Global Sections, the Minimal Model Program and Birational Tomography

ESI Workshop on the Unreasonable Effectiveness of Toric Geometry:

Bridging Mathematics, Computation, and String Theory

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Line Bundle Cohomology and the String Theory Incentive

Line bundle cohomology in string compactifications

- Compactifying the heterotic string on a smooth compact Calabi–Yau threefold requires the choice of a vector bundle V.
- The low-energy particle spectrum is determined by the bundle-valued cohomology groups

$$H^i(X,V),\ H^i(X,\wedge^2 V),\ H^i(X,V\otimes V^*)\ .$$

- For sums of line bundles,

$$V = L_1 \oplus \cdots \oplus L_r,$$

  this reduces to computing

$$h^i(X,L).$$

- Computing these dimensions efficiently became a major bottleneck for large string phenomenology scans (millions of line bundles).

The computational problem

- The Hirzebruch–Riemann–Roch theorem gives

$$\chi(X, L) = \sum_i (-1)^i h^i(X, L) = \int_X \mathrm{ch}(L) \cdot \mathrm{td}(X)$$

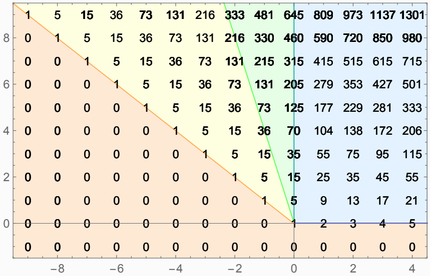
  This is helpful for computing chiral indices. However, full spectrum computations require cohomology dimensions.
- Vanishing theorems determine h^0 only in sufficiently positive regions.
- Away from these regions, there is no general closed formula for

$$h^i(X, L).$$

- Direct algorithms become computationally expensive when repeated millions of times.

Large-scale computations

- Extensive datasets of line bundle cohomology were generated on Calabi–Yau threefolds, motivated by string model building. [Anderson, AC, Gray, Lukas, Palti 13, AC, He, Lukas 18]
- The data exhibited remarkable regularity: piecewise quasipolynomial behaviour.



- Each coloured region is described by a single quasipolynomial.

From computation to geometry

- Initial hints from direct observation [AC, Lukas 18], [Larfors, Schneider 19] and ML [Klaewer, Schlechter 18], [Brodie, AC, Deen, Lukas 19]
- Theorems for surfaces, when X is toric, weak Fano and K3, for all line bundle cohomologies. [Brodie, AC 20], [Brodie, AC, Deen, Lukas 19]
- Theorems for global sections on Mori dream spaces and CY varieties. [AC, Lukas, Sheridan 26]
- Currently missing: understanding of the middle cohomologies.
- Generating functions for line bundle cohomology, encoding both zeroth and higher cohomologies [AC, 24].

This work: why do these piecewise formulae exist?

The answer is birational geometry.

Algebraic Geometry Recap

Line bundles and divisors

- The objects of interest are holomorphic line bundles L on complex projective varieties X for which wish to count:

$$H^0(X, L).$$

- A divisor is a collection of hypersurfaces with integer multiplicities,

$$D = \sum_i a_i D_i.$$

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- Every divisor defines a sheaf $\mathcal{O}_X(D)$, whose global sections are rational functions with zeros and poles controlled by D: positive coefficients allow poles, while negative coefficients require zeros.
- When X is smooth, $\mathcal{O}_X(D)$ is simply a line bundle.
- Throughout this talk I will use $\mathcal{O}_X(D) \leftrightarrow D$ interchangeably and discuss the cohomology function

$$D \longmapsto h^0(X, \mathcal{O}_X(D)).$$

Divisors and divisor classes

- Rational functions naturally define divisors (principal divisors):

$$(f) = \sum_i \mathrm{ord}_{D_i}(f)\, D_i,$$

  where $\mathrm{ord}_{D_i}(f)$ is the order of vanishing (positive) or pole (negative) of f along the hypersurface D_i.
- Divisors differing by the zeros and poles of a rational function,

$$D' = D + (f),$$

  define the same sheaf: $\mathcal{O}_X(D') \cong \mathcal{O}_X(D)$. This is the notion of linear equivalence, denoted by $\sim_{\mathrm{lin}}$.
- The divisor class group is $\mathrm{Cl}(X) = \mathrm{Div}(X)/\sim_{\mathrm{lin}},$.
- For example, on a curve, adding the divisor of a rational function simply moves the zeros and poles around since $\deg(f) = 0$.

