Digit expansions in Rational and Algebraic Basis

Lucía Rossi

TU WIEN

FWF Österreichischer Wissenschaftsfonds

RATIONAL BASE NUMBER SYSTEMS

(Akiyama, Frougny and Sakarovitch, '08)

$\frac{a}{b} \in \mathbb{Q}$, $\quad a > b \geq 2$ $\qquad$ base

$\mathcal{D} = \{0, 1, \dots, a-1\}$ $\qquad$ digit set

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Given $N \in \mathbb{N}$, they introduced an expansion

$$\boxed{N = \frac{1}{b} \sum_{j=0}^{k} \left(\frac{a}{b}\right)^j d_j}$$

($k \geq 0$, $d_j \in \mathcal{D}$) through the modified division algorithm.

$N = N_0$, $j \geq 0$, $\boxed{bN_j = aN_{j+1} + d_j}$

where $d_j \equiv bN_j \mod a$ (recall $\mathcal{D} = \{0, \dots, a-1\}$).

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Example: $\frac{a}{b} = \frac{3}{2}$, $\mathcal{D} = \{0, 1, 2\}$, $\underline{N = 4}$

$$2 \cdot \underline{4} = 3 \cdot N_1 + d_0 \implies d_0 = 2 \qquad N_1 = 2$$

$$2 \cdot 2 = 3 \cdot N_2 + d_1 \implies d_1 = 1 \qquad N_2 = 1$$

$$2 \cdot 1 = 3 \cdot N_3 + d_2 \implies d_2 = 2 \qquad \underline{N_3 = 0}$$

$$\underline{4} = \frac{3}{2} N_1 + \frac{2}{2} = \left(\frac{3}{2}\right)^2 N_2 + \frac{3}{2} \cdot \frac{1}{2} + \frac{2}{2}$$

$$= \frac{1}{2}\left(\left(\frac{3}{2}\right)^2 \cdot 2 + \left(\frac{3}{2}\right) \cdot 1 + 2 \right) = (212)_{3/2}$$

Reinterpreting the algorithm:

$N_j \in \mathbb{N}$

$$\boxed{b N_j = a N_{j+1} + d_j}$$

Now: $\frac{a}{b}$ can be negative,

$N_j \in b\mathbb{Z}$

$$\boxed{N_j = \frac{a}{b} \cdot N_{j+1} + d_j}$$

$$\Longrightarrow N = \left(\frac{a}{b}\right)^k d_k + \dots + \left(\frac{a}{b}\right) d_1 + d_0$$

(I can get rid of the $\frac{1}{b}$ factor)

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$$\boxed{\begin{aligned} T_{a/b} : b\mathbb{Z} &\longrightarrow b\mathbb{Z} \\ N &\longmapsto \frac{N-d}{a/b} \qquad d \in \mathcal{D} \end{aligned}}$$

backward division map restricted to a lattice

ALGEBRAIC NUMBER SYSTEMS

$\alpha \in \mathbb{Q}(i)$, $\quad |\alpha| > 1 \qquad \alpha \notin \mathbb{R}$

$P_\alpha(X) = a_2 X^2 + a_1 X + a_0$ minimal polynomial

$a_2 \neq 1$

$\mathcal{D} = \{0, 1, \dots, |a_0| - 1\}$

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We need a lattice $\Lambda_\alpha \subset \mathbb{C}$ such that

$$\boxed{\begin{aligned} T_\alpha : \Lambda_\alpha &\longrightarrow \Lambda_\alpha \\ x &\longmapsto \frac{x-d}{\alpha} \end{aligned}}$$

is well defined,

where d is the unique digit in $\mathcal{D}$ such that $\frac{x-d}{\alpha} \in \Lambda_\alpha$.

$$\Lambda_\alpha := \mathbb{Z}[\alpha] \cap \alpha^{-1}\mathbb{Z}[\alpha^{-1}]$$

Brunotte basis: $\Lambda_\alpha = a_2\mathbb{Z} + (a_2\alpha + a_1)\mathbb{Z}$

$$(a_2\alpha^2 + a_1\alpha + a_0 = 0)$$

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$$(a_2\alpha^2 + a_1\alpha + a_0 = 0)$$

Algorithm:

