Lower semicontinuity of integral functionals and applications

Martin Kružík

Czech Academy of Sciences, Institute of Information Theory and Automation,
Pod vodárenskou věží 4, CZ-182 00 Praha 8, Czechia

e-mail: kruzik@utia.cas.cz

Integral functionals of the form $I(y) = \int_\Omega W(x, y(x), \nabla y(x))\mathrm{d}x$ where $y : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ play an important role in variational methods applied to mathematical continuum mechanics and continuum physics, in general. Hyperelasticity, magnetism, magnetoelasticity, or electroelasticity can serve as genuine examples. The modern development has started by a seminal work by J.B. Morrey Jr. [18, 19] who identified a condition on $W(x, y, \cdot)$ making I lower semicontinuous in the weak* topology of $W^{1,\infty}(\Omega; \mathbb{R}^m)$ which is now called (Morrey's) *quasiconvexity*. In 1965, N.G. Meyers [17] significantly extended weak lower semicontinuity results for integral functionals depending on maps and their gradients available at that time to allow for integrands unbounded from below. While calculus of variations provides us with a rich toolkit of powerful approaches allowing to show lower semicontinuity of I in appropriate topology, many problems have been unsolved for decades if proper physics is taken into account. The will trace the development on this topic from that time on. Particular attention will be paid to signed integrands and to applications arising in continuum mechanics of solids. In this case, W stands for the elastic energy density of the body including work of external forces. We review existing results for polyconvex, simple as well as nonsimple materials, i.e., when W above depends also on $\nabla^2 y$, and related statements about sequential weak continuity of minors. These are non-coercive and belong precisely to the class of integrands studied by Meyers in his work. In spite of terrific progress in the mathematical theory, many questions, which only appear when we apply purely analytical results to mechanics, have remained unanswered. One such question is lower semicontinuity along sequences of orientation-preserving maps, in other words mappings satisfying $\det \nabla y > 0$ almost everywhere in Ω. We will also discuss this particular problem and mention recent partial results on this topic. We will mention various instances where lower semicontinuity is of major importance. This includes semidiscretization in time of evolutionary problems, relaxation (i.e. finding the largest lower semicontinuous envelope of I) and dimension reduction. Besides, we emphasize some recent progress in lower semicontinuity of functionals along sequences satisfying differential and algebraic constraints which have applications in continuum mechanics of solids to ensure injectivity and orientation-preservation of elastic deformations.

1. First-order problems.

Here we will discuss necessary and sufficient conditions for weak lower semicontinuity in case of coercive but also noncoercive integrands. Special attention will be paid to integrands and functionals bounded from below. We will identify situations when sufficient and necessary conditions coincide. Particular emphasize will be given to various convexity notions as e.g. polyconvexity and quasiconvexity. The situation becomes much more involved if we assume that $\det \nabla y > 0$. If y is an elastic deformation of a solid body this constraint represents an important physical requirement called orientation-preservation. Then the problem of weak lower semicontinuity of I is largely open [1]. We will discuss some partial results obtained e.g. in [2, 4] or in [7, 15, 14].

2. Higher-order problems and problems with general differential constraints.

Here we will deal with problems of the form $I(y) = \int_\Omega W(x, y(x), \nabla y(x), \nabla^2 y(x))\mathrm{d}x$ Besides studying necessary and sufficient conditions for weak lower semicontinuity of I in this setting, we will also investigate special cases including gradient polyconvexity [6] or the instance when $W = W(x, y, \nabla y, \nabla C(\nabla y))$ with $C(F) = F^\top F$. Another topic is represented by the notion of $\mathcal{A}$-quasiconvexity. Namely, we are concerned with lower semicontinuity of the functional $I(y) = \int_\Omega W(x, y(x))\mathrm{d}x$ with the constraint $\mathcal{A}y = 0$. Here $\mathcal{A}$ is a first order differential operator of the constant rank. Using this ansatz we can cover many constraints as, e.g., the "gradient case" setting $\mathcal{A} =$ curl, solenoidal fields, for $\mathcal{A} =$ div and many others; cf. [9, 12], for example.

