Generalizations of spin Sutherland models from Hamiltonian reductions of Heisenberg doubles

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To begin, recall that the classical Sutherland Hamiltonian, with coupling constant x^2,

$$H_{\text{trig}-\text{Suth}}(q,p) \equiv \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{8}\sum_{j\neq k}\frac{x^2}{\sin^2((q_j-q_k)/2)},$$

admits two kinds of spin extensions. The first one contains Lie algebraic ('collective') spin variables,

$$H_{\text{spin}-\text{Suth}}(q,p,\xi) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{8}\sum_{j\neq k}\frac{|\xi_{jk}|^2}{\sin^2((q_j-q_k)/2)},$$

where $\xi \in \mathfrak{u}(n)^*$, with zero diagonal part. These models exist for all simple Lie algebras,

$$H_{\text{spin}-\text{Suth}}(q,p,\xi) = \frac{1}{2}\langle p,p\rangle + \frac{1}{8}\sum_{\alpha\in\mathfrak{R}}\frac{2}{|\alpha|^2}\frac{|\xi_\alpha|^2}{\sin^2(\alpha(q)/2)},$$

and arise from Hamiltonian reduction of the cotangent bundle T^*G of a compact Lie group G. The 'spin variables' $\xi_\alpha \in \mathbb{C}$ $(\xi_{-\alpha} = \xi_\alpha^*)$ matter up to gauge transformation by the maximal torus $G_0 < G$ and $q,p \in \mathrm{i}\mathcal{G}_0$ with $\mathcal{G}_0 = \mathrm{Lie}(G_0)$. Here, we use the Killing form and the set of roots $\mathfrak{R} = \{\alpha\}$ of the complexified Lie algebra $\mathcal{G}^{\mathbb{C}}$.

The second kind of generalization is the Gibbons–Hermsen (1984) model

$$H_{\mathrm{G-H}} = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \frac{1}{8}\sum_{j\neq k} \frac{|(S_j S_k^\dagger)|^2}{\sin^2((q_j - q_k)/2)}.$$

The complex row-vector $S_j := [S_{j1}, \dots, S_{jd}] \in \mathbb{C}^d$, $d \geq 2$, is attached to the particle with coordinate q_j, representing internal degrees of freedom. The overall phases of the spin vectors S_j can be changed by gauge transformations. This model descends from the extended cotangent bundle $T^*U(n) \times \mathbb{C}^{n\times d}$.

The purpose of lecture 2 is to explain that generalizations of these models arise if one replaces the cotangent bundles by the so-called Heisenberg doubles, which are their Poisson–Lie analogues. We shall mainly focus on the first kind of models.

The lecture is based on the following papers:

- LF, *Poisson–Lie analogues of spin Sutherland models*, Nucl. Phys. B 949, 114807 (2019)
- LF, Poisson reductions of master integrable systems on doubles of compact Lie groups, Ann. Henri Poincaré 24, 1823-1876 (2023)
- LF, *Poisson–Lie analogues of spin Sutherland models revisited*, J. Phys A: Math. Theor. 57, 205202 (2024)

Reminder on the general picture

Let G be a (connected and simply connected) compact Lie group with simple Lie algebra $\mathcal{G}$. Denote $\mathcal{G}^{\mathbb{C}}$ and $G^{\mathbb{C}}$ the complexifications, and define $\mathfrak{P} := \exp(\mathrm{i}\mathcal{G}) \subset G^{\mathbb{C}}$. Example: $G = SU(n)$, $G^{\mathbb{C}} = SL(n,\mathbb{C})$, $\mathfrak{P} = \{X \in SL(n,\mathbb{C}) \mid X^\dagger = X,\ X\,\text{positive}\}$.

One has the following 3 'classical doubles' of G:

$$\text{Cotangent bundle} \quad T^*G \simeq G \times \mathcal{G}^* \simeq G \times \mathcal{G} =: \mathcal{M}_1$$

$$\text{Heisenberg double} \quad G^{\mathbb{C}}_{\mathbb{R}} \simeq G \times G^* \simeq G \times \mathfrak{P} =: \mathcal{M}_2$$

$$\text{Internally fused quasi-Poisson double} \quad G \times G =: \mathcal{M}_3$$

The pull-backs of the relevant rings of invariants

$$C^\infty(G)^G, \quad C^\infty(\mathcal{G})^G, \quad C^\infty(\mathfrak{P})^G$$

give rise to two 'master integrable systems' on each double.