$N = N_0 \in \Lambda_\alpha$

$$N_j = \alpha N_{j+1} + d_j$$

where d_j is the unique digit such that $N_{j+1} \in \Lambda_\alpha$

Example: $\alpha = \dfrac{-1+3i}{2}$ $\qquad P_\alpha(X) = 2X^2 + 2X + 5$

$\mathcal{D} = \{0, 1, 2, 3, 4\}$

$\Lambda_\alpha = a_2\mathbb{Z} + (a_2\alpha + a_1)\mathbb{Z} = 2\mathbb{Z} + (1+3i)\mathbb{Z}$

Let $N = \underline{1+3i} \in \Lambda_\alpha$

$$\underline{1+3i} = \frac{-1+3i}{2} \cdot 2 + 2$$

$$2 = \frac{-1+3i}{2} \cdot 0 + 2$$

$$\Longrightarrow \quad \underline{1+3i} = (22)_\alpha$$

Does the algorithm always terminate?
That is, is there always $k \geq 0$ s.t. $N_k = 0$?

No, and the set of bases α with this property is extremely difficult to describe.

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This set has been well studied in the context of shift radix systems.

We assume from now on that α has the finiteness property.

The tree of α-expansions for $\alpha = \frac{-1+3i}{2}$

The tree for $\alpha = \frac{-1+5i}{3}$, $P_\alpha = 9X^2 + 6X + 26$

$\mathcal{D} = \{0, 1, \dots, 25\}$

$\mathcal{L}_\alpha$ = language of expansions in base α

$\mathcal{L}_\alpha^k$ = expansions in base α of length k

Proposition:

$w \in \mathcal{L}_\alpha^k$

$$\{ d \in \mathcal{D} : wd \in \mathcal{L}_\alpha^{k+1} \} = \{ d \in \mathcal{D} : d \equiv r \bmod a_2 \}$$

for some r.

Corollary:

$$|\mathcal{L}_\alpha^k| \leq \left\lceil \frac{|\mathcal{D}|}{a_2} \right\rceil^k \quad \forall k \geq 0.$$

EXPANSION OF COMPLEX NUMBERS

<u>Def</u>: An α-expansion of $x \in \mathbb{C}$ is an expansion of the form

$$x = (d_k \cdots d_0 . d_{-1} d_{-2} \cdots)_\alpha = \sum_{j \le k} d_j \alpha^j$$

such that each finite prefix

$$d_k \cdots d_\ell$$

is the α-expansion of some $N \in \Lambda_\alpha$

(i.e., infinite paths on the tree).

FRACTAL TILES

$z \in \Lambda_\alpha$ $\qquad$ $z = (d_k \cdots d_0)_\alpha$

$$\mathcal{G}(z) := \left\{ x \in \mathbb{C} : x = (\overbrace{d_k \cdots d_0}\,.\, d_{-1} d_{-2} \cdots)_\alpha \right\}$$

Expansions whose integer part is z.

