Unimodularity and invariant volume forms on Poisson-Lie groups

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Integrability of a dynamical system $X \in \mathfrak{X}(M)$

Exact integrability

$dim(M) = n$

$\{f_1, \dots, f_{n-1}\} \in C^\infty(M)$

- $X(f_i) = 0$
- Functionally independent

$f_1, \dots, f_{n-2} \in C^\infty(M) \quad \& \quad \Phi \in \Omega^n(M)$

- $X(f_i) = 0$
- Functionally independent
- $\mathcal{L}_X \Phi = 0, \quad \Phi \neq 0$

Invariant volume form for a Hamiltonian system (M, Π, H)

(1) To describe the symplectic leaves of (M, Π)

and to apply Liouville's theorem obtaining invariant volume forms on the leaves.

Note that:

Symplectic leaves for some types of Poisson manifolds are hard to compute.

(2) To look for an invariant volume form on the whole manifold.

$\Downarrow$

Unimodularity of Poisson manifolds is related with the existence of invariant volume forms

Liouville's theorem

Given a Hamiltonian on a symplectic manifold, the flow of the Hamiltonian vector field preserves the symplectic volume.

Poisson-Lie groups

Poisson-Lie group (G, Π): Poisson structures on G compatible with multiplication

Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$: compatible pairs of Lie algebras in duality

$$\Downarrow$$

There exist $\delta : \mathfrak{g} \to \wedge^2 \mathfrak{g}$ such that:

(i) is a 1-cocycle on $\mathfrak{g}$ with values on $\mathfrak{g} \otimes \mathfrak{g}$, where $\mathfrak{g}$ acts on $\mathfrak{g} \otimes \mathfrak{g}$ by $ad^{(2)}$, i.e.

$$ad_X^{(2)}(\delta Y) - ad_Y^{(2)}(\delta X) - \delta[X, Y] = 0, \quad \forall X, Y \in \mathfrak{g}$$

(ii) $[\cdot, \cdot]^* := \delta^t : \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ defines a Lie bracket on $\mathfrak{g}^*$, i.e., is a skew-symmetric bilinear map on $\mathfrak{g}^*$ satisfying the Jacobi identity.

Why on Poisson-Lie groups?

$$[\![r, r]\!] \in \Lambda^3\mathfrak{g} \text{ is ad-invariant} \Longrightarrow r \in \wedge^2\mathfrak{g} \text{ is a solution of GYBE}$$

$$\Downarrow$$

$$\Pi = r^l - r^r \text{ is a Poisson-Lie structure on } G$$

An interesting connection between integrable systems and Poisson-Lie groups.[a]

[a] **M. Semenov-Tian-Shansky**, *Integrable systems: the r-matrix Approach*

Why on Poisson-Lie groups?

Hamiltonian systems on Poisson-Lie groups appear in the differential equation approach to the singular value decomposition (SVD) of a bidiagonal matrix.

- **M. Chu**, *A differential equation approach to the singular value decomposition of bidiagonal matrices*.
- **D. Percy, J. Demmel, L.-C. Li, C. Tomei**, *The Bidiagonal Singular Value Decomposition and Hamiltonian Mechanics*.

The system of differential equations underlying the SVD is Hamiltonian with respect to the (standard) Sklyanin bracket $\{\cdot,\cdot\}$ *defined on* $\mathrm{SL}(n,\mathbb{R})$.

Why on Poisson-Lie groups?

A method for obtaining integrable deformations

of Lie-Poisson bi-Hamiltonian systems is applied in

- **A. Ballesteros, J. C. Marrero and Z. Ravanpak**, *Poisson-Lie groups, bi-Hamiltonian systems and integrable deformations.* J. Phys. A Math. Theor. **50** (2017), 145204.

$$\Downarrow$$

We have interesting examples

from the Poisson-Lie deformation theory of Lie-Poisson bi-Hamiltonian systems

Aim of this talk

To discus about the existence of invariant volume form $\Omega^n(G)$: $\mathcal{L}_X\Omega = 0$

for Poisson-Lie Hamiltonian system (G, Π, H)

Previous result in

- **J.C. Marrero**, *Hamiltonian dynamics on Lie algebroids, Unimodularity and preservation of volumes*

- $\mathfrak{g}$ is a Lie algebra
- $\{\cdot,\cdot\}_{\mathfrak{g}^*}$ is Lie-Poisson bracket
- H a function as kinetic type
- $(\mathfrak{g}^*, \{\cdot,\cdot\}_{\mathfrak{g}^*}, H)$, Hamiltonian Lie-Poisson system