The group G acts on these phase spaces by 'diagonal conjugations', i.e., by the diffeomorphisms

$$A^i_\eta : (x,y) \mapsto (\eta x \eta^{-1}, \eta y \eta^{-1}), \quad \forall (x,y) \in \mathcal{M}_i \ (i = 1,2,3),\ \eta \in G.$$

The G-invariant functions form closed Poisson algebras, and thus the quotient space $\mathcal{M}_i^{\text{red}} \equiv \mathcal{M}_i/G$ becomes a (singular) Poisson space, which carries the corresponding reduced integrable systems.

Recall of degenerate integrability on symplectic and Poisson manifolds

Definition 1. Suppose that $\mathcal{M}$ is a **symplectic** manifold of dimension $2m$ with associated Poisson bracket $\{-,-\}$ and two distinguished subrings $\mathfrak{H}$ and $\mathfrak{F}$ of $C^\infty(\mathcal{M})$ satisfying the following conditions:

1. The ring $\mathfrak{H}$ has functional dimension r and $\mathfrak{F}$ has functional dimension s such that $r + s = \dim(\mathcal{M})$ and $r < m$.
2. Both $\mathfrak{H}$ and $\mathfrak{F}$ form Poisson subalgebras of $C^\infty(\mathcal{M})$, satisfying $\mathfrak{H} \subset \mathfrak{F}$ and $\{\mathcal{F}, \mathcal{H}\} = 0$ for all $\mathcal{F} \in \mathfrak{F}$, $\mathcal{H} \in \mathfrak{H}$.
3. The Hamiltonian vector fields of the elements of $\mathfrak{H}$ are complete.

Then, $(\mathcal{M}, \{-,-\}, \mathfrak{H}, \mathfrak{F})$ is called a **degenerate integrable system of rank** r. The rings $\mathfrak{H}$ and $\mathfrak{F}$ are referred to as the ring of Hamiltonians and constants of motion, respectively. (If $r = 1$, then this is the same as 'maximal superintegrability' of a single Hamiltonian.)

Definition 2. Consider a **Poisson** manifold $(\mathcal{M}, \{-,-\})$ whose Poisson tensor has maximal rank $2m \leq \dim(\mathcal{M})$ on a dense open subset. Then, $(\mathcal{M}, \{-,-\}, \mathfrak{H}, \mathfrak{F})$ is called a degenerate integrable system of rank r if conditions (1), (2), (3) of Definition 1 hold, and the Hamiltonian vector fields of the elements of $\mathfrak{H}$ span an r-dimensional subspace of the tangent space over a dense open subset of $\mathcal{M}$.

The example of the cotangent bundle: complement to lecture 1

The canonical Poisson bracket on the cotangent bundle

$$\mathcal{M} \equiv \mathcal{M}_1 = G \times \mathcal{G} = \{(g, J) \mid g \in G,\ J \in \mathcal{G}\} \text{ has the form}$$

$$\{\mathcal{F}, \mathcal{H}\}(g, J) = \langle \nabla_1 \mathcal{F}, d_2 \mathcal{H} \rangle - \langle \nabla_1 \mathcal{H}, d_2 \mathcal{F} \rangle + \langle J, [d_2 \mathcal{F}, d_2 \mathcal{H}] \rangle,$$

where the $\mathcal{G}$-valued derivatives are taken at (g, J). Here, $\langle X, Y \rangle$ is the Killing form on $\mathcal{G}$. The derivative $d_2 \mathcal{F} \in \mathcal{G}$ w.r.t. the second variable $J \in \mathcal{G}$ is the usual gradient, while the derivative $\nabla_1 \mathcal{F} \in \mathcal{G}$ w.r.t. first variable $g \in G$ is defined by

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}(e^{tX} g, J) =: \langle X, \nabla_1 \mathcal{F}(g, J) \rangle, \quad \forall X \in \mathcal{G}.$$

The equations of motion generated by the Hamiltonians $\mathcal{H}$ of the form $\mathcal{H}(g, J) = \varphi(J)$ with $\varphi \in C^\infty(\mathcal{G})^G$ read

$$\dot{g} = (d\varphi(J))g,\ \ \dot{J} = 0 \implies (g(t), J(t)) = (\exp(t d\varphi(J(0)))g(0), J(0)).$$

The constants of motions are arbitrary functions of J and $g^{-1} J g$.

