

Canonical DEQs and intersection matrices

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Differential Equations

One of the most powerful techniques in the evaluation of multi-loop Feynman integrals is the method of ε -factorized differential equations [Hen13]. Thereby, the master integrals \mathbf{I} of a family of Feynman integrals satisfy $d\mathbf{I} = \Omega(x, \varepsilon)\mathbf{I}$. If we find a rotation \mathbf{U} such that $\mathbf{J} = \mathbf{U}\mathbf{I}$ with

$$d\mathbf{J} = \varepsilon\mathbf{A}(x)\mathbf{J}, \quad \varepsilon\mathbf{A}(x) = \left(\mathbf{U}(x, \varepsilon)\Omega(x, \varepsilon) + d\mathbf{U}(x, \varepsilon) \right) \mathbf{U}(x, \varepsilon)^{-1},$$

we can solve for \mathbf{J} as a path ordered exponential

$$\mathbf{J}(x, \varepsilon) = \mathbb{P}_\gamma(x, \varepsilon)\mathbf{J}_0(\varepsilon), \quad \text{with} \quad \mathbb{P}_\gamma(x, \varepsilon) = \mathbb{P} \exp \left[\varepsilon \int_\gamma \mathbf{A}(x) \right].$$

Expanding in ε , the coefficients in each order of the Laurent expansion are given as iterated integrals.

ε -Factorized and Canonical DEQs

Besides ε -factorization, these bases often enjoy several additional properties. While an agreed-upon notion of a 'good'/canonical basis exists in the polylogarithmic case, this notion becomes much less clear beyond polylogs.

Polylogarithmic. If the associated geometry is a Riemann sphere, a canonical basis satisfies the following additional properties:

- \mathbf{A} consists of $d \log$ -forms.
- If $d \log$ -forms involve only rational functions (or appropriate combinations can be rationalized) \rightarrow the expansion coefficients are pure combinations of multiple polylogarithms (MPLs) of uniform transcendental weight.

Beyond Polylogs. For integrals that are connected to higher genus Riemann surfaces or higher dimensional Calabi-Yau varieties, there is no consensus yet on what the generalization of a canonical basis is. We define a canonical basis in these cases to be a basis for which the integration kernels in \mathbf{A}

- define linearly independent cohomology classes,
- have simple poles.

Motivation

Ref. [GNTW23] introduces an explicit construction to cast the differential equation into an ε -factorized form. Observationally, the arising basis is canonical in the above sense. The algorithm introduces so called ε -functions – iterated integrals, usually transcendental functions, determined by a suitable ansatz and a differential equation. These objects are not very well understood. We want to address the following questions

- When do they evaluate to simpler or known functions?
- How many independent ε -functions need to be introduced?

More generally, this is part of a broader effort to investigate (canonical) differential equations from the perspective of twisted cohomology and to connect concepts of both frameworks.

The Algorithm

The algorithm proposed in [GNTW23] constructs the transformation to canonical form. The first step relies on making a *good choice of initial basis*. Concretely, it is chosen such that it is compatible with the mixed Hodge structure (MHS) (cf., ref. [B⁺25] for a similar definition of filtration). Then this basis is rotated to canonical form by a sequence of rotations

$$\mathbf{U}(x, \varepsilon) = \mathbf{U}_t(x, \varepsilon) \mathbf{U}_\varepsilon(\varepsilon) \mathbf{U}_{\text{ss}}(x).$$

Ansatz for ε -functions
 ε -realignment
Inverse of semi-simple part of period matrix of underlying geometry

The ε -functions are determined by requiring \mathbf{U}_t to cancel all non-linear ε -dependence after the first two rotations

$$(\mathbf{U}_\varepsilon \mathbf{U}_{\text{ss}} \Omega + d(\mathbf{U}_\varepsilon \mathbf{U}_{\text{ss}})) \mathbf{U}_{\text{ss}}^{-1} \mathbf{U}_\varepsilon^{-1} = \sum_{k=k_{\text{min}}}^0 \tilde{\Omega}_k \varepsilon^k + \varepsilon \tilde{\Omega}_1 \quad \rightarrow \quad (\mathbf{U}(x, \varepsilon) \Omega(x, \varepsilon) + d(\mathbf{U}(x, \varepsilon))) \mathbf{U}(x, \varepsilon) = \varepsilon \mathbf{A}(x).$$

canceled by \mathbf{U}_t

Let us characterize this last rotation further.

- The ε -functions in \mathbf{U}_t are transcendental functions, which are typically iterated integrals over rational functions or (derivatives of) periods of the underlying geometry.
- \mathbf{U}_t respects the mixed Hodge structure and is of the form $\mathbf{U}_t = \begin{pmatrix} 1 & \cdots & 0 \\ * & \ddots & \vdots \\ * & * & 1 \end{pmatrix}$.

Twisted Cohomology

As twisted cohomology is a natural mathematical framework to describe Feynman integrals in dimensional regularization, we address the questions raised above from this perspective. We study integrals of the form

$$\int \Psi \varphi \quad \text{with } \Psi \text{ a multi-valued function.}$$

The twisted cohomology group is a vector space of (equivalence classes of) single valued differential forms φ . There exists a dual space and an inner product between differential forms and dual differential forms $\langle \varphi_i | \tilde{\varphi}_j \rangle$. The *intersection matrix* is

$$\mathbf{C}_{ij} = \langle \varphi_i | \tilde{\varphi}_j \rangle \sim \int \varphi_i \wedge \tilde{\varphi}_j.$$

It satisfies the differential equation

$$d\mathbf{C} = \Omega\mathbf{C} + \mathbf{C}\tilde{\Omega}^T. \quad (1)$$

Similarly the twisted homology group is the group of cycles (with coefficients in a local system). Then, the *period matrix* $\mathbf{P}_{ij} = \langle \varphi_i | \gamma_j \rangle \sim \int_{\gamma_j} \Psi \varphi_i$ is the fundamental solution of the differential equation of the master integrals. On the maximal cut, the bases can be chosen such that the system is *self-dual* $\tilde{\mathbf{P}}(x, \varepsilon) = \mathbf{P}(x, -\varepsilon)$.

