.}} B'$, B': $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_1, \dots, \mathbb{F}_{12}$

Corollary: Y, Calabi-Yau, $\dim_{\mathbb{C}} X = 3$, no multiple fiber

$$\begin{array}{c} Y \\ \downarrow \\ B_Y \end{array} \qquad \Delta \neq 0$$

$\rightarrow$ $\exists$ $X \overset{\text{bir}}{\sim} Y$, X CY $\quad \exists$ $B' \overset{\text{bir}}{\sim} B$

$$\begin{tikzcd} Y \arrow[r, dashed, leftrightarrow, "\text{bir.}"] \arrow[d] & X \arrow[d] \arrow[dr] & \\ B_Y \arrow[r, "\text{bir}"] & B \arrow[r, "\text{bir}"] & B' \end{tikzcd}$$

B': $\mathbb{P}^2$, $\mathbb{F}_n$, $0 \leq n \leq 12$

Toric

Finite, B rational.

Theorem (Gross): $\forall X$, CY elliptic, B rational. $\longrightarrow$ $\exists N$ s.t. $h^{1,1}(X) \leq N$

Theorem (AG- 92;95 ; AG-Wen 2021)

$\pi: Y \longrightarrow B_Y$, $/\mathbb{C}$ elliptic. Then:

(i) $\Delta_Y \neq 0 \Rightarrow h^1(Y, \mathcal{O}_Y) = h^1(B_Y, \mathcal{O}_{B_Y})$, $\mathrm{Kod}(Y) = \mathrm{Kod}(B_Y + \Delta_Y)$

If either: $\dim Y \leq 5$ / MMP flips exist and terminate in $\dim Y - 1$

(ii) $\exists$

$$\begin{array}{ccc} Y & \overset{\psi}{\dashrightarrow} & X \\ {\scriptstyle \pi_Y} \downarrow & & \downarrow {\scriptstyle \pi} \\ B & \xrightarrow{\Phi} & B \end{array}$$

Φ, ψ birational

$\Rightarrow$

X $\mathbb{Q}$-factorial terminal, (B, Δ) $\mathbb{Q}$-factorial, MMP $\to X$ minimal. / $\to X$ Mori Fiber space
bir.

$K_X \equiv \pi^*(K_B + \Delta)$, Δ supported on Σ /B

(iii) If $\mathrm{Kod}(X) = 0$, $\Delta \neq 0$ $\longrightarrow$ $B \dashrightarrow B'$ B' Mori fiber space.

(iv) If $\dim X = 3$, $\omega_X \sim \mathcal{O}_X$, $h^1(\mathcal{O}_X) = 0$, without multiple fibers, $\Delta \neq 0$

B smooth: $X \to B \overset{\text{bir.}}{\to} B'$, B' : $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_1, \dots, \mathbb{F}_{12}$

Another Technical slide.

Theorem (Gross 1994)

X elliptic Calabi-Yau 3-folds $X \to B$, B rational.
are birationally bounded

Theorem (Filipazzi - Hacon - Svaldi 2022)

There are only finitely many topological types of elliptic Calabi-Yau threefolds.

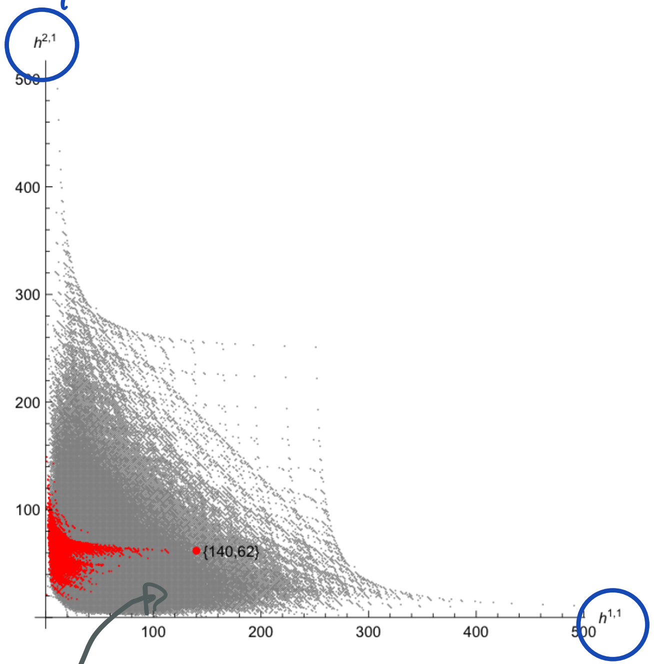
Theorem (Engel - Filipazzi - Greer - Mauri - Svaldi, 2025)

X elliptic Calabi-Yau are birationally bounded.
in any dimension.