We reduce by going to the orbit space of $\mathcal{M}$ w.r.t. the conjugation action of G.

We characterize the reduced system using a 'partial gauge fixing'. Define

$$\mathcal{M}^{\text{reg}} := \{(g, J) \in \mathcal{M} \mid g \in G^{\text{reg}}\}, \quad \mathcal{M}_0^{\text{reg}} := \{(Q, J) \in \mathcal{M} \mid Q \in G_0^{\text{reg}}\}.$$

Here, G^{reg} contains the group elements whose centralizer is a maximal torus, and G_0 is a fixed maximal torus. Let $\mathfrak{N}$ denote the normalizer of $G_0 < G$, which is the 'group of residual gauge transformations'.

Then, $\mathcal{M}^{\text{reg}}/G \equiv \mathcal{M}_0^{\text{reg}}/\mathfrak{N}$, and the restriction of functions yields the isomorphism

$$C^\infty(\mathcal{M}^{\text{reg}})^G \Longleftrightarrow C^\infty(\mathcal{M}_0^{\text{reg}})^{\mathfrak{N}},$$

By transferring the Poisson bracket from $C^\infty(\mathcal{M}^{\text{reg}})^G$ to $C^\infty(\mathcal{M}_0^{\text{reg}})^{\mathfrak{N}}$, we get

$$\{F, H\}_{\text{red}}(Q, J) = \langle \nabla_1 F, d_2 H\rangle - \langle \nabla_1 H, d_2 F\rangle + \langle J, [d_2 F, d_2 H]_{\mathcal{R}(Q)}\rangle,$$

with $[X, Y]_{\mathcal{R}} \equiv [\mathcal{R}X, Y] + [X, \mathcal{R}Y]$. The 'reduced evolution equations' generated by the invariant functions $\varphi \in C^\infty(\mathcal{G})^G$ can be written on $\mathcal{M}_0^{\text{reg}}$ as

$$\dot{Q} = (d\varphi(J))_0 Q, \qquad \dot{J} = [\mathcal{R}(Q)d\varphi(J), J].$$

Here, the subscript zero refers to the decomposition $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_\perp$, and $\mathcal{R}(Q) \in \text{End}(\mathcal{G})$ is the basic trigonometric solution of the modified classical dynamical Yang–Baxter equation. $\mathcal{R}(Q)$ vanishes on $\mathcal{G}_0$ and, writing $Q = \exp(\mathrm{i}q)$ with $\mathrm{i}q \in \mathcal{G}_0$, is given on $\mathcal{G}_\perp$ by $\mathcal{R}(Q) = \frac{1}{2}\coth(\frac{\mathrm{i}}{2}\text{ad}_q)$. ($\mathcal{G}_\perp$ is the orthogonal complement of $\mathcal{G}_0$.)

The (well known) spin Sutherland interpretation

Parametrize $J \in \mathcal{G}$ according to

$$J = -\mathrm{i}p + \sum_{\alpha\in\Delta_+}\left(\frac{\xi_\alpha}{e^{-\mathrm{i}\alpha(q)}-1}E_\alpha - \frac{\xi_\alpha^*}{e^{\mathrm{i}\alpha(q)}-1}E_{-\alpha}\right), \ p \in \mathrm{i}\mathcal{G}_0, \ ,$$

and take $\varphi(J) = -\frac{1}{2}\langle J, J\rangle$. Then (using $\langle E_\alpha, E_{-\alpha}\rangle = 2/|\alpha|^2$) we get

$$-\frac{1}{2}\langle J, J\rangle = \frac{1}{2}\langle p, p\rangle + \frac{1}{8}\sum_{\alpha\in\Delta}\frac{2}{|\alpha|^2}\frac{|\xi_\alpha|^2}{\sin^2(\alpha(q)/2)},$$

which is a standard spin Sutherland Hamiltonian $H_{\mathrm{spin-Suth}}(q,p,\xi)$. Here, we use the Killing form and the root space decomposition of the complexified Lie algebra $\mathcal{G}^{\mathbb{C}}$, with the set of roots $\Delta = \{\alpha\}$ and corresponding root vectors E_α.