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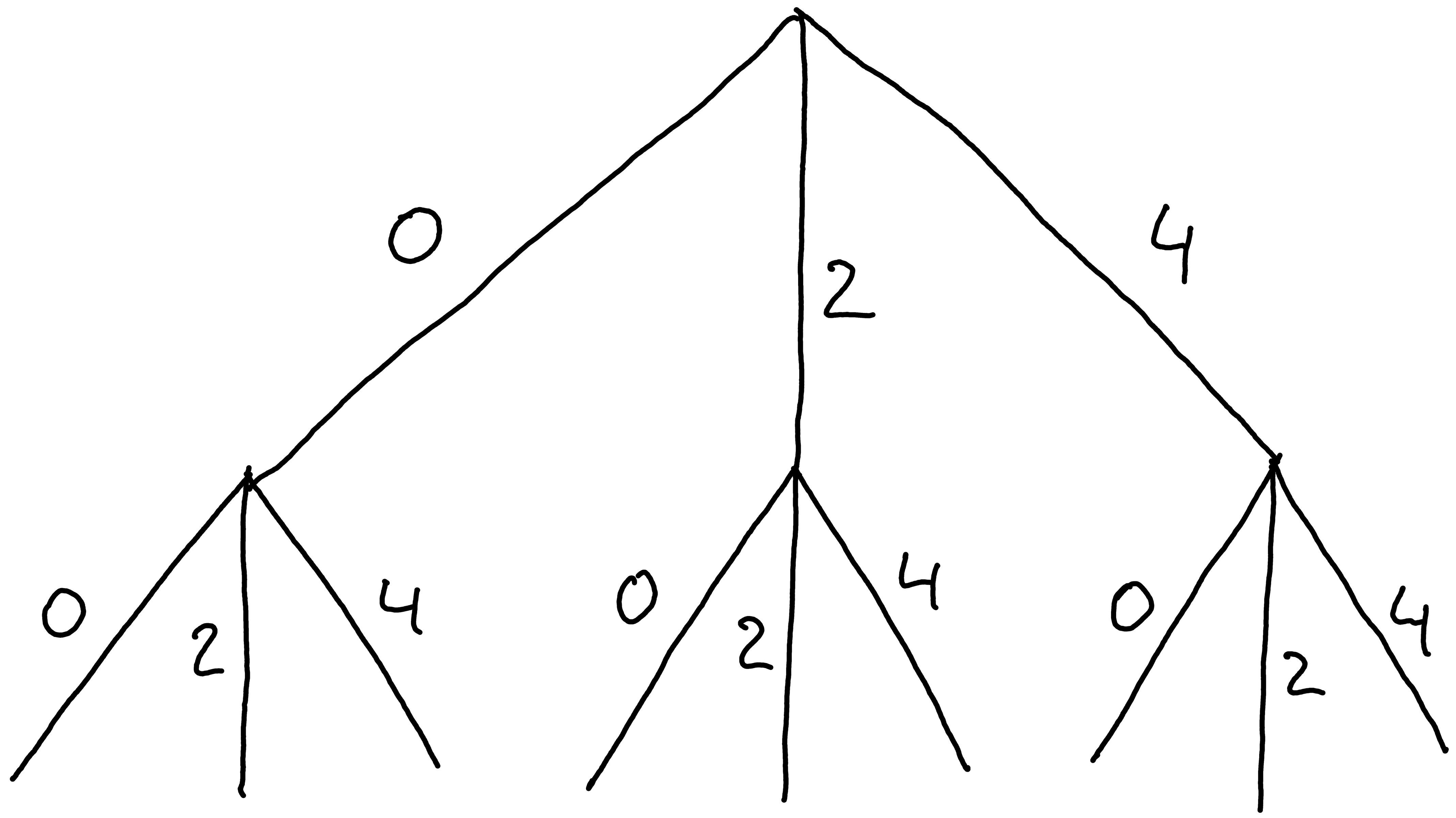
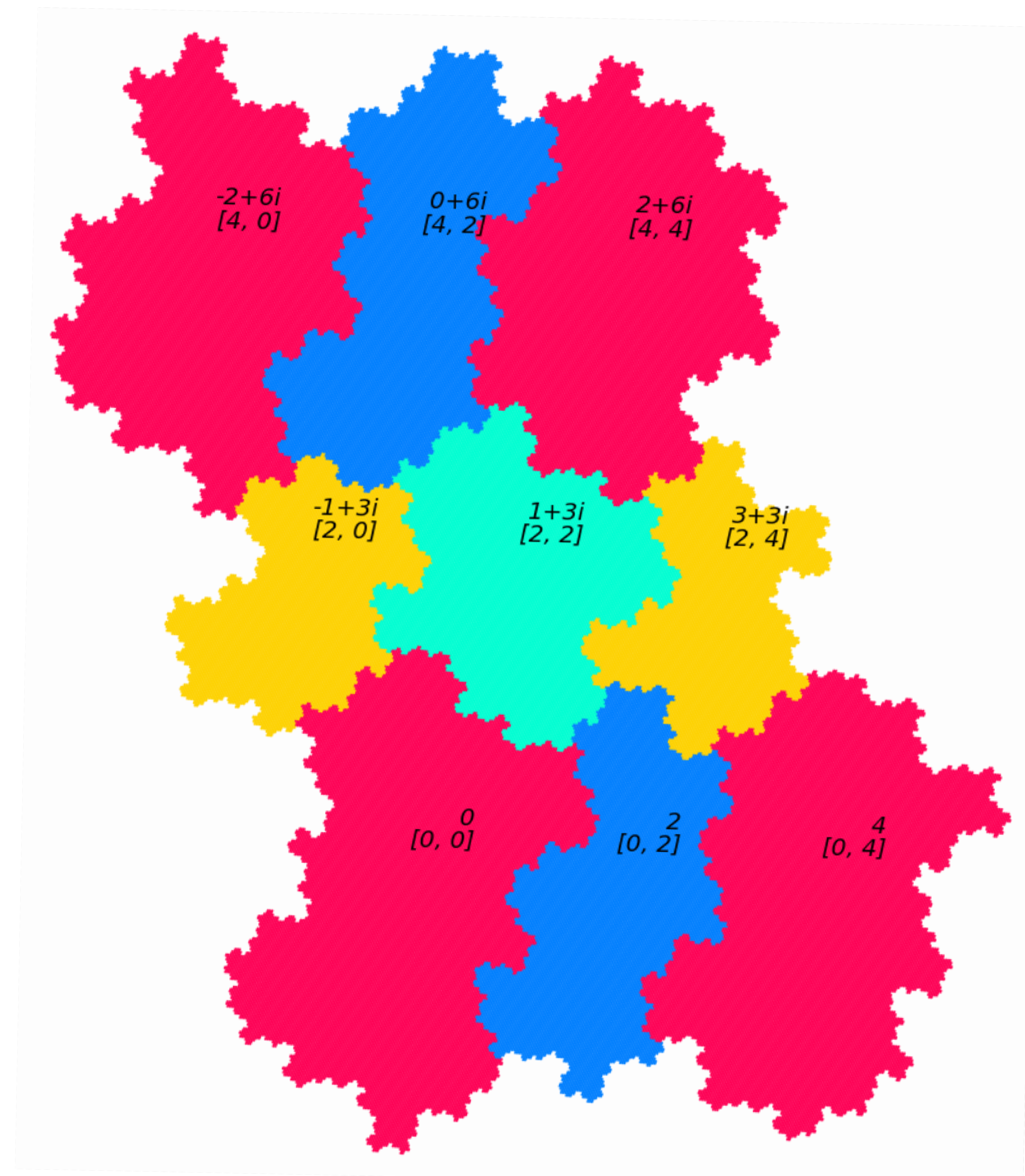
Expansions whose integer part is z.