X_H preserves a volume form

$$\Updownarrow$$

$\mathfrak{g}$ is unimodular

Poisson cohomology

(M,Π) Poisson manifold

- $\partial_\Pi : \nu^k(M) \to \nu^{k+1}(M)$

$$\partial_\Pi P = [\![\Pi, P]\!]$$

- The first cohomology group

$$H^1_\Pi(M) = \frac{P_\Pi(M)}{Ham_\Pi(M)}$$

$$X \in P_\Pi(M) \iff [\![X,\Pi]\!] = \mathcal{L}_X\Pi = 0$$

$$X^\Pi_H \in Ham_\Pi(M) \iff X^\Pi_H = \Pi^\#(\mathrm{d}H), \quad H \in C^\infty(M)$$

Unimodularity of Poisson manifolds

(M,Π) an orientable Poisson manifold

- The **modular vector field** $\mathcal{M}^{\Pi}_{\Phi} \in \mathfrak{X}(M)$ of (M,Π) with respect to Φ

$$\mathcal{L}_{X^{\Pi}_H}\Phi = \mathcal{M}^{\Pi}_{\Phi}(H)\Phi \quad \Longrightarrow \quad \mathcal{M}^{\Pi}_{\Phi} \quad \text{is a Poisson vector field}$$

- The divergence of the Hamiltonian vector field X^{Π}_H w.r.t the volume form Φ

$$\mathcal{M}^{\Pi}_{\Phi}(H) = \mathrm{div}_{\Phi}(X^{\Pi}_H), \ \text{for } H \in C^{\infty}(M)$$

- If $\Phi' = e^F\Phi$ is another positive volume form on M $\quad \Longrightarrow \quad \mathcal{M}^{\Pi}_{\Phi'} = \mathcal{M}^{\Pi}_{\Phi} - X^{\Pi}_F$

$\mathcal{M}^{\Pi}_{\Phi}$ induces a cohomology class $[\mathcal{M}^{\Pi}_{\Phi}] \in H^1_{\Pi}(M)$ in the first Poisson cohomology group of M, is called the modular class of M.

(M,Π) is unimodular if $[\mathcal{M}^{\Pi}_{\Phi}] = 0$, i.e., $\mathcal{M}^{\Pi}_{\Phi} = X^{\Pi}_F$, for $F \in C^{\infty}(M)$.

Obstruction

We can fine a volume form with $\mathcal{M}^{\Pi}_{\Phi} = 0 \Longleftrightarrow [\mathcal{M}^{\Pi}_{\Phi}] = 0$

$\mathcal{M}^{\Pi}_{\Phi} = 0 \Longleftrightarrow \Phi$ is invariant under all Hamiltonian flows

$$\Downarrow$$

The modular class is the obstruction to the existence of of a volume form in (M, Π) invariant under all Hamiltonian flows.

Unimodularity of Poisson manifolds

(M, Π) a Poisson manifold, $dim(M) = n$

M Unimodular

$$\Updownarrow$$

$\exists\, \Phi \in \Omega^n(M)$ volume form

$$\mathcal{L}_{X_H}\Phi = 0 \quad \forall\, H \in C^\infty(M)$$

$$M \text{ Unimodular} \Longrightarrow \exists F \in C^\infty(M),\ \Phi' \in \Omega^n(M) : \mathcal{M}^\Pi_{\Phi'} = X^\Pi_F$$

$$\Downarrow$$

$$\mathcal{L}_{X_H} e^F \Phi' = 0, \quad \forall\, H \in C^\infty(M)$$

Unimodularity of Lie-Poisson structure on the dual of a Lie algebra

$\mathfrak{g}$ Lie algebra, $dim(\mathfrak{g}) = n$, $(\mathfrak{g}^*, \{\cdot, \cdot\})$ Lie-Poisson structure

$$\Pi_{LP} = \frac{1}{2} c_{\alpha\beta}^{\gamma} x_{\gamma} \frac{\partial}{\partial x_{\alpha}} \wedge \frac{\partial}{\partial x_{\beta}} \;\&\; \Phi = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \implies \mathcal{M}_{\Phi}^{\Pi_{LP}} = c_{\alpha\beta}^{\beta} \frac{\partial}{\partial x_{\alpha}}$$

- The modular character $\mathcal{M}_{\mathfrak{g}} \in \mathfrak{g}^*$ of $\mathfrak{g}$

$$\mathcal{M}_{\mathfrak{g}}(\xi) = \mathrm{Tr}(\mathrm{ad}_{\xi}), \text{ for } \xi \in \mathfrak{g}$$