The 'spin variable' $\xi = \sum_{\alpha\in\Delta_+}(\xi_\alpha E_\alpha - \xi_\alpha^* E_{-\alpha})$ matters up to conjugations by the maximal torus G_0. After dividing by G_0, there remains a residual gauge symmetry under the Weyl group $W = \mathfrak{N}/G_0$. The pertinent dense open subset of the reduced phase space can be identified as $\left(T^*G_0^{\mathrm{reg}} \times (\mathcal{G}^*//_0 G_0)\right)/W$, with Darboux variables (q,p) on $T^*G_0^{\mathrm{reg}} \simeq G_0^{\mathrm{reg}} \times \mathcal{G}_0$ and spin variable $[\xi] \in \mathcal{G}^*//_0 G_0$.

Plan of the rest of the lecture

- Integrable ‘master system’ on the Heisenberg double
- Poisson reduction of the master system: reduced integrability
- Two descriptions of the reduced Poisson brackets
- Connection to the spin Sutherland models
- The dual system in a nutshell
- Conclusion

Preparations. Fix a maximal Abelian subalgebra, $\mathcal{G}_0 < \mathcal{G}$. A choice of positive roots with respect to the Cartan subalgebra $\mathcal{G}_0^{\mathbb{C}} < \mathcal{G}^{\mathbb{C}}$ leads to the triangular decomposition

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}_-^{\mathbb{C}} + \mathcal{G}_0^{\mathbb{C}} + \mathcal{G}_+^{\mathbb{C}}.$$

Equip the realification $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ of $\mathcal{G}^{\mathbb{C}}$ with bilinear form $\langle X, Y\rangle_{\mathbb{I}} := \Im\langle X, Y\rangle$, where $\langle -, -\rangle$ is the Killing form of $\mathcal{G}^{\mathbb{C}}$. Then one obtains the decomposition (a Manin triple)

$$\mathcal{G}_{\mathbb{R}}^{\mathbb{C}} = \mathcal{G} + \mathcal{B} \quad \text{with} \quad \mathcal{B} := \mathrm{i}\mathcal{G}_0 + \mathcal{G}_+^{\mathbb{C}} =: \mathcal{B}_0 + \mathcal{B}_+.$$

Let $G_{\mathbb{R}}^{\mathbb{C}}$ a connected and simply connected Lie group with Lie algebra $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$. We may write any $X \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ as $X = X_{\mathcal{G}} + X_{\mathcal{B}}$ or as $X = X_+ + X_0 + X_-$ or as $X = Y_1 + \mathrm{i}Y_2$ $(Y_1, Y_2 \in \mathcal{G})$. The complex conjugation θ with respect to $\mathcal{G}$ is a Cartan involution and it lifts to the involution Θ of $G_{\mathbb{R}}^{\mathbb{C}}$. We have the anti-automorphisms

$$Z \mapsto Z^\dagger := -\theta(Z), \qquad K \mapsto K^\dagger := \Theta(K^{-1}), \qquad \forall Z \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}},\ \forall K \in G_{\mathbb{R}}^{\mathbb{C}}.$$

By using the subgroups $G < G_{\mathbb{R}}^{\mathbb{C}}$ and $B := \exp(\mathcal{B}) < G_{\mathbb{R}}^{\mathbb{C}}$, every element $K \in G_{\mathbb{R}}^{\mathbb{C}}$ admits the unique (Iwasawa) decompositions:

$$K = g_L b_R^{-1} = b_L g_R^{-1} \quad \text{with} \quad g_L, g_R \in G,\ b_L, b_R \in B,$$

which yield the 'Iwasawa maps' $\Xi_L, \Xi_R : G_{\mathbb{R}}^{\mathbb{C}} \to G$ and $\Lambda_L, \Lambda_R : G_{\mathbb{R}}^{\mathbb{C}} \to B$,

$$\Xi_L(K) := g_L, \quad \Xi_R(K) := g_R, \quad \Lambda_L(K) := b_L, \quad \Lambda_R(K) := b_R.$$

