Geometry of Q-manifolds and Gauge Theories III

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A Lie algebroid $(E, \rho, [\cdot,\cdot])$ over M is

- a vector bundle $E \to M$
- a bundle map $E \to TM$, called the anchor map or the anchor
- a Lie bracket $[\cdot,\cdot] \colon \Lambda^2\Gamma(E) \to \Gamma(E)$ on the space of sections,

such that the following compatibility conditions hold

1. the Leibniz rule $[s, fs'] = \rho_s(f)s' + f[s, s']$
2. the morphism property $\rho_{[s,s']} = [\rho_s, \rho_{s'}]$

for all $f \in \mathcal{F}(M)$, $s, s' \in \Gamma(E)$.

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- an action Lie algebroid, uniquely determined for a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}-$space M as follows:
  - $E = \mathfrak{g} \times M$
  - for any $s \in \mathfrak{g}$, viewed as a section of E, $\rho_s \in \Gamma(TM)$ is the associated infinitesimal generator of the $\mathfrak{g}-$action
  - the bracket of $s, s' \in \mathfrak{g}$, regarded as sections of E, coincides with the corresponding bracket in $\mathfrak{g}$.

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- the Atiyah algebroid of a principal $G-$bundle $\mathrm{p} : P \to M$: $E = TP/G$, the anchor and the Lie bracket are induced by $\mathrm{d}p$ and the Lie bracket of vector fields on P, respectively

Jet Lie algebroid

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Example

$J^1(TM)$ is isomorphic to the Atiyah algebroid of $S^1(M)$, the GL-principal bundle of tangent frames.

Jet Lie algebroid as a Q-manifold

Let us identify a Lie algebroid $(E, \rho, [\cdot, \cdot])$ with the following Q-manifold $(E[1], Q_E)$, $\mathcal{F}(E[1]) \simeq \Gamma(\Lambda E^*)$, where

- $\langle Q_E f, s\rangle = \rho_s f$ for any $f \in \mathcal{F}(M) = \mathcal{F}^0(E[1])$, $s \in \Gamma(E)$
- $[\iota_s, [Q_E, \iota_{s'}]] = \iota_{[s,s']}$, for all $s, s' \in \Gamma(E)$.

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$$\Lambda^r\Gamma(E) \to \mathcal{F}(M)$$

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The corresponding Chevalley-Eilenberg differential equals to Q_E.

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$J^k(E)$ corresponds to the $k-$jet prolongation of $(E[1], Q_E)$, i.e.

- $J^k(E)[1] = J^k(\pi)$ for $\pi \colon E[1] \to M$, $Q_{J^k(E)} = Q_E^{(k)}$
- The canonical extension of $Q_E^{(\infty)}$ to $\mathcal{C}[1]$, which we will denote by the same letter, anti-commutes with d_h
- We have a bi-complex $\mathrm{d}_h + Q_E^{(\infty)}$, where the bi-grading is given by the two "independent" degrees - the degree of differential forms and the degree of functions on $J^\infty(\pi)$.

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- We have a bi-complex $\mathrm{d}_h + Q_E^{(\infty)}$, where the bi-grading is given by the two "independent" degrees - the degree of differential forms and the degree of functions on $J^\infty(\pi)$.
- Functions of degree r on $J^k(E)[1]$ can be regarded as $r-$multi-linear skew-symmetric differential operators on $\Gamma(E)$ with values in $\mathcal{F}(M)$: for every degree l function h on $J^k(\pi)$, viewed as a section of $\Lambda^l J^k(\pi)^*$, the associated multi-linear differential operator acts as follows:

$$\Gamma(E) \ni s_1, \dots s_l \mapsto \langle h, (s_1)_{(k)} \wedge \dots \wedge (s_l)_{(k)} \rangle$$

Jet Lie algebroid as a Q-manifold

- Similarly, functions on $\mathcal{C}[1]$ can be canonically identified with multi-linear skew-symmetric differential operators on $\Gamma(E)$ with values in $\Omega(M)$: given a function φ on $\mathcal{C}[1]$, regarded as a section of $\Lambda(T^*M) \otimes \Lambda^l J^k(\pi)^*$, the associated multi-linear differential operator acts as follows:

$$\Gamma(E) \ni s_1, \ldots s_l \mapsto \langle \varphi, (s_1)_{(k)} \wedge \ldots \wedge (s_l)_{(k)} \rangle \in \Omega(M)$$

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- Let us combine it with the integration over M (provided it is well-defined, eg. M is oriented, sections are compactly supported). We obtain a differential $l-$cochain c_φ on $\Gamma(E)$.