Example: $\alpha = \dfrac{-1+3i}{2}$

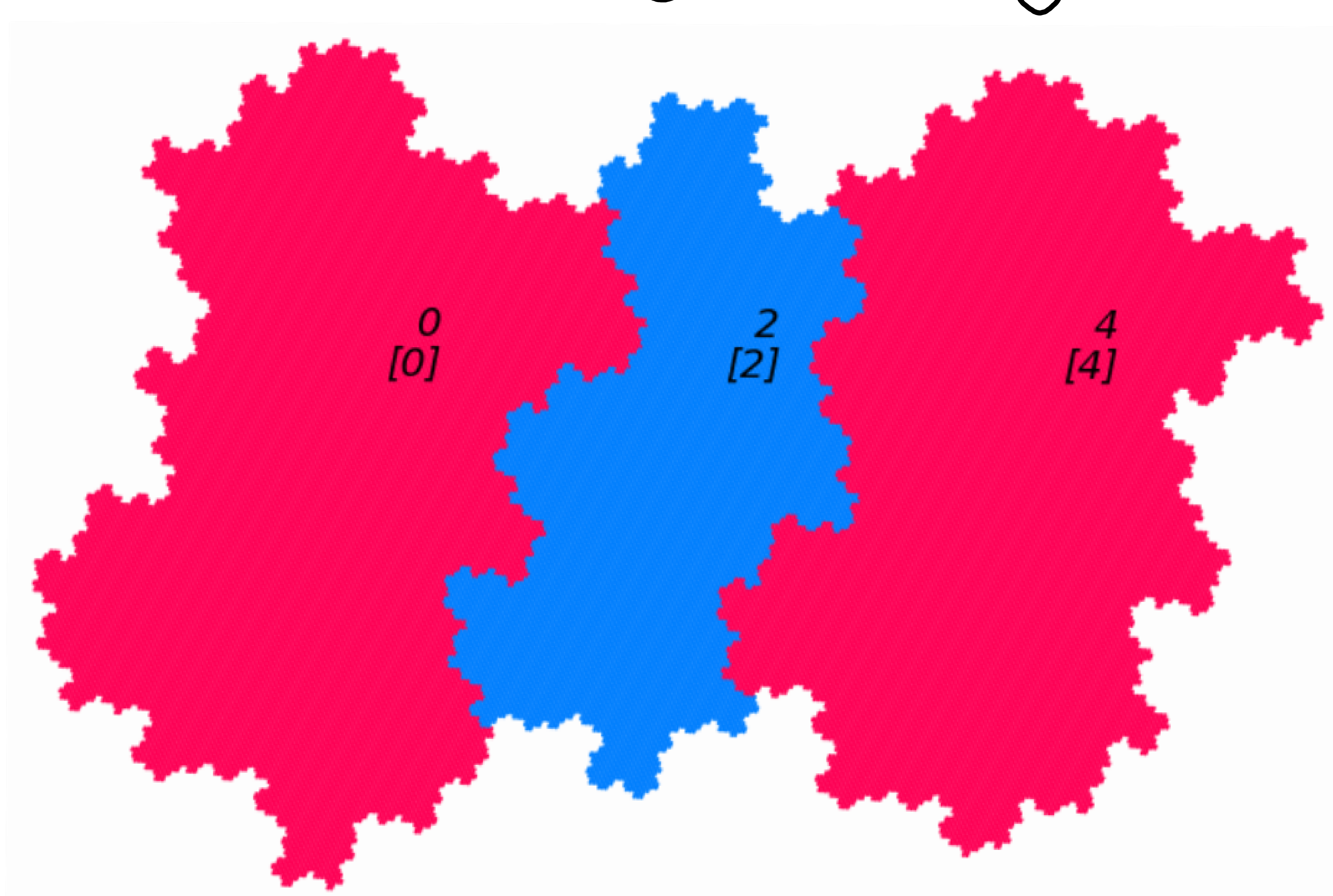
$$\sqrt{2} = (2.2341121\ldots)_\alpha \in \mathcal{G}(2)$$



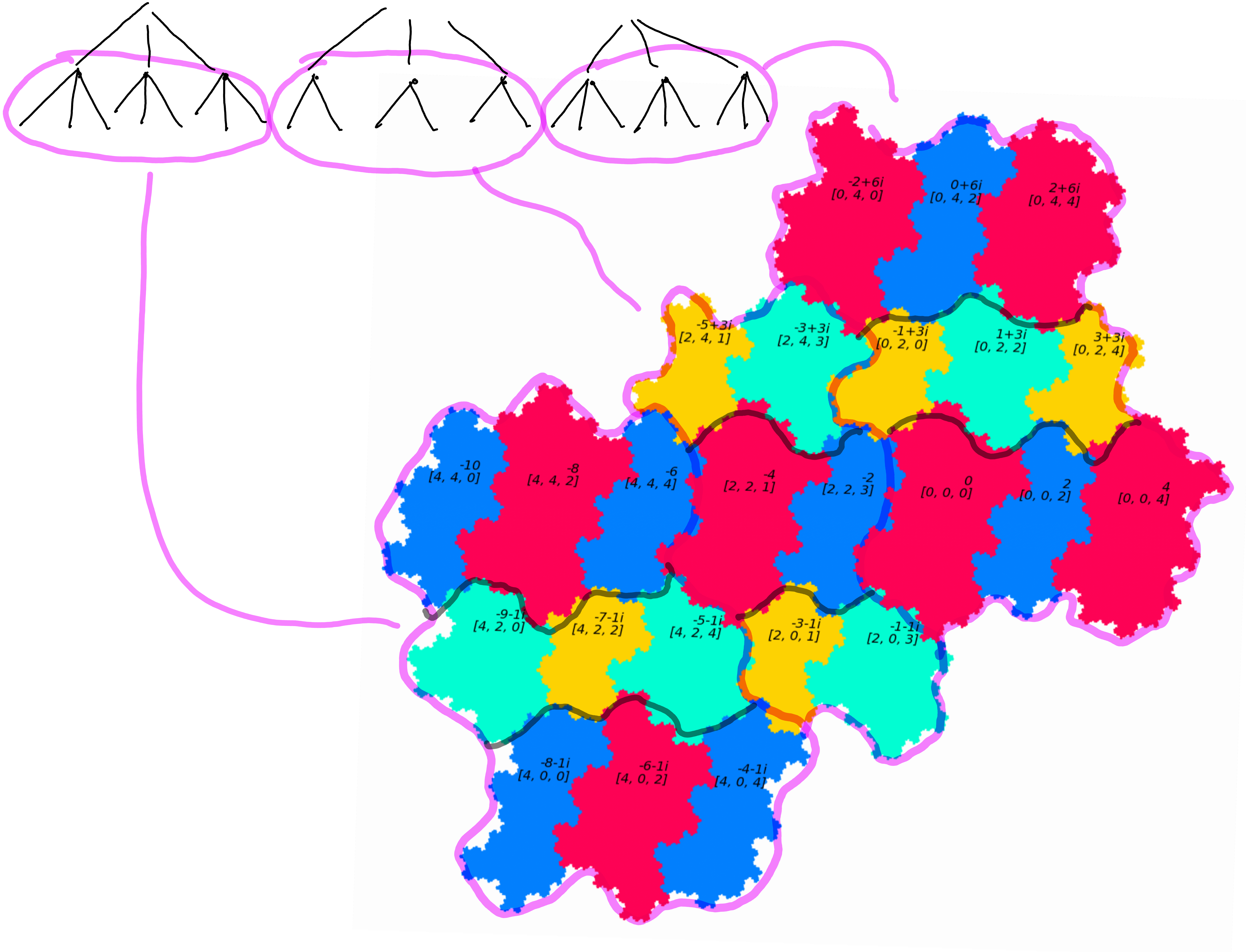
$\alpha = \frac{-1+3i}{2}$, $\{0,1,2,3,4\}$