We have the diffeomorphic manifolds $M := G_{\mathbb{R}}^{\mathbb{C}}$, $\mathfrak{M} := G \times B$ and $\mathbb{M} := G \times \mathfrak{P}$. Shall use the diffeomorphisms $m_1 := (\Xi_R, \Lambda_R) : M \to \mathfrak{M}$, that is, $m_1(K) = (g_R, b_R)$, and $m_2 : \mathfrak{M} \to \mathbb{M}$, $m_2(g, b) := (g, bb^\dagger)$.

The map $\nu : B \ni b \mapsto bb^\dagger \in \mathfrak{P} = \exp(\mathrm{i}\mathcal{G}) \subset G_{\mathbb{R}}^{\mathbb{C}}$ is a G-equivariant diffeomorphism if G acts on $\mathfrak{P}$ by conjugations and on B by 'dressing': $\mathrm{Dress}_\eta(b) := \Lambda_L(\eta b)$, $\forall \eta \in G$, $b \in B$.

The group manifold $M = G_{\mathbb{R}}^{\mathbb{C}}$ carries the following two Poisson brackets:

$$\{\Phi_1, \Phi_2\}_\pm := \langle \nabla\Phi_1, \rho\nabla\Phi_2\rangle_{\mathbb{I}} \pm \langle \nabla'\Phi_1, \rho\nabla'\Phi_2\rangle_{\mathbb{I}}, \quad \forall \Phi_1, \Phi_2 \in C^\infty(M).$$

Here, $\rho := \frac{1}{2}(\pi_{\mathcal{G}} - \pi_{\mathcal{B}})$ with $\pi_{\mathcal{G}}$ and $\pi_{\mathcal{B}}$ denoting the projections from $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ onto $\mathcal{G}$ and $\mathcal{B}$, which correspond to the direct sum $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}} = \mathcal{G} + \mathcal{B}$. For any real function $\Phi \in C^\infty(M)$, the $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$-valued 'left- and right-derivatives' are defined by

$$\langle X, \nabla\Phi(K)\rangle_{\mathbb{I}} + \langle X', \nabla'\Phi(K)\rangle_{\mathbb{I}} := \frac{d}{dt}\bigg|_{t=0} \Phi(e^{tX} K e^{tX'}), \quad \forall K \in M,\ X, X' \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}.$$

The minus bracket makes M into a Poisson–Lie group, of which G and B are Poisson–Lie subgroups, i.e., (embedded) Lie subgroups and Poisson submanifolds. Their inherited Poisson brackets take the form

$$\{\chi_1, \chi_2\}_G(g) = -\langle D'\chi_1(g), g^{-1}(D\chi_2(g))g\rangle_{\mathbb{I}},$$

$$\{\varphi_1, \varphi_2\}_B(b) = \langle D'\varphi_1(b), b^{-1}(D\varphi_2(b))b\rangle_{\mathbb{I}}.$$

The derivatives are $\mathcal{B}$-valued for $\chi_i \in C^\infty(G)$ and $\mathcal{G}$-valued for $\varphi_i \in C^\infty(B)$. The Poisson manifolds $(M, \{-,-\}_-)$ and $(M, \{-,-\}_+)$ are known, respectively, as the Drinfeld double and the Heisenberg double associated with the standard Poisson structures of G and B. The Poisson bracket $\{-,-\}_+$ is non-degenerate, its symplectic form reads

$$\Omega_+ = \frac{1}{2}\left\langle db_L b_L^{-1} \overset{\wedge}{,} dg_L g_L^{-1}\right\rangle_{\mathbb{I}} + \frac{1}{2}\left\langle db_R b_R^{-1} \overset{\wedge}{,} dg_R g_R^{-1}\right\rangle_{\mathbb{I}}.$$

The maps

$$(\Lambda_L, \Lambda_R) : M \to B \times B \quad \text{and} \quad (\Xi_L, \Xi_R) : M \to G \times G$$

are Poisson maps with respect to $(M, \{-,-\}_+)$ and the direct product Poisson structures on the targets obtained from $(B, \{-,-\}_B)$ and from $(G, \{-,-\}_G)$, respectively. (References: Semenov-Tian-Shansky [1985] and Alekseev–Malkin [1994]).