Jet Lie algebroid as a Q-manifold

Taking into account that the integral of a differential form ω depends only on its image in $\Omega^n(M)/\mathrm{d}_M\Omega^{n-1}(M)$ and, moreover, for all φ and $s_1, \ldots, s_l$ as above one has

$$\mathrm{d}_M\langle\varphi, (s_1)_{(k)} \wedge \ldots \wedge (s_l)_{(k)}\rangle = \langle \mathrm{d}_h\varphi, (s_1)_{(k)} \wedge \ldots \wedge (s_l)_{(k)}\rangle$$

we immediately conclude that:

- The differential cochains on $\Gamma(E)$, considered as a Lie algebra, are determined by the top-degree cohomology of d_h, i.e. the first term of the spectral sequence, associated to the above mentioned bi-complex;
- The Chevalley-Eilenberg cohomology of $(\Gamma(E), [,])$ are determined by the second term of the same spectral sequence.

More general case: let Q' be a homological degree 1 vector field on $\mathcal{C}[1]$ which anti-commutes with d_h, i.e. $\mathrm{d}_h + Q'$ is a degree 1 homological vector field again (Q' is not necessarily vertical).

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Let us split Q' into horizontal and vertical (evolutionary) parts

$$Q' = Q'_h + Q'_e$$

Then

- Q'_e is also homological (and it also anti-commutes with d_h as it is evolutionary)
- there exists a canonical degree 0 vector field ψ, such that $Q'_h = [\mathrm{d}_h, \psi]$ and

$$\exp \psi \left(\mathrm{d}_h + Q'\right) \exp\left(-\psi\right) = \mathrm{d}_h + Q'_e$$

Bott sequence

Every vector bundle $\pi\colon E \to M$ gives rise to the short exact sequence

$$0 \longrightarrow D^{k-1}(\pi, \underline{1}) \longrightarrow D^{k}(\pi, \underline{1}) \xrightarrow{\mathrm{symb}_k} S^k(TM) \otimes E^* \longrightarrow 0$$

where $D^k(\pi, \pi')$ the bundle whose sections are linear differential operators of order k acting from $\Gamma(\pi)$ to $\Gamma(\pi')$, $\underline{r}$ is a trivial vector bundle of rank r, and symb_k is the symbol map which associates to any differential operator of order k its principal symbol.

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One has $D^k(\pi, \underline{1}) \simeq J^k(\pi)^*$, where $J^k(\pi)$ is the bundle of $k-$jets of smooth sections of π.

By dualizing of the above exact sequence, we obtain for all $k \geq 1$ the short exact sequence of vector bundles, called the Bott sequence (R. Bott, 1963)

$$0 \longrightarrow S^k(T^*M) \otimes E \longrightarrow J^k(E) \longrightarrow J^{k-1}(E) \longrightarrow 0$$

In particular, $k = 1$ gives us

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The embedding of $T^*M \otimes E$ into $J^1(E)$ is determined for every $f, h \in C^\infty(M)$ and $s \in \Gamma(E)$ by the following formula

$$f\,\mathrm{d}h \otimes s \mapsto f\left(hs_{(1)} - (hs)_{(1)}\right)$$

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Every connection ∇ on E is in one-to-one correspondence with a splitting $\sigma \colon E \to J^1(E)$:

$$\sigma(s) = s_{(1)} + \nabla s$$

where $\nabla s \in \Gamma(T^*M \otimes E)$ is identified with its image in $\Gamma(J^1(E))$.

Jet Lie algebroid and the Bott sequence

Let $(E, \rho, [\cdot,\cdot])$ be a Lie algebroid over M. Now the Bott sequence

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becomes an exact sequence of Lie algebroids.