- If $\{e_{\alpha}\}$ is a basis of $\mathfrak{g}$ with dual basis $\{e^{\alpha}\}$, we have

$$\mathcal{M}_{\mathfrak{g}} = c_{\alpha\beta}^{\beta} e^{\alpha}, \quad [e_{\alpha}, e_{\beta}] = c_{\alpha\beta}^{\gamma} e_{\gamma}$$

$(\mathfrak{g}^*, \Pi_{LP})$ is unimodular if and only if the Lie algebra $\mathfrak{g}$ is unimodular, that is $\mathcal{M}_{g} = 0$.[a]

[a] **J.C. Marrero**, *Hamiltonian dynamics on Lie algebroids, Unimodularity and preservation of volumes*

Modular vector field w.r.t a left-invariant volume form on G

(G, Π) a connected Poisson-Lie group with Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, $dim(G) = n$

$\mathcal{M}_{\mathfrak{g}} \in \mathfrak{g}^*$ (resp. $\mathcal{M}_{\mathfrak{g}^*} \in \mathfrak{g}$) the modular character of $\mathfrak{g}$ (resp. $\mathfrak{g}^*$)
ν^l left-invariant volume form on G

$$\Downarrow$$

$$\mathcal{M}^{\Pi}_{\nu^l} = \frac{1}{2}(\mathcal{M}^l_{\mathfrak{g}^*} + \mathcal{M}^r_{\mathfrak{g}^*} + \Pi^\sharp(\mathcal{M}^r_{\mathfrak{g}}))^1$$

[1] **S. Evens J.-H. Lu and A. Weinstein**, Transverse measures, the modular class and a cohomology pairing for Lie algebroids

Unimodularity of a Poisson-Lie group

The 1-form $\mathcal{M}^r_{\mathfrak{g}}$ is exact $\Longrightarrow \mathcal{M}^r_{\mathfrak{g}} = \mathrm{d}(\log f_0)$

$$f_0(g) := \det\,(Ad^n_g) : \Lambda^n\mathfrak{g} \to \Lambda^n\mathfrak{g}, \ \text{for } g \in G$$

$$[\mathcal{M}^{\Pi}_{\nu^l}] = [\frac{1}{2}(\mathcal{M}^l_{\mathfrak{g}^*} + \mathcal{M}^r_{\mathfrak{g}^*} + \Pi^\sharp(\mathrm{d}(\log f_0)))] = [\frac{1}{2}(\mathcal{M}^l_{\mathfrak{g}^*} + \mathcal{M}^r_{\mathfrak{g}^*})]$$

(G, Π) unimodular

$\Updownarrow$

$\mathcal{M}_{\mathfrak{g}^*} = 0$, i.e. $\mathfrak{g}^*$ is unimodular

(G, Π) is unimodular $\Longrightarrow \mathcal{M}^{\Pi}_{\nu^l} = \frac{1}{2}X^{\Pi}_{(\log f_0)}$

Preservation of volume forms

(G, Π) connected Poisson-Lie group & $H : G \to \mathbb{R}$ Hamiltonian function

Theorem

X_H^Π preserves a volume form on G

$$\Updownarrow$$

$$\exists \sigma \in C^\infty(G) : \quad \mathcal{L}_{X_H^\Pi}\Phi = X_H^\Pi(\sigma - log\sqrt{f_0}) + \frac{1}{2}(\mathcal{M}^l_{\mathfrak{g}^*}(H) + \mathcal{M}^r_{\mathfrak{g}^*}(H)) = 0$$

If $\nu \in \wedge^n \mathfrak{g}^*$, with $\nu \neq 0$, then the volume form $e^\sigma \nu^l$ is preserved by X_H^Π.

Consequences

$$\mathcal{L}_{X_H^\Pi}\Phi = X_H^\Pi(\sigma - log\sqrt{f_0}) + \frac{1}{2}(\mathcal{M}^l_{\mathfrak{g}^*}(H) + \mathcal{M}^r_{\mathfrak{g}^*}(H)) = 0$$

Consequence 1

$g \in G$ is a singular point of X_H^Π

&

X_H^Π preserves a volume form on G

$\Downarrow$

$$\mathcal{M}^l_{\mathfrak{g}^*}(g)(H) + \mathcal{M}^r_{\mathfrak{g}^*}(g)(H) = 0$$

Consequence 2

$H \in C^\infty(G)$ a first integral of vector field $\mathcal{M}^l_{\mathfrak{g}^*} + \mathcal{M}^r_{\mathfrak{g}^*}$