Example: divisors on $\mathbb{P}^1$

- Let $D = 2\{\infty\}$. Then

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(D)) = \{f \in \mathbb{C}(x) \mid (f) + D \geq 0\},$$

  that is, rational functions with at most a double pole at ∞.
- A basis is

$$1, \qquad x, \qquad x^2,$$

  so

$$h^0(\mathbb{P}^1, \mathcal{O}(2)) = 3.$$

- Adding the divisor of a rational function moves the poles and zeros:

$$2\{\infty\} \sim P + Q,$$

  for any two (generic) points $P, Q \in \mathbb{P}^1$.
- The complete linear system

$$|D| = \{D' \geq 0 \mid D' \sim_{\text{lin}} D\}$$

  is therefore the 2-dim family consisting of all effective divisors of degree 2.

Fixed components

- A prime divisor Γ is a fixed component if

$$\Gamma \subseteq D' \qquad \text{for every } D' \in |D|.$$

- Removing a fixed component does not change the space of sections:

$$H^0(X, \mathcal{O}_X(D)) \cong H^0(X, \mathcal{O}_X(D - \Gamma)).$$

Example: fixed components on $\mathbb{F}_1$

- The Hirzebruch surface

$$\mathbb{F}_1 = \mathrm{Bl}_p(\mathbb{P}^2)$$

  is the blow-up of $\mathbb{P}^2$ at one point.

- Let

$$H = \pi^*(\text{class of a line in } \mathbb{P}^2), \qquad E = \text{exceptional curve.}$$

  Then

$$H \cdot E = 0, \qquad E^2 = -1.$$

- Consider $D = dH + eE$. If $e > 0$, then

$$D \cdot E = -e < 0.$$

  Any effective divisor not containing E must intersect E non-negatively. Therefore every divisor in $|D|$ contains E.

- Hence

$$H^0(\mathbb{F}_1, \mathcal{O}(dH + eE)) \cong H^0(\mathbb{F}_1, \mathcal{O}(dH)).$$

Briational Geometry Recap

From algebra to geometry: curves and numerical equivalence

- Curves are the basic one-dimensional subvarieties of X.
- A divisor and a curve admit a natural pairing:

$$D \cdot C,$$

which, intuitively, counts the number of intersection points (with appropriate multiplicities).

- More formally,

$$D \cdot C = \deg(\mathcal{O}_X(D)|_C)\,.$$

- The various notions of divisor positivity are measured by these intersection numbers. As such, it is natural to identify divisors that intersect every curve in the same way: this is the notion of numerical equivalence.

Numerical equivalence

- Two divisors are numerically equivalent if

$$D \equiv D' \quad \Longleftrightarrow \quad D \cdot C = D' \cdot C \quad \text{for all curves } C \subset X.$$

- Numerical equivalence is coarser than linear equivalence:

$$D \sim_{\text{lin}} D' \Longrightarrow D \equiv D'.$$

- Taking the quotient by numerical equivalence and tensoring with $\mathbb{R}$ gives the finite-dimensional real vector space

$$N^1(X)_{\mathbb{R}}.$$

This is the space in which divisor cones live.

Three cones of divisors

Inside $N^1(X)_{\mathbb{R}}$, we consider three positivity cones.

- Effective cone

  $\mathrm{Eff}(X) =$ cone generated by numerical classes of effective divisors.