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The compatibility of a Lie algebroid structure with a connection is governed by the vanishing of the compatibility tensor S, the curvature of the splitting, defined for all $s, s' \in \Gamma(E)$ by the formula

$$S(s, s') = [\sigma(s), \sigma(s')] - \sigma\left([s, s']\right)$$

Given that $\rho\left(S(s, s')\right) = 0$, it is obvious that S can be identified with a section of $T^*M \otimes E \otimes \Lambda^2 E^*$.

Cartan-Lie algebroid

(E, ∇) is called a Cartan-Lie algebroid (Blaom, 2004) over M, if E is a Lie algebroid, ∇ a connection in $E \to M$, and its induced splitting $\sigma \colon E \to J^1(E)$ is a Lie algebroid morphism, i.e. if $S = 0$.

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Examples

- Let $E = M \times \mathfrak{g}$ be an action Lie algebroid. Then the canonical flat connection ∇ is compatible. Furthermore, every Lie algebroid with a flat compatible connection is locally an action Lie algebroid;
- If E is a bundle of Lie algebras, then ∇ on E is compatible if and only if it preserves the fiber-wise Lie algebra bracket;

Examples

- If $E = TM$ is the standard Lie algebroid, a connection on TM is compatible if and only if the dual connection is flat. However, any torsion-free connection on TM gives rise to a compatible connection on $J^1(TM)$.
- Let $\pi\colon P \to M$ be a principal H-bundle, $\mathfrak{h}$ be the Lie algebra of H, $\mathfrak{g}$ be a Lie algebra, such that $\mathfrak{h} \subset \mathfrak{g}$. Consider a Cartan structure on P, i.e. an H-equivariant 1-form $\omega\colon TP \to \mathfrak{g}$, the restriction of which onto the fibers of π is the MC form for H, such that ω is a linear isomorphism $T_pP \simeq \mathfrak{g}$ for each $p \in P$. Then the Atiyah algebroid of P is canonically a Cartan-Lie algebroid; it is flat if and only if the corresponding Cartan structure is flat.

Cartan-Lie algebroids via Q-manifolds

The choice of a connection on E gives us the decomposition of differential forms on $E[1]$ into the product of vertical and horizontal forms:

$$\Omega^m(E[1]) = \bigoplus_{p+q=m} \Omega^{p,q}(E[1])$$

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The connection is compatible with the Lie algebroid structure if and only if L_Q preserves the corresponding horizontal distribution on $E[1]$ ($Q = Q_E$).

This means that $L_Q = \bar{Q} + \hat{\rho}$, where $\bar{Q} \colon \Omega^{p,q}(E[1]) \to \Omega^{p,q}(E[1])$ and $\hat{\rho} \colon \Omega^{p,q}(E[1]) \to \Omega^{p+1,q-1}(E[1])$

Observe that the Lie derivative of any tensor field χ along a vector field depends on the first jet-prolongation of this vector field only, which allows to introduce a natural Lie algebroid representation of $J^1(TM)$ on arbitrary tensor fields, such that the jet prolongations of a vector field acts by the Lie derivative. This idea can be generalized to arbitrary Lie algebroids

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The representations of $J^1(E)$ on E and TM, combined with a Cartan connection ∇ on E, give us representations of E on E and TM - ${}^{\alpha}\nabla$ and ${}^{\tau}\nabla$ on E and TM, respectively, such that for all $s, s' \in \Gamma(E)$, $X \in \Gamma(TM)$

$$\begin{aligned} {}^{\alpha}\nabla_s s' &= [s, s'] + \nabla_{\rho_{s'}} s \\ {}^{\tau}\nabla_s X &= [\rho_s, X] + \rho_{(\nabla_X s)} \end{aligned} \tag{1}$$

The anchor map $\rho\colon E \to TM$ obeys the property ${}^{\tau}\nabla \circ \rho = \rho \circ {}^{\alpha}\nabla$.

Curved Yang-Mills-Higgs Gauge Theory

Let Σ be a d-dimensional Lorentzian manifold, (M, g) be a Riemannian manifold, (E, ρ) be a Lie algebroid over M supplied with a fiber metric κ, and B be an $E-$valued 2-form on M.