$\Updownarrow$

$e^{log\sqrt{f_0}}v^l$ is preserved by X_H^Π

$\mathcal{M}^r_{\mathfrak{g}} = \mathrm{d}(\log f_0)$

Unimodularity of Poisson-Lie groups & preservation of volume forms

(G, Π) connected Poisson-Lie group, $\dim(G) = n$

Corollary

If (G, Π) is unimodular, $\nu \in \wedge^n \mathfrak{g}^*$ with $\nu \neq 0$

$$\Downarrow$$

The volume form $\sqrt{f_0}\nu^l$ is preserved by all Hamiltonian vector fields

$f_0 : G \to \mathbb{R}$: $\mathcal{M}^r_{\mathfrak{g}} = \mathrm{d}(\log f_0)$

Ex 1: Integrable deformation of the Euler top

Integrable deformation of the Euler top

$$\begin{aligned}
\dot{x} &= e^{\eta x}(y^2 - z^2), \\
\dot{y} &= \eta e^{\eta x} y z^2 - \frac{1}{2}\eta e^{\eta x} y(y^2 + z^2) + \frac{\sinh(\eta x)}{\eta}(2z - y), \\
\dot{z} &= -\eta e^{\eta x} y^2 z + \frac{1}{2}\eta e^{\eta x} z(y^2 + z^2) + \frac{\sinh(\eta x)}{\eta}(z - 2y).
\end{aligned}$$

On the so-called "book" Lie group G_η, with the Lie algebra $\mathfrak{g}$

$$[X, Y] = -\eta Y, \quad [X, Z] = -\eta Z, \quad [Y, Z] = 0$$

G_η is diffeomorphic to $\mathbb{R}^3$, we can choose global coordinates, group law

$$g(x, y, z) \cdot g'(x', y', z') = g\left(x + x', y + y' e^{-\eta x}, z + z' e^{-\eta x}\right)$$

Ex 1: Poisson-Lie structure on G_η

The system is bi-Hamiltonian w.r.t two Poisson-Lie structures on G_η

$$\Pi_{G,0} = -z\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} + \frac{1}{2}\left(-\eta(y^2+z^2) + \frac{e^{-2\eta x}-1}{\eta}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}$$

$$\Pi_{G,1} = -y\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + z\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} + \left(-\eta yz + \frac{e^{-2\eta x}-1}{\eta}\right)\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z}$$

and two functions

$$\mathcal{H}_0 = yze^{\eta x} + 2\left(\frac{\cosh(\eta x)-1}{\eta^2}\right)$$

$$\mathcal{H}_1 = -\frac{1}{2}(y^2+z^2)e^{\eta x} + \frac{\cosh(\eta x)-1}{\eta^2}$$

- The dynamical system defined by the Hamiltonian vector field $\Pi^\sharp_{G,0}(d\mathcal{H}_0) = \Pi^\sharp_{G,1}(d\mathcal{H}_1)$.

Ex 1: Euler top system

- $\eta \to 0$ limit, Poisson bivectors on the 3-dimensional abelian Lie group $\mathbb{R}^3$

$$\Pi_{\mathbb{R}^3,0} = -z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + y\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} - x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$$
$$\Pi_{\mathbb{R}^3,1} = -y\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial z} - 2x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$$

- They are the Lie-Poisson structures associated with the Lie algebras $\mathfrak{g}^* \simeq \mathfrak{so}(3)$ and $\mathfrak{g}^* \simeq \mathfrak{sl}(2,\mathbb{R})$, respectively. The Hamiltonian functions $\mathcal{H}_0$ and $\mathcal{H}_1$ go to

$$\mathcal{H}_0^{\eta=0} = x^2 + yz, \quad \mathcal{H}_1^{\eta=0} = -\frac{1}{2}(x^2 + y^2 + z^2).$$

- The dynamical system defined by $\Pi^\sharp_{\mathbb{R}^3,0}(d\mathcal{H}_0^{\eta=0}) = \Pi^\sharp_{\mathbb{R}^3,1}(d\mathcal{H}_1^{\eta=0})$

$$\dot{x} = y^2 - z^2, \qquad \dot{y} = x(2z - y), \qquad \dot{z} = x(z - 2y)$$

It is equivalent to a particular case of the Euler top.