  Equivalently, a divisor class contributes if

  $$h^0(X, \mathcal{O}_X(D)) > 0.$$

- The **big cone** is the interior of the pseudo-effective cone:

  $$\mathrm{Big}(X) = \mathrm{Int}\,\overline{\mathrm{Eff}}(X).$$

- Nef cone

  $$\mathrm{Nef}(X) = \{D \mid D \cdot C \geq 0 \text{ for every curve } C \subset X\}.$$

- Movable cone

  $\mathrm{Mov}(X) =$ cone generated by divisors whose $|D|$ has no fixed divisorial components.

$$\mathrm{Nef}(X) \subseteq \mathrm{Mov}(X) \subseteq \mathrm{Eff}(X).$$

Why is the nef cone special?

- The Kawamata–Viehweg vanishing theorem states that if

$$D - K_X$$

is nef and big, then

$$H^i(X, \mathcal{O}_X(D)) = 0, \qquad i > 0.$$

- Consequently,

$$h^0(X, \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X(D)).$$

- Since the Euler characteristic is given by the Hirzebruch–Riemann–Roch theorem,

$$\chi(X, \mathcal{O}_X(D)) = \int_X \mathrm{ch}(\mathcal{O}_X(D))\,\mathrm{td}(X),$$

the zeroth cohomology is an explicit polynomial on the nef cone (shifted by $-K_X$).

What happens outside the nef cone?

- Outside the nef cone, higher cohomology need no longer vanish, so

$$h^0 \neq \chi.$$

- If

$$D \in \mathrm{Eff}(X) \setminus \mathrm{Mov}(X),$$

then the linear system $|D|$ has fixed divisorial components.

These may be subtracted, or contracted birationally without changing h^0.

- If

$$D \in \mathrm{Mov}(X) \setminus \mathrm{Nef}(X),$$

then D has no fixed divisors, but fails to be positive on some curves:

$$D \cdot C < 0.$$

These curves are modified birationally by flips (small birational modifications).

Small birational modifications

- A birational map

$$\varphi : X \dashrightarrow X'$$

  is a small $\mathbb{Q}$-factorial modification (SQM) if it is an isomorphism in codimension one.
- This means that divisors are not contracted and we identify divisor classes across the birational map:

$$\mathrm{Div}(X) \cong \mathrm{Div}(X').$$

- But curves may be changed. As such, a movable divisor which is not nef on X may become nef on a small birational model:

$$D' \in \mathrm{Nef}(X'),$$

  where D' is the transform of D on X'.
- A movable divisor which is not nef on X may become nef on a small birational model X'.

Formulae for Counting Global Sections. Contraction

The D-minimal model program

Starting from an effective divisor D:

1. If D has fixed divisorial components, contract them.
2. If D is movable but not nef, perform a sequence of flips until the transform of D becomes nef.

- The resulting birational model

$$f : X \dashrightarrow Y$$

is called the D-minimal model. On Y, the resulting divisor is nef

$$D' = f_* D \in \mathrm{Nef}(Y)$$

- The goal is to find a birational model on which the transform of D is nef. This is precisely the objective of the D-minimal model program.

Ideally: apply a vanishing theorem on Y and compute $h^0(Y, \mathcal{O}_Y(D')) = \chi(Y, \mathcal{O}_Y(D'))$. Can this idea be made to work?

The birational philosophy

Under suitable hypotheses, the birational strategy can be carried out:

- The D-minimal model program preserves

$$h^0(X, \mathcal{O}_X(D))$$

  while transforming D into a nef divisor on a birational model.
- If X has klt singularities, then every D-minimal model Y also has klt singularities. If, moreover, $-K_Y \in \mathrm{Mov}(Y) \cap \mathrm{Big}(Y)$, Kawamata–Viehweg can be applied on an SQM Y' of Y giving

$$h^0(Y, \mathcal{O}_Y(D')) = \chi(Y', \mathcal{O}'_Y(D'')).$$

- When Y' is smooth, we know that

$$\chi(Y, \mathcal{O}_Y(D'))$$

  is polynomial, but this is a very strong assumption. However, by a general theorem due to Snapper, on any complete $\mathbb{Q}$-factorial normal variety, the Euler characteristic is quasipolynomial on the divisor class group.

When is the birational strategy explicit?