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Definition (Fields of the CYMH)

The Higgs field(s) is a smooth map $X \colon \Sigma \to M$, while the gauge field(s) A is a section of $X^*(E) \otimes T^*\Sigma$.

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The fields (X, A) together can be viewed as a bundle map $TM \to E$ or, equivalently, to a degree preserving map of the corresponding graded superspaces $\phi\colon T[1]\Sigma \to E[1]$.

Let ∇^E be a vector bundle connection on E; it determines an $E-$connection on E by the formula $\nabla^E \rho(s) s'$, the $E-$torsion of which is defined for all $s, s' \in \Gamma(E)$ as

$$t(s, s') = [s, s'] - \nabla^E_{\rho_s} s' + \nabla^E_{\rho_{s'}} s$$

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Let us denote by DX the canonical section of $X^* TM \otimes T^* \Sigma$ defined as

$$DX : = dX - \rho(A)$$

and by F the following section of $X^* E \otimes \Lambda^2 T^* \Sigma$:

$$F : = DA + t(A, A)$$

where D is covariant derivative $\Omega^{\cdot}(\Sigma, X^* E) \to \Omega^{\cdot + 1}(\Sigma, X^* E)$ determined by the pull-back of ∇^E by the formula

$$D = d + DX^i \nabla_i$$

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$\phi\colon T\Sigma \to E$ is a Lie algebroid morphism if the induced pullback map $\Phi\colon = \phi^*$ on superfunctions is a chain map, i.e. is satisfies

$$\mathcal{F} = Q_{DR}\Phi - \Phi Q_E = 0$$

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Proposition

$\phi\colon T\Sigma \to E$ is a Lie algebroid morphism if and only if both DX and F vanish.

THE CYMH ACTION

The coupled (curved) YMH functional is of the form $S_{CYMH}[X, A] = S_{CYM}[X, A] + S_{Higgs}[X, A]$ where

$$
\begin{aligned}
S_{Higgs}[X, A] &= \frac{1}{2}\int_{\Sigma} g(DX, *DX) \\
S_{CYM}[X, A] &= \frac{1}{2}\int_{\Sigma} \kappa(G, *G)
\end{aligned}
$$

Here $*$ is the Hodge operator on Σ determined by the Lorentzian metric and G is of the form $F + B(DX, DX)$.

Example (Yang-Mills-Higgs Theory)

Let $\mathfrak{g}$ be a Lie algebra, V be a unitary representation of $\mathfrak{g}$, and let $E = \mathfrak{g} \times V$ be the corresponding action Lie algebroid together with the canonical flat Cartan connection. Then the CYMH reduces to the usual Yang-Mills-Higgs action (provided $B = 0$).

Gauge transformations

Let us view bundle maps $T\Sigma \to E$ as sections of the following bundle (an "exact sequence" in the category of Lie algrbroids): $T\Sigma \times E \to T\Sigma$. Apparently, (X, A) is a Lie algebroid morphism if an only if so is the corresponding section.

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Denote by $\tilde{E}$ the pull-back of E, i.e. $\tilde{E} = E \times \Sigma$. The jet algebroid $J^1(\tilde{E})$ acts on the space of bundle maps as follows:
given $\epsilon \in \Gamma(\tilde{E})$, $h \in C^\infty(\Sigma \times M)$ one has

$$\begin{aligned} \delta_{\epsilon_{(1)}}\Phi &:= \Phi \circ L_\epsilon \\ \delta_{\mathrm{d}h \otimes \epsilon}\Phi &:= \mathcal{F}(h)\Phi \circ \iota_\epsilon \end{aligned}$$

Proposition

1. *For any* $\lambda \in \Gamma\left(J^1(\tilde{E})\right)$*,* δ_λ *is an infinitesimal symmetry of the Lie algebroid morphism equation* $\mathcal{F} = 0$*;*
2. *For any* $\lambda, \lambda \in \Gamma\left(J^1(\tilde{E})\right)$ *one has* $[\delta_\lambda, \delta_{\lambda'}] = \delta_{[\lambda,\lambda']}$*.*

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Let us choose a local system of coordinates $\{X^i\}_{i=1}^n$ and a local frame $\{e_a\}_{a=1}^r$ of E, in which

$$\begin{aligned}
\nabla(e_a) &:= \omega_{ai}^b(X)\mathrm{d}X^i \otimes e_b \\
\rho(e_a) &:= \rho_a^i(X)\partial_{X^i} \\
[e_a, e_b] &:= C_{ab}^c(X)e_c \\
A &:= A^a(x, \mathrm{d}x)e_a \\
\epsilon &:= \epsilon^a(x, X)e_a
\end{aligned}$$

where $x \in \Sigma$.