Ex 1: Unimodularity of $(G_\eta, \Pi_{G,0})$ & $(G_\eta, \Pi_{G,1})$

- The dual Lie algebras

$$\mathfrak{g}_0^* \simeq \mathfrak{so}(3): \quad [\bar{X}, \bar{Y}]_{\mathfrak{g}_0^*} = -\bar{Z}, \qquad [\bar{X}, \bar{Z}]_{\mathfrak{g}_0^*} = \bar{Y}, \qquad [\bar{Y}, \bar{Z}]_{\mathfrak{g}_0^*} = -\bar{X}$$

$$\mathfrak{g}_1^* \simeq \mathfrak{sl}(2, \mathbb{R}): \quad [\bar{X}, \bar{Y}]_{\mathfrak{g}_1^*} = -\bar{Y}, \qquad [\bar{X}, \bar{Z}]_{\mathfrak{g}_1^*} = \bar{Z}, \qquad [\bar{Y}, \bar{Z}]_{\mathfrak{g}_1^*} = -2\bar{X}$$

$$\mathcal{M}_{\mathfrak{g}_1^*} = \mathcal{M}_{\mathfrak{g}_2^*} = 0 \Longrightarrow \text{Both are unimodular Lie algebras}$$

$$\Downarrow$$

Both Poisson-Lie structures are unimodular

$$\Downarrow$$

<u>The dynamical system admits invariant volume forms</u>

Ex 1: Volume form preseved by the flow of the Hamiltonian vector fields

- The modular character of Lie algebra $\mathfrak{g}$: $\mathcal{M}_{\mathfrak{g}} = -2\eta\bar{X}$
- $\mathcal{M}^r_{\mathfrak{g}} = -2\eta \mathrm{d}x = \mathrm{d}(\log f_0)$
- $f_0 = e^{-2\eta x}$

The volume form $\Phi = \sqrt{f_0}\nu^l = e^{\eta x} dx \wedge dy \wedge dz$
is preserved by all Hamiltonian vector fields, for $v = \bar{X} \wedge \bar{Y} \wedge \bar{Z} \in \wedge^3 \mathfrak{g}^*$.

If Π is unimodular

(G, Π) a connected Poisson-Lie group, $\dim(G) = n$

If Π is unimodular

$\Downarrow$

For $\nu \in \wedge^n \mathfrak{g}^*$ with $\nu \neq 0$,
the volume form $\sqrt{f_0}\nu^l$ is preserved
by all Hamiltonian vector fields,
$\mathcal{M}^r_{\mathfrak{g}} = \mathrm{d}(\log f_0)$

What if Π is not unimodular!

If Π is not unimodular

$$X_H^\Pi \text{ preserves volume form} e^\sigma v^l \iff X_H^\Pi(\sigma - log\sqrt{f_0}) + \frac{1}{2}(\mathcal{M}^l_{\mathfrak{g}^*}(H) + \mathcal{M}^r_{\mathfrak{g}^*}(H)) = 0$$

(1) $g \in G$ is a singular points of X_H^Π

&

$$\mathcal{M}^l_{\mathfrak{g}^*}(g)(H) + \mathcal{M}^r_{\mathfrak{g}^*}(g)(H) \neq 0$$

$\Downarrow$

X_H^Π does not preserve a global volume form

(2) $H \in C^\infty(G)$ a first integral of vector field $\mathcal{M}^l_{\mathfrak{g}^*} + \mathcal{M}^r_{\mathfrak{g}^*}$

$\Downarrow$

$e^{log\sqrt{f_0}} v^l$ is preserved by X_H^Π

$\mathcal{M}^r_{\mathfrak{g}} = \mathrm{d}(\log f_0)$

Ex 2: A Hamiltonian system on the Poisson-Lie group $\mathrm{SL}(2,\mathbb{R})$

Consider the special linear group $\mathrm{SL}(2,\mathbb{R})$ with Lie algebra $\mathfrak{sl}(2,\mathbb{R})$

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{GL}(2,\mathbb{R}) \mid \det A = 1 \right\}$$

$$\mathfrak{sl}(2,\mathbb{R}) = \{ A \in \mathfrak{gl}(2,\mathbb{R}) \mid \operatorname{Tr} A = 0 \}$$

- Poisson-Lie structure on $\mathrm{GL}(2,\mathbb{R})$

$$\{a_{11}, a_{12}\} = a_{11}a_{12}, \quad \{a_{11}, a_{21}\} = a_{11}a_{21}, \quad \{a_{11}, a_{22}\} = 2a_{12}a_{21},$$
$$\{a_{12}, a_{21}\} = 0, \quad \{a_{12}, a_{22}\} = a_{12}a_{22}, \quad \{a_{21}, a_{22}\} = a_{21}a_{22}$$

- $\det A = a_{11}a_{22} - a_{12}a_{21}$ is a Casimir $\Longrightarrow \{\cdot,\cdot\}$ a Poisson structure on $\mathrm{SL}(2,\mathbb{R})$