$g(0)$ $g(2)$ $g(4)$



$\alpha g(0) = g(0) \cup g(2) \cup g(4)$



-2+6i
[0, 4, 0]
0+6i
[0, 4, 2]
2+6i
[0, 4, 4]
-5+3i
[2, 4, 1]
-3+3i
[2, 4, 3]
-1+3i
[0, 2, 0]
1+3i
[0, 2, 2]
3+3i
[0, 2, 4]
-10
[4, 4, 0]
-8
[4, 4, 2]
-6
[4, 4, 4]
-4
[2, 2, 1]
-2
[2, 2, 3]
0
[0, 0, 0]
2
[0, 0, 2]
4
[0, 0, 4]
-9-1i
[4, 2, 0]
-7-1i
[4, 2, 2]
-5-1i
[4, 2, 4]
-3-1i
[2, 0, 1]
-1-1i
[2, 0, 3]
-8-1i
[4, 0, 0]
-6-1i
[4, 0, 2]
-4-1i
[4, 0, 4]

P-ADIC COMPLETIONS

Steiner, Thuswaldner. "Rational self-affine tiles"

$p \in \mathbb{Q}(i)$ Gaussian prime

K_p p-adic completion of $\mathbb{Q}(i)$

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K_p p-adic completion of $\mathbb{Q}(i)$

$\alpha \in \mathbb{Q}(i) \longrightarrow$ unique factorization into Gaussian primes

$$\alpha = \frac{\mathrm{num}(\alpha) \in \mathbb{Z}[i]}{\mathrm{den}(\alpha) \in \mathbb{Z}[i]} \quad \text{(coprime)}$$

$$\mathrm{den}(\alpha) = p_1^{r_1} \cdot \ldots \cdot p_s^{r_s}$$

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Theorem: a series $\sum_{j \leq k} d_j \alpha^j$ is an α-expansion if and only if it converges to 0 in K_p $\forall p \mid \mathrm{den}(\alpha)$

Theorem:

$\{g(z) : z \in \Lambda_\alpha\}$ forms a tiling of $\mathbb{C}$

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Corollary:

The α-expansion of a complex number is unique almost everywhere.

OPEN QUESTIONS

- What are exactly the points with multiple α-expansions?

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- What are exactly the points with multiple α-expansions?

- Let $\mathcal{L}_\alpha^\omega$ be the language of all (infinite) α-expansions.
  Do all digits (resp. finite words) appear with the same frequency?

THANK

YOU!