To obtain explicit formulae for $D \longmapsto h^0(X, \mathcal{O}_X(D))$, the birational geometry must be sufficiently well behaved.

Ideally,

- every divisor in the big cone admits a D-minimal model;
- the big cone is partitioned into regions on which the D-minimal model is constant.

Optional extra:

- only finitely many D-minimal models occur.

These regions are the Mori chambers.

Mori dream spaces and beyond

Mori dream spaces (Hu–Keel)

- every divisor admits a D-minimal model;
- finitely many D-minimal models occur;
- the effective cone is partitioned into finitely many Mori chambers, each corresponding to a one birational model.

For a Mori dream space, the Mori chamber decomposition plays the role of the secondary fan of a toric variety. Each chamber corresponds to an SQM or contraction and the birational models are finite in number.

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Calabi–Yau varieties

- every divisor admits a D-minimal model;
- the movable cone is covered by nef cones of birational models;
- in general, infinitely many chambers may occur.

Global sections via the D-minimal model program

Theorem (Mori dream spaces)

Let X be a Mori dream space with klt singularities such that every D-minimal model Y satisfies

$$-K_Y \in \operatorname{Mov}(Y) \cap \operatorname{Big}(Y).$$

Then $h^0(X, \mathcal{O}_X(D))$ is a piecewise quasipolynomial on the big cone with domains given by the Mori chambers (possibly shifted by anticanonical divisors).

Theorem (Mori dream space CY varieties)

Let X be a Calabi–Yau Mori dream space with klt singularities. Then $h^0(X, \mathcal{O}_X(D))$ is a piecewise quasipolynomial on the big cone with domains given by the Mori chambers.

Calabi–Yau varieties beyond Mori dream spaces

Theorem (Calabi–Yau varieties)

Let X be a projective $\mathbb{Q}$-factorial klt Calabi–Yau variety, with $K_X \sim_{\mathbb{Q}} 0$. For every big divisor D, the D-minimal model program produces a birational contraction

$$f : X \dashrightarrow Y$$

such that

$$D' = f_* D$$

is nef on Y. Moreover, the effective cone admits a chamber decomposition according to the D-minimal model. On each chamber,

$$h^0(X, \mathcal{O}_X(D)) = \chi(Y, \mathcal{O}_Y(D')),$$

where Y is the corresponding D-minimal model.

Formulae for Counting Global Sections. Subtraction

Why not always use contraction?

The contraction method is conceptually clean:

$$D \quad \rightsquigarrow \quad f_*D \in \mathrm{Nef}(Y), \qquad h^0(X, \mathcal{O}_X(D)) = \chi(Y, \mathcal{O}_Y(f_*D)).$$

But in practice it has drawbacks.

- One must explicitly construct the D-minimal model

$$f : X \dashrightarrow Y.$$

- The birational models may be harder to describe than the original variety X. For instance, the model Y may be singular and the Euler characteristic on Y may require singular Riemann–Roch data.

Can we stay on X instead?

Subtraction

The key observation is elementary.

- If Γ is a fixed component of $|D|$, then every section of $\mathcal{O}_X(D)$ vanishes along Γ.
- Therefore
$$H^0(X, \mathcal{O}_X(D)) = H^0(X, \mathcal{O}_X(D - \Gamma)).$$
- Repeating this process removes fixed divisorial components:
$$D \longmapsto D - \Gamma_1 \longmapsto D - \Gamma_1 - \Gamma_2 \longmapsto \cdots$$

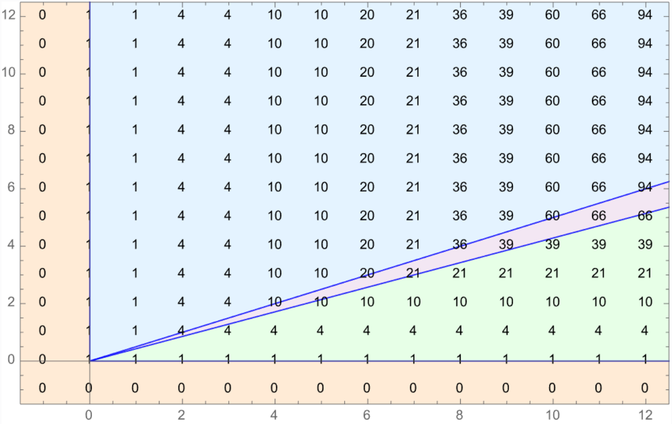
Subtraction preserves h^0 while improving positivity.