This Poisson-Lie structure is the one defined by the so-called Drinfel'd-Jimbo r-matrix.[a]

[a] **A. G. Reyman**, *Poisson structures related to quantum groups.*

Ex 2: Drinfel'd-Jimbo r-matrix

- With respect to the basis $\{J_3, J_+, J_-\}$

$$J_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

- The (standard) Drinfel'd-Jimbo r-matrix is given by

$$r = J_- \wedge J_+ \in \wedge^2 \mathfrak{sl}(2, \mathbb{R})$$

$$\delta(J_3) = 0, \qquad \delta(J_\pm) = J_3 \wedge J_\pm$$

$$[J^3, J^\pm]_{\mathfrak{g}^*} = J^\pm, \qquad [J^+, J^-]_{\mathfrak{g}^*} = 0$$

Interestingly, this Poisson structure is the same as the Sklyanin bracket appeared in the singular value decomposition in Toda-SVD.

Ex 2: Unimodularity of Poisson-Lie structure on $\mathrm{SL}(2,\mathbb{R})$

$$\mathcal{M}_{\mathfrak{g}^*} = 2J_3 \neq 0 \ \ \& \ \ \mathcal{M}_{\mathfrak{g}} = 0$$

$$\Downarrow$$

$$\frac{1}{2}(\mathcal{M}^r_{\mathfrak{g}^*} + \mathcal{M}^l_{\mathfrak{g}^*}) = 2a_{11}\frac{\partial}{\partial a_{11}} - 2a_{22}\frac{\partial}{\partial a_{22}}$$

$$\Downarrow$$

The Poisson manifold $(\mathrm{SL}(2,\mathbb{R}), \{\cdot,\cdot\})$ is not unimodular.

Ex 2: Preservation of a volume form by $X_H^{\sqcap}$ on $\mathrm{SL}(2,\mathbb{R})$

$$H = \frac{1}{2}Tr(A^TA) = \frac{1}{2}(a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2) = \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}^2, \quad H \in C^\infty(\mathrm{GL}(2,\mathbb{R}))$$

This Hamiltonian function generates the Toda-SVD flow defined by the Poisson bracket.

$$A = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}), \quad a \neq 0$$

is a singular point for the Hamiltonian vector field $X_H^{\sqcap}$ on $\mathrm{SL}(2,\mathbb{R})$.

$$\mathcal{M}_{\mathfrak{g}^*}^l(A)(H) + \mathcal{M}_{\mathfrak{g}^*}^r(A)(H) = 4a^2 - \frac{4}{a^2}$$

$$\text{For } a \neq \pm 1 \Longrightarrow \mathcal{M}_{\mathfrak{g}^*}^l(A)(H) + \mathcal{M}_{\mathfrak{g}^*}^r(A)(H) \neq 0$$

$$\Downarrow$$

$X_H^{\sqcap}$ does not preserve a global volume form on $\mathrm{SL}(2,\mathbb{R})$.

Ex 3: A Hamiltonian system on the Poisson-Lie group S^3

S^3 the unit sphere in $\mathbb{R}^4 - \{(0,0,0,0)\}$

- In $\mathbb{R}^4 - \{(0,0,0,0)\}$

$$\begin{aligned}(x,y,z,t)(x',y',z',t') &= (xx' - yy' - zz' - tt', xy' + yx' - zt' + tz',\\ &\quad zx' - ty' + xz' + yt', zy' + tx' + xt' - yz')\end{aligned}$$

- This is the Lie group structure identifying the quaternions $\mathbb{H}$ with $\mathbb{R}^4 - \{(0,0,0,0)\}$.
- If $\{e_1, e_2, e_3, e_4\}$ is the canonical basis in $\mathbb{R}^4 - \{(0,0,0,0)\}$, non-zero commuting relations

$$[e_2, e_3]_{\mathfrak{g}} = -2e_4, \quad [e_2, e_4]_{\mathfrak{g}} = 2e_3, \quad [e_3, e_4]_{\mathfrak{g}} = -2e_2.$$

- S^3 is a closed normal Lie subgroup of $\mathbb{R}^4 - \{(0,0,0,0)\}$.
- $\{e_2, e_3, e_4\}$ is a basis of the Lie algebra of the Lie subgroup $S^3 \cong \mathrm{SU}(2,\mathbb{C})$.