'Master system' on $\mathbb{M}$. For any real function $\phi \in C^\infty(\mathfrak{P})$, define its $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$-*valued derivative* $\mathcal{D}\phi$ as follows:

$$\langle X, \mathcal{D}\phi(L)\rangle_{\mathbb{I}} := \frac{d}{dt}\bigg|_{t=0} \phi(e^{tX} L e^{tX^\dagger}),\ \langle Y, \mathcal{D}\phi(L)\rangle_{\mathbb{I}} := \frac{d}{dt}\bigg|_{t=0} \phi(e^{tY} L e^{-tY}),\ \forall X \in \mathcal{B},\ Y \in \mathcal{G}.$$

Using the diffeomorphism $m := m_2 \circ m_1 : M \to \mathbb{M} = G \times \mathfrak{P}$, we transfer the Heisenberg double Poisson bracket to $\mathbb{M} = G \times \mathfrak{P}$. This gives

$$\{\mathcal{F}, \mathcal{H}\}_{\mathbb{M}}(g, L) = \langle \mathcal{D}_2\mathcal{F}, (\mathcal{D}_2\mathcal{H})_{\mathcal{G}}\rangle_{\mathbb{I}} - \left\langle g\mathcal{D}_1'\mathcal{F}g^{-1}, \mathcal{D}_1\mathcal{H}\right\rangle_{\mathbb{I}} + \langle \mathcal{D}_1\mathcal{F}, \mathcal{D}_2\mathcal{H}\rangle_{\mathbb{I}} - \langle \mathcal{D}_1\mathcal{H}, \mathcal{D}_2\mathcal{F}\rangle_{\mathbb{I}},$$

where the derivatives of $\mathcal{F}, \mathcal{H} \in C^\infty(\mathbb{M})$ are evaluated at $(g, L) \in \mathbb{M}$; and $\mathcal{D}_1\mathcal{F} \in \mathcal{B}$.

Define the map $\Psi : \mathbb{M} \to \mathfrak{P} \times \mathfrak{P}$ by $\Psi(g, L) := (g^{-1}Lg, L)$, which in terms of the model M reads $(\nu \circ (\Lambda_L)^{-1}, \nu \circ \Lambda_R)$; remember $\nu(b) = bb^\dagger$.

Proposition 1. *The two subrings of $C^\infty(\mathbb{M})$ defined by*

$$\mathfrak{H} := \pi_2^*\left(C^\infty(\mathfrak{P})^G\right) \quad \textit{and} \quad \mathfrak{F} := \Psi^*\left(C^\infty(\mathfrak{P} \times \mathfrak{P})\right)$$

engender a degenerate integrable system on the symplectic manifold $(\mathbb{M}, \{-,-\}_{\mathbb{M}})$. The rank of this integrable system is equal to the rank $r = \dim(\mathcal{G}_0)$ of Lie algebra $\mathcal{G}$.

Proof. One calculates that any Hamiltonian $\mathcal{H} = \pi_2^*(\phi)$ with a function $\phi \in C^\infty(\mathfrak{P})^G$ has the integral curves

$$(g(t), L(t)) = (\exp\left(t\mathcal{D}\phi(L(0))\right) g(0), L(0)).$$

Since the derivative $\mathcal{D}\phi : \mathfrak{P} \to \mathcal{G}$ is G-equivariant, Ψ **is constant along these curves, and it is a Poisson map** for the ν-transferred Poisson bracket on $\mathfrak{P}_- \times \mathfrak{P}$. One can verify that the derivative $D\Psi$ has constant rank, equal to $\dim(\mathbb{M}) - r$, at every point of $G \times \mathfrak{P}^{\text{reg}}$. This implies that $\mathfrak{F}$ has functional dimension $\dim(\mathbb{M}) - r$. It is obvious that $\mathfrak{H} \subset \mathfrak{F}$, and its functional dimension is r, which completes the proof.