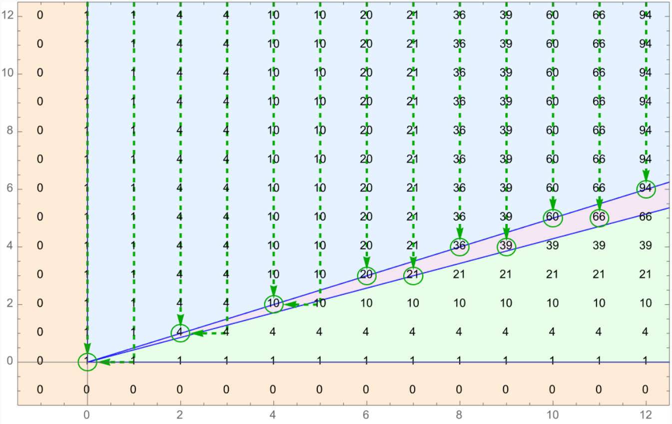
Subtraction chamber by chamber

On a Mori dream space, the fixed components are controlled by the Mori chamber decomposition.

- If D lies in a Mori chamber $\mathcal{C}$, the associated D-minimal model contracts divisors
$$E_1, \ldots, E_r.$$
- The fixed divisorial part of D is an integral combination
$$\mathrm{Fix(D)} = \sum_i a_i E_i, \qquad a_i \in \mathbb{Z}_{\geq 0}.$$
- Subtraction gives
$$h^0(X, \mathcal{O}_X(D)) = h^0(X, \mathcal{O}_X(D - \mathrm{Fix}(D))).$$
- After subtraction, the new divisor may lie in a different chamber, so the process is repeated.

Because the coefficients decrease, this terminates after finitely many steps.

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Asymptotic subtraction

Chamberwise, the fixed part is governed by a linear Zariski decomposition

$$D = P_{\mathcal{C}}(D) + N_{\mathcal{C}}(D), \qquad N_{\mathcal{C}}(D) = \sum_i \lambda_i(D) E_i.$$

- The functions $\lambda_i(D)$ are linear on the chamber $\mathcal{C}$.
- For large multiples mD, the fixed part is obtained by rounding:

$$\mathrm{Fix}(mD) = \sum_i \lceil m\lambda_i(D) \rceil E_i.$$

- Hence one asymptotic subtraction gives

$$h^0(X, \mathcal{O}_X(mD)) = h^0\left(X, \mathcal{O}_X\left(mD - \sum_i \lceil m\lambda_i(D) \rceil E_i\right)\right).$$

- The resulting divisor is movable, so vanishing and Euler characteristics apply on an SQM.

Thus each chamber gives an asymptotic quasipolynomial.

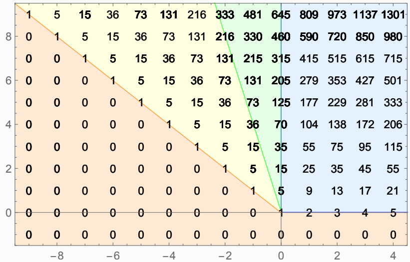
From asymptotic to chamberwise formulae

- The asymptotic subtraction formula is a chamberwise quasipolynomial.
- The contraction method identifies the same quasipolynomial with h^0 throughout the chamber, whenever the vanishing hypotheses hold on the corresponding D-minimal models.

Subtraction gives the formula; contraction extends its validity.

If two quasipolynomials agree on all sufficiently large integers in each residue class, then they are identical.