Corollary: X elliptic, Calabi Yau $\longrightarrow$ $h^{1,1}(X)$ is bounded

Corollary: X elliptic Calabi-Yau 3fold $\longrightarrow$ $h^{2,1}$ is bounded.

$h^{2,1} = b^3/2 - 1$, b_3 : Betti number, topological.



$h^{1,1} = b_2$: Betti number topological.

elliptic CY 3folds in Toric ambient space.

Can we find explicit bounds for

$h^{1,1}(X)$, if X 3fold $h^{2,1}(X)$?

- Partial answer for elliptic fibration with section.

§: Elliptic fibration with section.

• Elliptic curve, Marked.

Complex torus. Marked

• ~~Ellipses~~ ?

← Elliptic integrals.

(Abel : date 1825)

Take ①

E : Plane curve

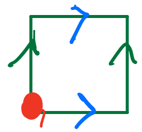
• $y^2 = x^3 + ax + b$

$a, b \in K$

(Flex point)

Take ②

E : Quotient , $K = \mathbb{C}$



$\mathbb{C} / \mathbb{Z} \oplus (\tau)\mathbb{Z}$

T^2

$\exists$ structure of an abelian group on E, over K.

Question : what is the structure of $E(K)$ the group of K-rational points ?

Theorem : (Mordell ~ 1920) $E(\mathbb{Q}) \simeq \underbrace{E(\mathbb{Q})_{tors}}_{\text{finite, abelian.}} \oplus \mathbb{Z}^r$

Theorem : (Weil ~ 1928) $E(K) \simeq E(K)_{tors} \oplus \mathbb{Z}^r$, $[K:\mathbb{Q}] < \infty$

rank

Question [Classical] Describe $E(K)$, other K

Still a question . $MW(E/K)$: Mordell-Weil group.

Def: • X is <u>elliptic</u> with section if .

$$\begin{tikzcd} X \arrow[d, "\pi"'] & \supseteq & E(\mathbb{C}) \arrow[d] \\ B \arrow[u, bend left=60, dashed, "\sigma"] & \supsetneq U \ni & p \end{tikzcd}$$

, $\mathcal{U}$ dense open

• $\sigma : B \dashrightarrow X$ rational , $\pi \circ \sigma = \mathrm{Id}$ <u>is a section</u>.

$MW(X/B) = E(K) = E(k(B))$ <u>Mordell-Weil</u> group of sections.

$\{\text{sections}\} \longleftrightarrow U(1)^r$ Abelian-Gauge Symmetries

$\mapsto$ • $\exists\, W$, $X \overset{\text{bir}}{\sim} W$, <u>Weierstrass model</u>: $y^2 = x^3 + a(\ )x + b(\ )$

$a(\ ), b(\ ) \in k(B)$

- $\operatorname{rk} MW(X/B) \leq h^{1,1}(X) \leq$??

Theorem (W. Taylor), $\dim X = 3$

$X \to B$, elliptic with section, X C-Y $\to h^{2,1}(X) \leq 491$

Shuetuneth.