Ex 3: Poisson-Lie structure on S^3

Poisson structure Π on $\mathbb{R}^4 - \{(0,0,0,0)\}$

$$\{x,y\} = -(z^2+t^2), \quad \{x,z\} = yz$$
$$\{y,z\} = -xz, \quad \{x,t\} = yt, \quad \{y,t\} = -xt$$

$$\Pi = -(z^2+t^2)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + yz\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial z} + yt\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial t} - xz\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} - xt\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial t}$$

$$\|\cdot\| : \mathbb{R}^4 - \{(0,0,0,0)\} \to \mathbb{R}, \quad (x,y,z,t) \to \|(x,y,z,t)\|^2 = x^2+y^2+z^2+t^2$$

is a Casimir function

$$\Downarrow$$

Poisson-Lie structure on $S^3 \cong \mathrm{SU}(2,\mathbb{C})$.

Ex 3: Unimodularity of (S^3, Π)

$$[e^2, e^3]_{\mathfrak{g}^*} = -[e^3, e^2]_{\mathfrak{g}^*} = -e^3,$$
$$[e^2, e^4]_{\mathfrak{g}^*} = -[e^4, e^2]_{\mathfrak{g}^*} = -e^4$$

$$\mathcal{M}_{\mathfrak{g}^*} = -2e_2 \neq 0 \ \& \ \mathcal{M}_{\mathfrak{g}} = 0$$

$$\Downarrow$$

$$\frac{1}{2}(\mathcal{M}^l_{\mathfrak{g}^*} + \mathcal{M}^r_{\mathfrak{g}^*}) = -2(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x})$$

$$\Downarrow$$

$(S^3, \{\cdot, \cdot\})$ is not unimodular.

Ex 3: Preservation of a volume form by X_H^{Π} on S^3

$$\begin{array}{rcl} H\colon S^3 & \to & \mathbb{R} \\ (x,y,z,t) & \mapsto & P(z,t), \quad P \in C^{\infty}(\mathbb{R}^2) \end{array}$$

is a first integral of $\frac{1}{2}(\mathcal{M}^l_{\mathfrak{g}^*} + \mathcal{M}^r_{\mathfrak{g}^*})_{|S^3}$

$$X_H^{\Pi}(\sigma - log\sqrt{f_0}) + \frac{1}{2}(\mathcal{M}^l_{\mathfrak{g}^*}(H) + \mathcal{M}^r_{\mathfrak{g}^*}(H)) = 0$$

$$\mathcal{M}_{\mathfrak{g}} = 0 \Rightarrow f_0 = Constant \Rightarrow \Phi = \nu^l \quad \text{for} \quad \nu \in \wedge^n \mathfrak{g}^*$$

$$\Downarrow$$

X_H^{Π} preserves any left-invariant volume form Φ on S^3

Unimodularity in Poisson-Lie groups vs preservation of volume forms

Let (G, Π) be a connected Poisson-Lie group, $\dim(G) = n$

If Π is unimodular

$\Downarrow$

For $\nu \in \wedge^n \mathfrak{g}^*$ with $\nu \neq 0$,
the volume form $\sqrt{f_0}\nu^l$ is preserved
by all Hamiltonian vector fields,
for $f_0 \in C^\infty(G)$ & $\mathcal{M}^r_{\mathfrak{g}} = \mathrm{d}(\log f_0)$

If a volume form is preserved
by a Hamiltonian vector field
with respect to certain functions

$\Downarrow$

Π is unimodular!

Morse function

G a Lie group with identity element e, $H \in C^\infty(G)$

Definition

A function H is said to be **Morse at** e if

i) $\mathrm{d}H(e) = 0$

ii) $(\mathrm{Hess}\, H)(e) \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is nondegenerate

$H : G \to \mathbb{R}$ is Morse if the Hessian of H at each singular point of H is nondegenerate.

Preservation of volume forms and unimodularity of Poisson-Lie groups

(G, Π) a connected Poisson-Lie group, $\dim(G) = n$

Theorem

$H \in C^\infty(G)$ a Morse function at e & X_H^Π preserves a volume form Φ on G

$$\Downarrow$$

The dual Lie algebra $\mathfrak{g}^*$ is unimodular ($i.e.$ Π is unimodular)

$$\Updownarrow$$

The volume form $\sqrt{f_0}\nu^l$ is preserved by all Hamiltonian vector fields

$\nu \in \wedge^n \mathfrak{g}^*$ & $\nu \neq 0$ & $\mathcal{M}^r_{\mathfrak{g}} = \mathrm{d}(\log f_0)$

Ex 4

Π_{LP} the Lie-Poisson structure on dual Lie algebra $\mathfrak{g}^*$ of Lie algebra $\mathfrak{g}$