A fairly innocent CY threefold example



$$X = \begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^4 \end{array} \left[\begin{array}{cc} 1 & 1 \\ 4 & 1 \end{array} \right]^{2,86}$$

$$L = \mathcal{O}_X(k_1 D_1 + k_2 D_2)$$

$$\chi(X, L) = 2k_1(1 + k_2^2) + \frac{5}{6}k_2(5 + k_2^2)$$

region in eff. cone	$h^0(X, L = \mathcal{O}_X(D = k_1 D_1 + k_2 D_2))$
$\mathcal{K}(X)$	$\chi(X, \mathcal{O}_X(D))$
$\overline{\mathcal{K}}(X') \setminus \{\mathcal{O}_X\}$	$\chi(X', \mathcal{O}_{X'}(D'))$
$\overline{\Sigma}$	$\chi\left(X', \mathcal{O}_{X'}\left(D' - \left\lceil \frac{D' \cdot \tilde{C}_2'}{\Gamma' \cdot \tilde{C}_2'} \right\rceil \Gamma'\right)\right)$
$k_1 \geq 0,\ k_2 = 0$	$\chi(\mathbb{P}^1, (D \cdot C_1) H_{\mathbb{P}^1})$

Birational Tomography

Birational tomography

So far we have used birational geometry to compute

$$D \longmapsto h^0(X, \mathcal{O}_X(D)).$$

How much birational geometry data is encoded in this function?

The piecewise (quasi)polynomial formula records

- the chamber decomposition,
- the birational models,
- the contracted divisors,
- the Euler characteristics on each model.

Can one reconstruct the birational geometry from these formulae?

Information visible in the formulae

Examples of information visible in the formulae:

- chamber walls appear as changes of quasipolynomial;
- subtraction terms detect exceptional divisors;
- Euler polynomials determine intersection-theoretic data;
- generating functions package the entire section ring.
- more information in the periodic part: it may be able to detect Cartier indices, torsion in divisor class groups, or singularity structure.

The cohomology formulae behave like a tomographic image of the birational geometry. Perhaps the polynomial part encodes the birational geometry, while the periodic part records its arithmetic.

Generating Functions for Cohomology

Conjecture 3. *Let X be a general hypersurface of bi-degree (d,e) in $\mathbb{P}^1 \times \mathbb{P}^{n\geq 3}$ with $d \leq n$ and e arbitrary or d arbitrary and $e=1$. Denote $H_1 = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(1,0)|_X$ and $H_2 = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(0,1)|_X$. Then in the basis $\{H_1, H_2\}$,*

$$
\begin{aligned}
CS^0(X,\mathcal{O}_X) &= \left(\frac{(1-t_2^e)^{d+1}}{(1-t_1)^2(1-t_2)^{n+1}(1-t_1^{-1}t_2^e)^d}\,,\ \begin{matrix} t_2 & t_1 \\ 0 & 0 \end{matrix}\right) = \sum_{m_1,m_2\in\mathbb{Z}} h^0(X,\mathcal{O}_X(m_1H_1+m_2H_2))t_1^{m_1}t_2^{m_2} \\
CS^1(X,\mathcal{O}_X) &= \left(\frac{(1-t_2^e)^{d+1}}{(1-t_1)^2(1-t_2)^{n+1}(1-t_1^{-1}t_2^e)^d}\,,\ \begin{matrix} t_2 & t_1 \\ 0 & \infty \end{matrix}\right) = \sum_{m_1,m_2\in\mathbb{Z}} h^1(X,\mathcal{O}_X(m_1H_1+m_2H_2))t_1^{m_1}t_2^{m_2} \\
(-1)^nCS^{n-1}(X,\mathcal{O}_X) &= \left(\frac{(1-t_2^e)^{d+1}}{(1-t_1)^2(1-t_2)^{n+1}(1-t_1^{-1}t_2^e)^d}\,,\ \begin{matrix} t_2 & t_1 \\ \infty & 0 \end{matrix}\right) = \sum_{m_1,m_2\in\mathbb{Z}} h^{n-1}(X,\mathcal{O}_X(m_1H_1+m_2H_2))t_1^{m_1}t_2^{m_2} \\
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\end{aligned}
\tag{1.10}
$$

and all intermediate line bundle cohomologies vanish.

