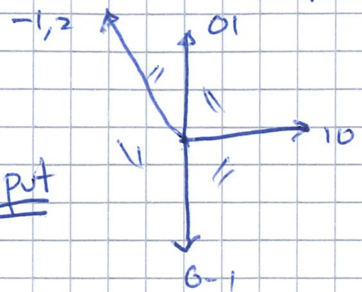
LECTURE 1:
- Quotients in algebraic geometry (toric version)
- Nef + Mori cones, intersection theory (toric version)

- Parable (Philip last week) $\mathbb{P}^2$ → affine charts $\mathbb{A}^2 \cup \mathbb{A}^2 \cup \mathbb{A}^2$ • road atlas
  → global quotient: $\mathbb{C}^3 \setminus 0 / \mathbb{C}^*$ • globe (almost)
- we'll go from example to definition = backwards.

§1 Toric Variety as a quotient: Σ a fan (SCRPC) in $N_{\mathbb{R}}$

input



$$0 \to \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 2 \end{bmatrix}} \mathbb{Z}^4 \xrightarrow{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}} \mathbb{Z}^2 \to 0$$

ray matrix Y

coker of this is $Cl(X_\Sigma)$

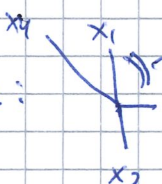
output: X_Σ a toric variety, constructed as follows: → $\mathbb{C}^{|\Sigma(1)|=d}$ #rays; → $V(B_\Sigma)$ "bad locus"; → $G \subseteq (\mathbb{C}^*)^d$

① $\mathbb{C}^d$ = big space (example: $\mathbb{C}^4$)

② B_Σ = "toric irrelevant ideal" $\subseteq S = \mathbb{C}[x_1, \dots, x_d]$ ← variable for each $\rho \in \Sigma(1)$

:= label rays $x_1, \dots, x_d$, $B_\Sigma = \langle$monomials corresponding to Complement of max faces$\rangle$

(example:



complement $= x_3 x_4$

$\Rightarrow B_\Sigma = \langle x_3 x_4, x_1 x_4, x_1 x_2, x_2 x_3 \rangle$

(Primitive relation) → $= \langle x_1, x_3 \rangle \cap \langle x_2, x_4 \rangle$)

(remark/definition: $\{u_{i_1}, \dots, u_{i_k}\} \subseteq \Sigma(1)$ is a Primitive Collection if it is not a face of Σ, but all (proper) subsets are)

③ G := apply $\mathrm{Hom}_{\mathbb{Z}}(Cl(X_\Sigma), \mathbb{C}^*)$ to the sequence

$0 \to M \to \mathbb{Z}^d \to Cl(X_\Sigma) \to 0$, get $0 \to G \to (\mathbb{C}^*)^d \to (\mathbb{C}^*)^n \to 0$

rays as rows

$$G = \left\{ (t_\rho) \in (\mathbb{C}^*)^d \text{ such that } \prod_{\rho \in \Sigma(1)} t_\rho^{\langle e_i, u_\rho \rangle} = 1, \ i = 1, \dots, n \right\}$$

$$\boxed{\text{THM}} \quad X_\Sigma = (\mathbb{C}^*)^d \setminus V(B_\Sigma) / G$$

What happened in our example? Bad locus $= \begin{matrix} V(x_1, x_3) \\ V(x_2, x_4) \end{matrix}$

$\Rightarrow V(B_\Sigma) = (*, 0, *, 0) \cup (0, \underbrace{*, 0}_{[0,1,0,-1]}, *)$ not closed orbits

because $G = \prod t^{\langle e_i \cdot \text{rays} \rangle} = 1 \Rightarrow t_2 t_4^{-1} = 1, \ \overbrace{t_1 t_3^{-1} t_4^{2}}^{[1\,0\,-1\,2]} = 1$

$\Rightarrow t_2 = t_4, \ t_3 = t_1 t_4^2 = t_1 t_2^2 \Rightarrow \begin{bmatrix} t_1 & & & \\ & t_2 & & \\ & & t_1 t_2^2 & \\ & & & t_2 \end{bmatrix} \circlearrowright \mathbb{C}^4$ or $\langle \lambda, \mu, \lambda\mu^2, \mu \rangle$

the $(\mathbb{C}^*)^2 = G$

General Zen To get a "nice" quotient, we'd like the $\boxed{\text{points to correspond to closed orbits}}$ hence may need to remove "bad points" (GIT: foreshadowing of last lecture)