Reduction. Define the 'conjugation action' $A : G \times \mathbb{M}$ by $A_\eta(g, L) := (\eta g \eta^{-1}, \eta L \eta^{-1})$. All $\mathcal{H} \in \mathfrak{H}$ and their Hamiltonian vector fields are G-invariant, and the invariant functions, $C^\infty(\mathbb{M})^G$, form a Poisson subalgebra. Therefore, we may take the Poisson quotient

$$\mathbb{M}^{\text{red}} := \mathbb{M}/G, \quad C^\infty(\mathbb{M}^{\text{red}}) := C^\infty(\mathbb{M})^G.$$

$$\text{We have} \qquad \mathfrak{H} \subset \mathfrak{F}^G := \Psi^*(C^\infty(\mathfrak{P}_- \times \mathfrak{P})^G) \subset C^\infty(\mathbb{M})^G.$$

For $\mathbb{M}^{\text{red}}$ is a not a smooth manifold, we restrict to its dense open subset $\mathbb{M}^{\text{red}}_* = \mathbb{M}_*/G$, where $\mathbb{M}_* \subset \mathbb{M}$ is the submanifold of principal orbit type:

$\mathbb{M}_* := \{(g, L) \in \mathbb{M} \mid G_{(g,L)} = Z(G)\}$. Note: $\mathbb{M}_*$ is stable w.r.t. the flows of $C^\infty(\mathbb{M})^G$.

The' space of constants of motion' $\mathfrak{C} := \Psi(\mathbb{M}) \subset \mathfrak{P} \times \mathfrak{P}$ is also not a smooth manifold, but $\mathfrak{C}_{\text{reg}} := \{\tilde{L}, L) \in \mathfrak{C} \mid L \in \mathfrak{P}^{\text{reg}}$ is a smooth, embedded submanifold of $\mathfrak{P}^{\text{reg}} \times \mathfrak{P}^{\text{reg}}$. Here, $\mathfrak{P}^{\text{reg}}$ consists of the points of $\mathfrak{P}$ whose isotropy group in G is a maximal torus.

A key technical point is to consider

$$\mathfrak{C}_* := \{(\tilde{L}, L) \in \mathfrak{C}_{\text{reg}} \mid G_{(\tilde{L},L)} = Z(G)\} \quad \text{and} \quad \mathbb{M}_{**} := \Psi^{-1}(\mathfrak{C}_*).$$

The restriction of Ψ yields the G-equivariant submersion $\psi : \mathbb{M}_{**} \to \mathfrak{C}_*$, and we get the diagram of **smooth** Poisson submersions (where $\mathbb{M}^{\text{red}}_{**} = \mathbb{M}_{**}/G$ and $\mathfrak{C}^{\text{red}}_* = \mathfrak{C}_*/G$):

$$\begin{tikzcd}
\mathbb{M}_{**} \arrow[r, "\psi"] \arrow[d, "p_1"'] & \mathfrak{C}_* \arrow[d, "p_2"] \\
\mathbb{M}^{\text{red}}_{**} \arrow[r, "\psi_{\text{red}}"] & \mathfrak{C}^{\text{red}}_*
\end{tikzcd}$$

The rings $\mathfrak{H}$ and $\mathfrak{F}^G$ yield the subrings $\mathfrak{H}_{\text{red}}$ and $\mathfrak{F}_{\text{red}}$ of $C^\infty(\mathbb{M}^{\text{red}})$, and we denote their restrictions on $\mathbb{M}^{\text{red}}_*$ and $\mathbb{M}^{\text{red}}_{**}$ by $\mathfrak{H}^*_{\text{red}}$, $\mathfrak{H}^{**}_{\text{red}}$ and $\mathfrak{F}^*_{\text{red}}$, $\mathfrak{F}^{**}_{\text{red}}$, respectively. Moreover, we define the restricted reduced Poisson manifold by

$$(C^\infty(\mathbb{M}^{\text{red}}_{**}), \{-,-\}^{\text{red}}_{**}) \simeq (C^\infty(\mathbb{M}_{**})^G, \{-,-\}_{\mathbb{M}_{**}}).$$