Generating functions for line bundle cohomology

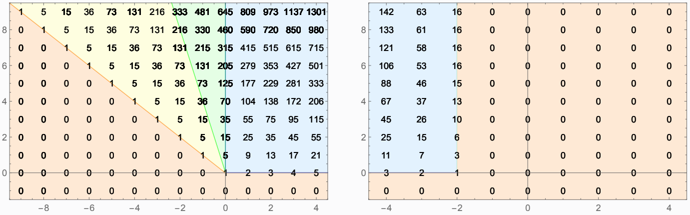

Figure 3: *Zeroth and first line bundle cohomology data (left plot and, respectively, right plot) for a general Calabi-Yau complete intersection in the deformation family* (1.16). *The numbers indicate cohomology dimensions while their locations indicate first Chern classes of line bundles.*

Generating functions for line bundle cohomology

Conjecture 5. *Let X be a general complete intersection of two hypersurfaces of bi-degrees $(1,1)$ and $(1,4)$ in $\mathbb{P}^1 \times \mathbb{P}^4$, belonging to the deformation family with configuration matrix*

$$\begin{array}{c} \mathbb{P}^1 \\ \mathbb{P}^4 \end{array} \left[\begin{array}{cc} 1 & 1 \\ 1 & 4 \end{array} \right] . \tag{1.16}$$

The effective, movable and nef cones of X are given by

$$\begin{gathered} \mathrm{Eff}(X) = \mathbb{R}_{\geq 0} H_1 + \mathbb{R}_{\geq 0}(H_2 - H_1),\ \mathrm{Mov}(X) = \mathbb{R}_{\geq 0} H_1 + \mathbb{R}_{\geq 0}(4H_2 - H_1) \\ \mathrm{Nef}(X) = \mathbb{R}_{\geq 0} H_1 + \mathbb{R}_{\geq 0} H_2\ , \end{gathered} \tag{1.17}$$

where $H_1 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^4}(1,0)|_X$ and $H_2 = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^4}(0,1)|_X$. All cohomology dimensions of all line bundles in the Picard group of X are encoded by the following generating function, relative to the basis $\{H_1, H_2\}$ of $\mathrm{Pic}(X)$*:*

$$\begin{aligned} CS^0(X, \mathcal{O}_X) &= \left(\frac{(1-t_2)^2 \left(1-t_2^4\right)^2}{(1-t_1)^2 (1-t_2)^5 \left(1-t_1^{-1}t_2\right) \left(1-t_1^{-1}t_2^4\right)}, \begin{array}{cc} t_2 & t_1 \\ 0 & 0 \end{array} \right) \\ CS^1(X, \mathcal{O}_X) &= \left(\frac{(1-t_2)^2 \left(1-t_2^4\right)^2}{(1-t_1)^2 (1-t_2)^5 \left(1-t_1^{-1}t_2\right) \left(1-t_1^{-1}t_2^4\right)}, \begin{array}{cc} t_2 & t_1 \\ \infty & 0 \end{array} \right) \\ CS^2(X, \mathcal{O}_X) &= \left(\frac{(1-t_2)^2 \left(1-t_2^4\right)^2}{(1-t_1)^2 (1-t_2)^5 \left(1-t_1^{-1}t_2\right) \left(1-t_1^{-1}t_2^4\right)}, \begin{array}{cc} t_2 & t_1 \\ 0 & \infty \end{array} \right) \\ CS^3(X, \mathcal{O}_X) &= \left(\frac{(1-t_2)^2 \left(1-t_2^4\right)^2}{(1-t_1)^2 (1-t_2)^5 \left(1-t_1^{-1}t_2\right) \left(1-t_1^{-1}t_2^4\right)}, \begin{array}{cc} t_2 & t_1 \\ \infty & \infty \end{array} \right) \end{aligned} \tag{1.18}$$