- If $I : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathbb{R}$ is a symmetric $\mathbb{R}$-bilinear form then we can consider the Hamiltonian function $H_I : \mathfrak{g}^* \to \mathbb{R}$ given by

$$H_I(\mu) = \frac{1}{2} I(\mu, \mu), \text{ for } \mu \in \mathfrak{g}^*$$

- If $\{e_\gamma\}$ is a basis of $\mathfrak{g}$ with dual basis $\{e^\gamma\}$ for $\mathfrak{g}^*$ and $\{x_\gamma\}$ the corresponding global coordinates of $\mathfrak{g}^*$, we have that

$$H_I(x) = \frac{1}{2} I^{\alpha\beta} x_\alpha x_\beta$$

- The identity element of $\mathfrak{g}^*$ as an abelian Poisson-Lie group is a singular point of H_I.

$$H_I \text{ is Morse at } 0 \iff I \text{ is nondegenerate}$$

Ex 4

Hypothesis in the theorem: H_I is Morse at 0 & $X_{H_I}^{\Pi_{LP}}$ preserves a volume form

$$\cong$$

I is nondegenerate & $\mathfrak{g}$ is unimodular

$$\Downarrow$$

$$f_0 : \mathfrak{g}^* \to \mathbb{R},\ f_0 = 1$$

$$\Downarrow$$

The volume form $\Phi = dx_1 \wedge \cdots \wedge dx_n$ is preserved by all flows

This extends a previous result by Kozlov [a] for the particular case when H_I is a Hamiltonian function of kinetic type, i.e. I is positive definite.

[a] **V. V. Kozlov**, *Invariant measures of the Euler-Poincaré equations on Lie algebras*

Morse function and contrast function

G a Lie group with identity element e, $H \in C^{\infty}(G)$

A smooth function $H\colon G \to \mathbb{R}$ is a **contrast function** if $H(e) = 0$ & $\mathrm{d}H(e) = 0$.

$$X_H = X_{H-H(e)}$$

$$\Downarrow$$

If H is Morse $\Longrightarrow$ $H - H(e)$ is contrast

This kind of functions plays an important role in information geometry.

Information geometry and Morse functions

$\iota : G \to \mathbb{R}^n$ an embedding of the Lie group G in $\mathbb{R}^n$, $\iota(e) = 0$

$\langle \cdot, \cdot \rangle$ a non-degenerate bilinear symmetric form on $\mathbb{R}^n$

$$H_{\langle \cdot,\cdot \rangle} : \mathbb{R}^n \to \mathbb{R} : \; H_{\langle \cdot,\cdot \rangle}(x) = \tfrac{1}{2}\langle x, x\rangle, \; \text{for } x \in \mathbb{R}^n$$

$$H = H_{\langle \cdot,\cdot \rangle} \circ \iota : G \to \mathbb{R}$$

$$\Downarrow$$

$$\mathrm{d}H(e) = 0 \,\&\, H(e) = 0 \Longrightarrow H \text{ is contrast}$$

If $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ with $\mathfrak{g} = T_e G \subseteq T_0\mathbb{R}^n \cong \mathbb{R}^n$ is non-degenerate

$$\Downarrow$$

H is Morse at e

Contrast functions on Lie groups

If G is a matrix Lie group, the smooth function $H : G \to \mathbb{R}$

$$H(A) = \mathrm{Tr}\left((Id - A)(Id - A^t)\right) \qquad A \in G$$

is Morse at $Id \in G$. Here Id is the identity matrix in $\mathrm{GL}(n, \mathbb{R})$.

The previous function H is used[a] *as a metric contrast function on* $\mathrm{GL}(n, \mathbb{R})$.

[a] **K. Grabowska, J. Grabowski, M. Kuś and G. Marmo**, *Lie groupoids in information geometry*

Future work

- **P. Xu**, *Gerstenhaber Algebras and BV-Algebras in Poisson Geometry.*

Information geometry & Discrete geometric mechanics

Divergence functions & Discrete Lagrangian functions

Future work

To study the relation between
Hamiltonian dynamics on Poisson-Lie groups
&
Discrete geometric mechanics and information geometry

Future work

- **M. Semenov-Tian-Shansky**, *Dressing transformations and Poisson group actions*.

Poisson homogeneous spaces are given by the quotient G/H of a Poisson-Lie group G with a closed Lie subgroup H of G

We have a description of the modular class of the Poisson structure on G/H[2].

Future work

What is relation between the unimodularity of the Poisson structure on G/H

&

the existence of invariant volume forms for the Hamiltonian system on G/H?

[2] **S. Evens J.-H. Lu and A. Weinstein**, *Transverse measures, the modular class and a cohomology pairing for Lie algebroids*

Thank you!