Proof: $\longrightarrow$

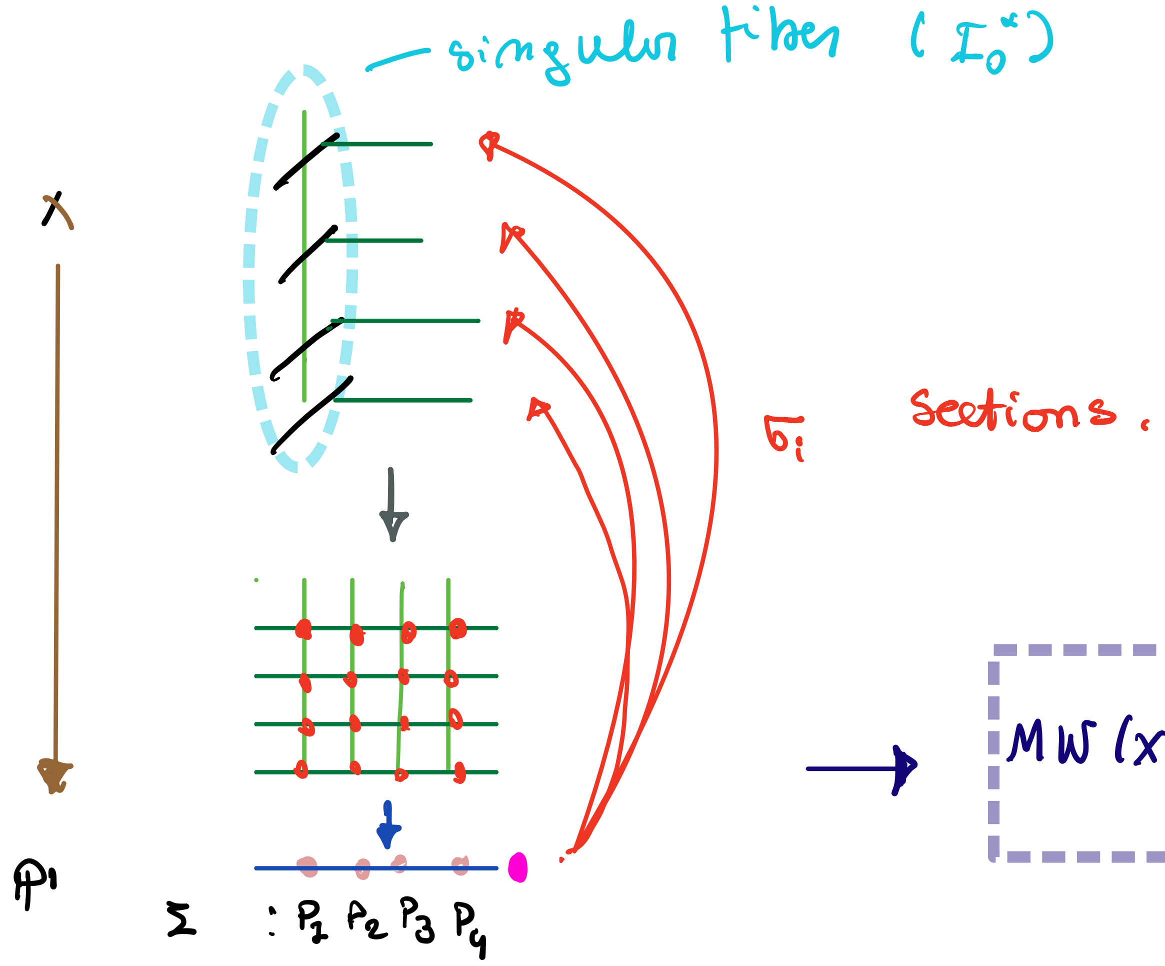
$$\begin{array}{ccc} X & & \\ \downarrow & & \\ B & \longrightarrow & B' \end{array}$$

, $B' = \mathbb{P}^2$, $\mathbb{F}_n$, $0 \leq n \leq 12$

- $h^{2,1}(X) \leq h^{2,1}(Y \to \mathbb{F}_{12}) = 491$ ∎

Example ;

$$\begin{array}{ccl} X & , & \text{K3, Kummer surface} \\ \downarrow & & \\ S = E\times E/(\pm 1) & & \text{singular at 16 points. } = \{p_1, \dots, p_{16}\} \\ \downarrow \pi_1 & & A_1 \text{ singularities: locally : } s^2+t^2+u^2=0 \\ B = \mathbb{P}^1 = E/(\pm 1) & & \end{array}$$



$\longrightarrow$ $\boxed{MW(X/\mathbb{P}^1) = \mathbb{Z}^4 \oplus (\mathbb{Z}/2\mathbb{Z})^2}$

- $c_k \, MW(x/B) \le h^{1:d}(x)$

bound.

Physics Predictions :

Lee - Regalado - Weigand, 2018
Martucci - Risso - Weigand, 2023
Lee - Oehlmann 2023

$X \longrightarrow B$ X, Calabi-Yau elliptic, smooth, B smooth.