Generating functions for line bundle cohomology

Conjecture 6. *Let X be a general hypersurface of bidegree $(3,3)$ in $\mathbb{P}^2 \times \mathbb{P}^2$. The effective, movable and nef cones coincide*

$$\mathrm{Eff}(X) = \mathrm{Mov}(X) = \mathrm{Nef}(X) = \mathbb{R}_{\geq 0} H_1 + \mathbb{R}_{\geq 0} H_2 \ ,$$

where $H_1 = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,0)|_X$ and $H_2 = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(0,1)|_X$. The first line bundle cohomology is non-trivial within the two disconnected cones $\mathbb{R}_{>0}H_2 + \mathbb{R}_{\geq 0}(-H_1+H_2)$ and $\mathbb{R}_{>0}H_1 + \mathbb{R}_{\geq 0}(H_1-H_2)$. Defining

$$G(x,y) = \frac{(x^{-1}y)^3((1+x-y)^3 - 1 + 3x(1-y))}{(1-x^{-1}y)^3(1-y)^3} \ , \tag{1.20}$$

all line bundle cohomology dimensions on X are encoded in the following generating function

$$\begin{aligned}
CS^0(X,\mathcal{O}_X) &= 1 + \left(G(t_1,t_2)\,, \begin{array}{cc} t_1 & t_2 \\ 0 & 0 \end{array} \right) + \left(G(t_2,t_1)\,, \begin{array}{cc} t_1 & t_2 \\ 0 & 0 \end{array} \right) \\
CS^1(X,\mathcal{O}_X) &= 0 + \left(G(t_1,t_2)\,, \begin{array}{cc} t_1 & t_2 \\ \infty & 0 \end{array} \right) + \left(G(t_2,t_1)\,, \begin{array}{cc} t_1 & t_2 \\ 0 & \infty \end{array} \right) \\
-CS^2(X,\mathcal{O}_X) &= 2 + \left(G(t_1,t_2)\,, \begin{array}{cc} t_1 & t_2 \\ 0 & \infty \end{array} \right) + \left(G(t_2,t_1)\,, \begin{array}{cc} t_1 & t_2 \\ \infty & 0 \end{array} \right) \\
-CS^3(X,\mathcal{O}_X) &= 1 + \left(G(t_1,t_2)\,, \begin{array}{cc} t_1 & t_2 \\ \infty & \infty \end{array} \right) + \left(G(t_2,t_1)\,, \begin{array}{cc} t_1 & t_2 \\ \infty & \infty \end{array} \right)
\end{aligned} \tag{1.21}$$

The interpretation of the generating function $G(x,y)$ is not transparent. However, the three terms that give the zeroth cohomology dimensions combine to

$$CS^0(X,\mathcal{O}_X) = \left(\frac{(1-t_1^3t_2^3)}{(1-t_1)^3(1-t_2)^3}\,, \begin{array}{cc} t_1 & t_2 \\ 0 & 0 \end{array} \right) \ , \tag{1.22}$$

Cohomology formulae & generating functions

- Our original motivation came from string compactifications: explicit cohomology formulae make spectrum computations essentially instantaneous.

Cohomology formulae & generating functions

- Our original motivation came from string compactifications: explicit cohomology formulae make spectrum computations essentially instantaneous.
- More conceptually, line bundle cohomology formulae and generating functions appear to encode remarkably rich birational information:
    - chamber decompositions,
    - birational models,
    - exceptional divisors,
    - intersection-theoretic invariants,
    - section rings.

Cohomology formulae & generating functions

- Birational tomography: Line bundle cohomology formulae and generating functions encode a remarkable amount of birational information. How much of the birational geometry (and ultimately of the Cox ring) can be reconstructed from this data?

Cohomology formulae & generating functions

- Birational tomography: Line bundle cohomology formulae and generating functions encode a remarkable amount of birational information. How much of the birational geometry (and ultimately of the Cox ring) can be reconstructed from this data?
- For higher cohomology: can the chamber decomposition and the Cox ring replace the fan and Laurent monomials in the construction of explicit higher cohomology representatives on toric varieties?

Thank you for listening!