Then

$$\begin{aligned}\delta_\epsilon X^i &= \rho^i_a \epsilon^a \\ \delta_\epsilon A^a &= \mathrm{d}\epsilon^a + C^c_{ab} A^b \epsilon^c + \omega^a_{bi} \epsilon^b D X^i\end{aligned}$$

where $DX^i = \mathrm{d}X^i - \rho^i_a(X) A^a$.

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Using the same local data, we have

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F^a \colon = \mathrm{d}A^a + \omega_{bi}(X) DX^i \wedge A^b + \frac{1}{2} t^a_{bc}(X) A^b \wedge A^c
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Remark

If ∇ is a Cartan connection then the above defined gauge transformations are closed off-shell, i.e. for any $\epsilon, \epsilon' \in \Gamma(\tilde{E})$ one has

$$[\delta_\epsilon, \delta_{\epsilon'}] = \delta_{[\epsilon, \epsilon']}$$

THEOREM (A.K., T. STROBL, 2015)

S_{CYMH} is gauge invariant if and only if the following conditions hold:

1. ∇^E *- Cartan connection, i.e.* $S = 0$
2. ${}^{\tau}\nabla(g) = 0$
3. ${}^{\alpha}\nabla(\kappa) = 0$
4. $R_{\nabla} + [\nabla^E, \rho](B) + \langle t, B\rangle = 0$, *where* ρ *is viewed as an operator acting from* E *to* TM *naturally extended to* $\Lambda^{\cdot} T^*M \otimes E$ *by the Leibniz property.*

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Remark

The auxiliary 2-form B *is not explicitly defined yet: we only know that it must satisfy the 3d equation.*

The compatibility condition for B

The choice of a connection on E gives us the decomposition of differential forms on $E[1]$ into the product of vertical and horizontal forms:

$$\Omega^m(E[1]) = \bigoplus_{p+q=m} \Omega^{p,q}(E[1])$$

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$$\Omega^m(E[1]) = \bigoplus_{p+q=m} \Omega^{p,q}(E[1])$$

The connection is compatible with the Lie algebroid structure if and only if L_Q preserves the corresponding horizontal distribution on $E[1]$ ($Q = Q_E$).

This means that $L_Q = \bar{Q} + \hat{\rho}$, where $\bar{Q}\colon \Omega^{p,q}(E[1]) \to \Omega^{p,q}(E[1])$ and $\hat{\rho}\colon \Omega^{p,q}(E[1]) \to \Omega^{p+1,q-1}(E[1])$

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The de Rham differential splits into the vertical, horizontal and curvature parts, where the latter $\hat{R}$ acts as follows

$$\hat{R}\colon \Omega^{p,q}(E[1]) \to \Omega^{p-1,q+2}(E[1])$$

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The compatibility between the Q-field and the de Ram operator

$$[L_Q, \mathrm{d}] = 0$$

implies

$$\bar{Q}(\hat{R})\colon = [\bar{Q}, \hat{R}] = 0$$

Thus the curvature of the connection is a $\bar{Q}-$cocycle.

The compatibility condition for B

Now the the compatibility condition for B reads as

$$\hat{R} = \bar{Q}(\hat{B})$$

where $\hat{B}\colon \Omega^{p,q}(E[1]) \to \Omega^{p-1,q+2}(E[1])$ is induced by B.

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This idea can generalized to more arbitrary N-graded Q-manifolds with compatible splittings of differential forms, which gives new higher Yang-Mills type gauge theories (T. Strobl, 2016 and 2018 and A.K., T.Strobl, 2018)

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Thank you for your attention!