- $\dim_{\mathbb{C}} X = 2, 3$ $\longrightarrow$ $\operatorname{rk} MW(X/B) \leq 28$ if $B = \mathbb{P}^2_{\mathbb{C}}$, ($\dim X = 3$)
  $\longrightarrow$ $\operatorname{rk} MW(X/B) \leq 18$ otherwise.

- $\dim_{\mathbb{C}} X = 4$ No explicit bound.

  ? $\exists$ a movable, rational smooth curve $C \subset B$ "s.t.
  $\operatorname{rk} MW(X/B) \leq 10(-K_B).C - 2$.

200 years after Abel

Theorem (AG, Miranda, Paranjape, Srinivas, Weigand - 2020)

$$\begin{array}{l} X \\ \downarrow \pi \\ B \end{array}$$ Calabi-Yau, π: w/ section

Then:

- $\dim X = 3 \longrightarrow \begin{cases} \mathrm{rk}\, MW(X/B) \le 28 & B = \mathbb{P}^2 \\ \mathrm{rk}\, MW(X/B) \le 18 & \text{otherwise.} \end{cases}$

- $\dim X = 4 \longrightarrow \mathrm{rk}\, MW(X/B) \le 38$

  (+) mild assumptions

Theorem (AG, Miranda, Paranjape, Srinivas, Weigand - 2026)

$$\begin{array}{l} X \\ \downarrow \pi \\ B \end{array} \quad \text{Calabi-Yau}, \quad \pi\text{: w/ section}$$

Then:

- $\dim X = 3 \longrightarrow \begin{cases} \mathrm{rk}\, MW(X/B) \leq 28 & B = \mathbb{P}^2 \\ \mathrm{rk}\, MW(X/B) \leq 18 & \text{otherwise.} \end{cases}$

$\longrightarrow$ It confirms the physics predictions. [Toric case]

- $\dim X = 4 \longrightarrow \mathrm{rk}\, MW(X/B) \leq 38$..

  ⊕ mild assumptions

$\longrightarrow$ Beyond the physics prediction. [Non Toric cases]

Theorem: $\exists$ X is Calabi-Yau, elliptic (quasi-Weigand ✱) $MW(X/\mathbb{P}^1) \simeq \mathbb{Z}^{10}$

Remark. This example has the highest known rk $MW(X/B)$.

Theorem (AG, Miranda, Paranjape, Srinivas, Weigand - 2026)

X Calabi-Yau, π: w/ section
$$\begin{array}{c} X \\ \downarrow \pi \\ B \end{array}$$

Then:

- $\dim X = 3 \longrightarrow \begin{cases} \operatorname{rk} MW(X/B) \leq 28 & B = \mathbb{P}^2 \\ \operatorname{rk} MW(X/B) \leq 18 & \text{otherwise.} \end{cases}$

- $\dim X = 4 \longrightarrow \operatorname{rk} MW(X/B) \leq 38$
  + mild assumptions

Proof: $\exists\, C \subset B^{\text{smooth}}$, curve s.t. $\pi^{-1}(C) = Z$ $\to$ smooth / $\xrightarrow{\text{bir}}$ K3 / $\xrightarrow{\text{bir}}$ RES of dP_9

$C \underset{\neq}{\subseteq} \Sigma_{X/B}$

$$\mathrm{MW}(X/B) \leq \mathrm{MW}(Z/C)$$

$$\Downarrow$$

$$\mathrm{rk}\ \mathrm{MW}(X/B) \leq 10(-K_B)\cdot C + 2g(C) - 2$$

— ~

If $\dim X = 3$, X Calabi-Yau, $\pi: X \longrightarrow B$ with section:

B smooth: $X \to B \xrightarrow{\text{bir.}} B'$, B': $\mathbb{P}^2$, $\underbrace{\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_1, \ldots, \mathbb{F}_{12}}$

Toric.

C: extremal ray of $NE(B)$, C not isolated curve.

$$\Downarrow$$

C is rational; $C \simeq \mathbb{P}^1$

(Physics analysis - C always rational?)

" B Toric, smooth ; C nef $\mapsto$ C base point free "

$\Updownarrow$

C movable, "EFT - Physics"

- To show

$MW(X/B) \leq MW(Z/C)$ is delicate, otherwise.