3. Applications.

Obtained results will be exploited in problems arising in physics. A rich source of mathematical models is continuum mechanics of solids. Here we will explore classical models of elasticity with various extensions as, e.g., electromagnetism.

References

[1] Ball, J.M: Progress and puzzles in nonlinear elasticity. *In: J. Schröder and P. Neff, editors, Proceedings of the CISM Course on Poly-, Quasi- and Rank-One Convexity in Applied Mechanics, Udine, 24.9.2007 - 28.9.2007: Poly-, Quasi- and Rank-One Convexity in Applied Mechanics* Springer, Wien, 2010, pp. 1–15.

[2] Benešová, B., Kampschulte, M., Gradient Young measures generated by quasiconformal maps in the plane , *SIAM J. Math. Anal.* **47** (2015), 4404–4435.

[3] Benešová, B., Krömer S., Kružík, M.: Boundary effects and weak* lower semicontinuity for signed integral functionals on BV. *ESAIM: Control Optim. Calc. Var.* **21** (2015), 513–534.

[4] Benešová, B., Kružík, M., Characterization of gradient Young measures generated by homeomorphisms the plane, *ESAIM: Control Optim. Calc. Var.* **22** (2016), 267–288.

[5] Benešová, B., Kružík, M., Weak lower semicontinuity of integral functionals and applications, *SIAM Rev.* **59** (2017), 703–766.

[6] Benešová B., Kružík M., Schlömerkemper A., A note on locking materials and gradient polyconvexity. *Math. Mod. Meth. Appl. Sci.* **28** (2018), 2367–2401.

[7] Conti, S., Dolzmann, G.: On the theory of relaxation in nonlinear elasticity with constraints on the determinant. *Arch. Rat. Mech. Anal.*, **217** (2015), 413–437.

[8] Dacorogna, B. *Direct Methods in the Calculus of Variations.* 2nd ed., Springer, New York, 2008.

[9] Fonseca, I., Kružík, M.: Oscillations and concentrations generated by $\mathcal{A}$-free mappings and weak lower semicontinuity of integral functionals. *ESAIM Control. Optim. Calc. Var.* **16** (2010), 472–502.

[10] Fonseca, I., Müller, S.: Quasi-Convex Integrands and lower Semicontinuity in L^1. *SIAM J. Math. Anal.* **23** (1992): 1081–1098.

[11] Fonseca, I., Müller, S.: Relaxation of quasiconvex functionals in $BV(\Omega;\mathbb{R}^p)$ for integrands $f(x, u, \nabla u)$. *Arch. Ration. Mech. Anal.* **123** (1993), 1–49.

[12] Fonseca, I., Müller, S.: $\mathcal{A}$-quasiconvexity, lower semicontinuity, and Young measures. *SIAM J. Math. Anal.* **30** (1999), 1355–1390.

[13] Friedrich, M., Kružík, M: On the passage from nonlinear to linearized viscoelasticity. *SIAM J. Math. Anal.* **50** (2018), 4426–4456.

[14] Koumatos, K., Rindler, F., Wiedemann, E.: . Differential inclusions and Young measures involving prescribed Jacobians. *SIAM J. Math. Anal.*, **47(2)** (2015), 1169–1195.

[15] Koumatos, K., Rindler, F., Wiedemann, E.: Orientation-preserving Young measures. *Quart. J. Math.* **67** (2016), 439–466.

[16] Kružík, M. Roubíček, T.: *Mathematical Methods in Continuum Mechanics of Solids.* Springer 2019.

[17] Meyers, N.G.: Quasi-convexity and lower semicontinuity of multiple integrals of any order, *Trans. Am. Math. Soc.* **119** (1965), 125–149.

[18] Morrey, C.B.: Quasi-convexity and the lower semicontinuity of multiple integrals. *Pacific J. Math.* **2** (1952), 25–53.

[19] Morrey, C.B.: *Multiple Integrals in the Calculus of Variations.* Springer, Berlin, 1966.