Canonical Intersection Matrix

The iterated integrals, arising from a canonical basis as defined above are linearly independent as shown in ref. [DPSS25]. As a consequence of this independence, together with the *twisted Riemann bilinears*

$$\mathbf{P}(\varepsilon, x) \mathbf{H}(\varepsilon)^{-1T} \mathbf{P}(x, -\varepsilon)^T = \mathbf{C}(x, \varepsilon),$$

it follows that, in a canonical, self-dual basis the intersection matrix is constant

$$\mathbf{U}(x, \varepsilon) \mathbf{C} (\mathbf{U}(x, -\varepsilon))^T = f(\varepsilon) \mathbf{\Delta},$$

where $\mathbf{\Delta}$ is a constant, symmetric or antisymmetric matrix. Therefore, the constancy of the intersection matrix in a given ε -factorized basis, provides a necessary (though not sufficient) criterion for the linear independence of the iterated integrals.

Constraints on ε -functions

In a canonical basis, the differential equation of the intersection matrix (1) reduces to

$$\varphi_\Delta(\mathbf{A}(x)) = \mathbf{\Delta} \mathbf{A}^T(x) \mathbf{\Delta}^{-1} = \mathbf{A}(x), \quad (2)$$

where we refer to $\varphi(\cdot)$ as the $\mathbf{\Delta}$ -transposition.

Observation [DMP⁺25]: Let UT be the group of lower triangular, unipotent matrices. \mathbf{U}_t is element of a subgroup $G_{\text{par}} \subseteq \text{UT}$ that respects the initial ordering (filtration) of the basis. This subgroup is closed under $\mathbf{\Delta}$ -transposition $\varphi_\Delta(\mathbf{U}) \in G_{\text{par}}$.

Theorem [DMP⁺25]: There are unique (unipotent, lower-triangular) matrices \mathbf{O}, \mathbf{R} such that

$$\mathbf{U}_t = \mathbf{O}\mathbf{R}, \quad \text{with} \quad \varphi_\Delta(\mathbf{O}) = \mathbf{O}^{-1} \quad \text{and} \quad \varphi_\Delta(\mathbf{R}) = \mathbf{R}.$$

As a consequence of (2), the matrix \mathbf{R} is fully fixed by

$$\mathbf{R}^2 \tilde{\Omega}_1 = \varphi_\Delta(\tilde{\Omega}_1) \mathbf{R}^2,$$

in terms of entries of $\tilde{\Omega}_1$ which are (derivatives of) periods and algebraic functions. The matrix \mathbf{O} cannot be constrained by (2).

This yields an upper bound on the number of independent ε -functions! Since \mathbf{R} is lower-triangular, solving the constraint amounts to solving a linear system. Observationally, the entries of \mathbf{O} define genuine ε -functions that cannot be expressed in terms of periods or algebraic functions.

Obtaining $\mathbf{\Delta}$

To make use of the constraint (2), it is necessary to know the canonical intersection matrix. It is typically a very simple matrix and there are multiple approaches to obtain $\mathbf{\Delta}$. Assume the rotation \mathbf{U}_t is known in terms of a formal ansatz for the ε -functions. One can pursue one of the following strategies [DMP⁺25]:

- Solve the differential equation for the ε -functions numerically near some point in kinematic space. Then solve eq. (2) in the form

$$\mathbf{\Delta} \mathbf{A}^T(x) - \mathbf{A}(x) \mathbf{\Delta} = 0,$$

for $\mathbf{\Delta}$. Since $\mathbf{\Delta}$ is constant, this is the full solution.

- Compute \mathbf{C} in the original basis and explicitly perform the rotation to canonical basis. The result will depend on the ε -functions. Since $\mathbf{\Delta}$ is constant (and the ε -functions are only determined up to a constant), we require the entries that depend on linearly independent combinations of the ε -functions to vanish. This yields the relations between the ε -functions as well as $\mathbf{\Delta}$ at the same time.
- Solve for the intersection matrix before the last rotation

$$\tilde{\Omega}_1 \tilde{\mathbf{C}} = \tilde{\mathbf{C}} \tilde{\Omega}_1^T,$$

and use the closure under $\mathbf{\Delta}$ -transposition to fix remaining degrees of freedom. Rotate to canonical form, and obtain $\mathbf{\Delta}$ by requiring kinematic independence.

Examples

Calabi-Yau Operators.

Based on the above results, we conjecture that for deformed Calabi-Yau operators [DMP⁺25] (cf. e.g. Ref. [DMN⁺25]), the canonical intersection matrix takes the universal anti-diagonal form

$$\mathbf{\Delta} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & & 1 & 0 \\ \vdots & & \ddots & \vdots & \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The number of independent ε -functions is bounded by $\begin{cases} \frac{1}{4}N(N-2), & \text{if } N \text{ even,} \\ \frac{1}{4}(N-1)^2, & \text{if } N \text{ odd.} \end{cases}$

Banana Integrals. For the four-loop banana integral with two unequal masses, a priori the number of ε -functions in the ansatz of \mathbf{U}_t is 23 (21 on the maximal cut). Using the constraint from the canonical intersection matrix this number reduces to 8 (6).

For the three-loop unequal mass banana integral the number of ε -functions reduces from 23 (19) to 13 (9).



References

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