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§2 General Quotient construction (alg geom) $G \circlearrowright X \leftarrow$ variety

want: X/G to be a variety Y such that if

$X \xrightarrow{\varphi} Z$ s.t. $\varphi(gx) = \varphi(x)$, then

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "\pi"'] & \arrow[d, phantom, "\circlearrowleft"] & Z \\ & Y \arrow[ur, "\exists ! \Phi"'] & \end{tikzcd}$$

~~maybe~~ also X irr $\to Y$ irr,
X normal $\to Y$ normal ★

$\boxed{\text{THM if } R^G \text{ f'gend } Y = \operatorname{spec}(R^G) \text{ works!}}$

Notice (1) we'd want $g \cdot x \Longleftrightarrow \varphi_g(x)$ a morphism

(2) $X = \operatorname{spec}(R)$, function f on X i.e. $f \in R$ G invariant

Exercise action of g on R is $g \cdot f(x) = f\, g^{-1} x$.

★ Footnote (on board, hidden) R Normal $\Rightarrow R$ integrally closed in $K(R)$
R is R_1 and S_2
↳ regular in codim 1 $\Rightarrow$ can define $V(f)$ $f \in K(R)$
↳ holo fns on sm. locus extend
↳ sings in codim ≥ 2

! TORIC $\Leftrightarrow$ built from SCRPC !

Example 1 $\mathbb{Z}/2 \circlearrowright \mathbb{C}^2$ via $(-1)(x,y) = (-x,-y) \Rightarrow$ invariants $\{x^2, xy, y^2\}$

$\frac{\mathbb{C}[a,b,c]}{ac-b^2} \longrightarrow \mathbb{C}[x^2, xy, y^2]$, $V(ac-b^2) \cong \mathbb{C}^2/G$ orbits all have 2 pts except 0, orbits closed ☺

(ok for all finite gps)

Example 2: $\mathbb{C}^*$ acts on $\mathbb{C}^4$: $\lambda(a_1 \ldots a_4) = (\underset{x_1}{\lambda a_1}, \underset{x_2}{\lambda a_2}, \underset{x_3}{\lambda^{-1} a_3}, \underset{x_4}{\lambda^{-1} a_4})$

invariants $= \mathbb{C}[x_1x_3, x_1x_4, x_2x_3, x_2x_4] \longleftarrow \mathbb{C}[y_1 \ldots y_4]/_{y_1y_4 - y_2y_3}$, we'd like $\mathbb{C}^4/\mathbb{C}^* \cong V(y_1y_4 - y_2y_3)$. But consider orbit of $p = (\alpha, 0, \frac{1}{\alpha}, 0)$, $\alpha \in \mathbb{C}^*$. $p \longmapsto (1,0,0,0)$, but not closed ☹

SOLN: Restrict to a "nice" subset (semistable, stable^{GIT})

Definition (Hide on board) $G \circlearrowright X = \mathrm{Spec}(R)$, ~~then~~ then $\pi : X \to Y$ const on G orbits is a good categorical quotient if

- $U \subseteq Y$ open $\Rightarrow \mathcal{O}_Y(U) \cong \mathcal{O}_X(\pi^{-1}(U))^G$
- $W \subseteq X$ closed + G-inv $\Rightarrow \pi(W)$ closed
- W_1, W_2 " and disjoint $\Rightarrow \pi(W_1) \cap \pi(W_2) = \emptyset$

Written $X/\!/G$

- Good cat quotient is good geom quotient if $\exists$ bijection $X/\!/G \longleftrightarrow G$ orbits, i.e. all G orbits in X closed.

{still on board} $\underbrace{\text{Affine alg group}}_{\text{subgp + subvar of } GL}$, $\underbrace{\text{acting algebraically}}_{\cdot g = \text{morp}}$, reductive $\Rightarrow$ Good cat quotient $\Rightarrow$ remove bad locus, get geom. quot

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§3 Nef + Mori cones. X a normal variety.

Definition $N^1(X) \triangleq$ Cartier divisors $/ \equiv \otimes_{\mathbb{Z}} \mathbb{R}$, $\equiv \Leftrightarrow D \cdot C = 0$ $\forall$ irr complete curves C

$N_1(X) \triangleq$ irr complete curves $/ \equiv \otimes_{\mathbb{Z}} \mathbb{R} \Leftrightarrow D \cdot C = 0$ $\forall$ Cartier D.

Thm: These are dual vector spaces, finite dimensional$/\mathbb{R}$, i.e. $\exists$ nondegenerate pairing!

<u>Definition</u> • $Nef(X)$ = cone gen'd by $D \in N^1(X)$ | $D \cdot C \geq$ ∀ irr complete curves C

• $NE(X)$ = cone gen'd by C irr complete curve $\subseteq N_1(X)$, <u>Mori cone</u> $= \overline{NE}(X)$

$\boxed{\text{THM}}$ $Nef(X) = \overline{NE}(X)^\vee$, $\overline{NE}(X) = Nef(X)^\vee$ < $Nef(X)$ is strongly convex / $NE(X)$ is full dim.

§ <u>Toric Case</u>: First, recall <u>Cartier Divisor</u> is defined via <u>local</u> <u>data</u> $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ is Cartier if $\forall \sigma \in \Sigma(1)$, $\exists\, m_\sigma$ s.t. $\langle m_\sigma, u_\rho \rangle = -a_\rho$ $\forall \rho \in \sigma$

and $Pic(X) = CDiv(X)/\sim$, so $Pic(X) \subseteq Cl(X)$

$\boxed{\text{Thm}}$ with $Pic(X)$ free abelian if Σ contains a full dim cone

• $Pic\, X_\Sigma = Cl(X_\Sigma) \iff \Sigma$ smooth, $Pic(X_\Sigma)$ finite index in $Cl(X_\Sigma)$ if Σ <u>simplicial</u>.

$\boxed{\text{Thm}}$ • $Nef(X_\Sigma)$ is rational polyhedral in $Pic(X_\Sigma)_{\mathbb{R}}$

wall = sep two max cones

• $NE(X_\Sigma) = \overline{NE}(X_\Sigma)$ " in $N_1(X_\Sigma)_{\mathbb{R}}$, gen'd by $V[\tau]$

→ Every irr. complete curve in X_Σ is num ~ to $\sum a_i V[\tau_i]$, $a_i > 0$

• and if X_Σ is <u>projective</u>, $Nef(X_\Sigma)$, $NE(X_\Sigma)$ <u>dual</u> SCRPC of full dimension.

$\boxed{\text{Thm}}$ <u>Toric Kleiman</u>: D nef $\iff D \cdot C \geq 0$ ∀ T-inv curves ⟺ <u>ϕ_D convex</u>

D ample $\iff D \cdot C > 0$ ∀ T-inv. curves ⟺ ϕ_D strictly convex ⊛

☆

$D = \sum a_i D_i$, $\phi_D(u_i) = -a_i$ PL fn.

$\boxed{\text{Thm}}$ X_Σ proj + simplicial, then [Batyrev]

D nef $\iff \phi_D(\sum u_i) \geq \sum \phi_D(u_i)$

ample " $>$ "

∀ prim. collections.

⊛ <u>Rmk</u>: ϕ_D piecewise linear, $\Sigma(u_i)$ in <u>some</u> max. cone.

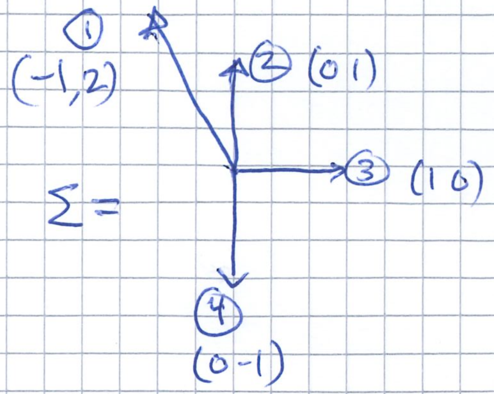
careful.

§ Toric Example

Hirzebruch$_2$

$$\boxed{X_\Sigma = \mathcal{H}_2}$$

$\Sigma =$



Lets find $\mathrm{Nef}(X_\Sigma)$, $\mathrm{Mori}(X_\Sigma)$.

(1) $$\mathbb{Z}^2 \xrightarrow[\begin{bmatrix}-1 & 2\\ 0 & 1\\ 1 & 0\\ 0 & -1\end{bmatrix}]{A} \mathbb{Z}^4 \xrightarrow[\begin{bmatrix}1 & -2 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}]{B} \mathbb{Z}^2 = Cl(X_\Sigma) = Pic(X_\Sigma) \to 0$$

$B_1\ B_2\ B_3\ B_4$ ← these will generate the effective cone in $N^1(X_\Sigma)$

(2) Primitive Collections: $\{u_1, u_3\}$, $\{u_2, u_4\}$

(3) Pick basis to make computation easy: $B_3 + B_4$ $(D_3 + D_4)$

From linear equivalence (A matrix) $-D_1 + D_3 = 0 \Rightarrow D_1 = D_3$

$2D_1 + D_2 - D_4 = 0$

$D_2 = D_4 - 2D_1 = D_4 - 2D_3$

arbitrary D can be written as

$0D_1 + 0D_2 + aD_3 + bD_4$

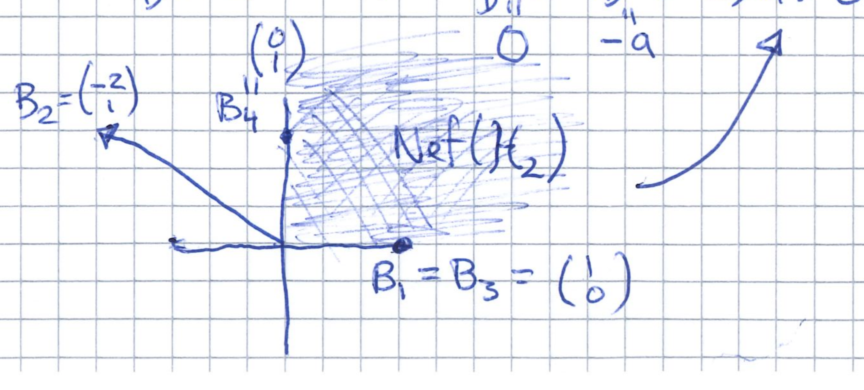
(4) Use primitive relations: • $v_2 + v_4 = (0,0)$ so need

$$\phi_D(0,0) = 0 \geq \underset{0}{\underset{\shortparallel}{\phi_D(v_2)}} + \underset{-b}{\underset{\shortparallel}{\phi_D(v_4)}} \Rightarrow b \geq 0$$

$$\boxed{\text{recall: } \phi_D(u_i) = -a_i \quad D = \textstyle\sum a_i D_i}$$

• $v_1 + v_3 = 2v_2$

$$\phi_D(2v_2) = 2\cdot 0 \geq \underset{0}{\underset{\shortparallel}{\phi_D(v_1)}} + \underset{-a}{\underset{\shortparallel}{\phi_D(v_3)}} \Rightarrow a \geq 0$$



But there is more to be mined from the primitive relation. By definition, we can write $\sum_{p\in P} u_p = \sum_{p\in\sigma(1)} c_p u_p$ for *some* max cone σ

$p\in P \leftarrow$ the prim. reln

$\Rightarrow \sum_{p\in P} u_p - \sum_{p\in\sigma(1)} c_p u_p = 0 \leftarrow$ write coeffs as a vector b_p.

$\boxed{\text{Theorem}}$ The relations $(b_p) =: r(P)$ generate $\overline{NE}(X_\Sigma)$

i.e. $NE(X_\Sigma) = \sum \mathbb{R}_{\geq 0}\, r(P)$

$\boxed{\text{Theorem}}$ The pairing $D = \sum a_i D_i$ and $C = (b_p)$ is $\sum a_p b_p$!!!

<u>Mori cone of $\mathcal{H}_2$</u> (rmk: curves = divisors for a surface, bit confusing)

$D_3^\perp$ For reln $\{v_2, v_4\}$ $b = (0\,1\,0\,1)$ since $v_2 + v_4 = (0,0)$ on the nose

$D_4^\perp$ = For reln $\{v_1, v_3\}$ $b = (1\,{-2}\,1\,0)$ since $v_1 + v_3 = 2v_2$.

$\uparrow$

D_3, D_4 basis for Nef

<u>Prim Rlns</u> (write rays in terms of) (rays $\Leftrightarrow$ walls $\Leftrightarrow$ gen Mori cone)

$v_1 \leftrightarrow (0\,1\,0\,1)$

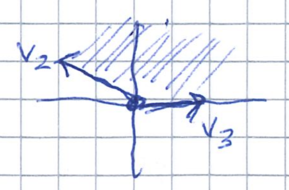
$v_2 \leftrightarrow (1\,{-2}\,1\,0)$

$v_3 \leftrightarrow (0\,1\,0\,1)$

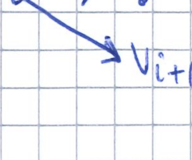
$v_4 \leftrightarrow (1\,0\,1\,2)$

$v_4 = v_2 + 2v_3$ dont need

$\Rightarrow v_2, v_3$ gen Mori $\Rightarrow$



<u>Exercise</u>: Show on a toric surface that



$D_i \cdot D_j = 0 \quad j \notin [i-1, i+1]$

$D_i \cdot D_i = -\alpha, \; \alpha v_i = v_{i-1} + v_{i+1}$ (<u>convex</u>)

$D_{i-1} \cdot D_i = \dfrac{1}{\text{mult}(\text{cone}\, v_{i-1}, v_i)} = \dfrac{1}{\det \sigma_{i,i+1}}$

- e.g. $\angle^{D_i}_{D_{i+1}} = 1 / \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 1/2$