Theorem 2. *Suppose that $r := \dim(\mathcal{G}_0) \neq 1$. Then, the 'restricted reduced system' $(C^\infty(\mathbb{M}^{\text{red}}_{**}), \{-,-\}^{\text{red}}_{**}), \mathfrak{H}^{**}_{\text{red}})$ is a degenerate integrable system of rank r with constants of motion provided by the ring of functions*

$$\mathfrak{F}^\sharp_{\text{red}} := \psi^*_{\text{red}}\left(C^\infty(\mathfrak{C}^{\text{red}}_*)\right).$$

That is, the quadruple $(\mathbb{M}^{\text{red}}_{**}, \{-,-\}^{\text{red}}_{**}, \mathfrak{H}^{**}_{\text{red}}, \mathfrak{F}^\sharp_{\text{red}})$ satisfies Definition 2, with the co-dimension of the generic symplectic leaves being equal to r. The reduced Hamiltonian vector fields associated with $\mathfrak{H}^{**}_{\text{red}}$ span an r-dimensional subspace of the tangent space at every point of $\mathbb{M}^{\text{red}}_{**}$, and the differentials of the elements of $\mathfrak{F}^\sharp_{\text{red}}$ span a co-dimension r subspace of the cotangent space.

The symplectic leaves in $\mathbb{M}^{\text{red}}_*$ as well as in $\mathbb{M}^{\text{red}}_{**}$ are (the connected components of) the joint level surfaces of the Casimir functions, which are obtained from

$$\Lambda^*(C^\infty(B)^G) \quad \text{with the Poisson–Lie moment map} \quad \Lambda : \mathbb{M} \to B.$$

The map Λ is defined by transferring to $\mathbb{M}$ the moment map $\Lambda := \Lambda_L \Lambda_R : G^{\mathbb{C}}_{\mathbb{R}} \to B$. The conjugation action of G is orbit-equivalent to the Poisson–Lie action generated by the moment map.

Corollary 3. *The restriction of the system $(\mathbb{M}^{\text{red}}_{**}, \{-,-\}^{\text{red}}_{**}, \mathfrak{H}^{**}_{\text{red}}, \mathfrak{F}^\sharp_{\text{red}})$ of Theorem 2 to any symplectic leaf of $\mathbb{M}^{\text{red}}_{**}$ of co-dimension r is a degenerate integrable system in the sense of Definition 1.*

Remark: The $r = 1$ case arises for $G = \mathrm{SU}(2)$, and in this case we obtain 'only' Liouville integrability.

The integrability statement can be extended to $\mathbb{M}_*^{\mathrm{red}}$ by using that at each $y \in \mathbb{M}_{**}^{\mathrm{red}}$ the differentials of the elements of $\mathfrak{F}_{\mathrm{red}}^{**} \subset \mathfrak{F}_{\mathrm{red}}^{\sharp}$ span the same subspace of $T_y\mathbb{M}_{**}^{\mathrm{red}}$ as do the differentials of the elements of $\mathfrak{F}_{\mathrm{red}}^{\sharp}$. This can be shown utilizing the fact that for any smooth action of a compact Lie group on a connected manifold the dimension of the differentials of the smooth invariant functions at a point of principal orbit type is equal to the co-dimension of the principal orbits. (We apply this to $C^\infty(\mathfrak{P}_- \times \mathfrak{P})^G$ and use pull-back.) The point is that the elements of $\mathfrak{F}_{\mathrm{red}}$ belong to $C^\infty(\mathbb{M}^{\mathrm{red}})$ and their restrictions give smooth function on $\mathbb{M}_*^{\mathrm{red}}$.

Theorem 4. *Suppose that $r = \mathrm{rank}(G) > 1$ and consider the restriction of the master system of free motion on the dense, open submanifold $\mathbb{M}_* \subset \mathbb{M}$ of principal orbit type with respect to the G-action. Then, this system descends to the degenerate integrable system $(\mathbb{M}_*^{\mathrm{red}}, \{-,-\}_*^{\mathrm{red}}, \mathfrak{H}_{\mathrm{red}}^*, \mathfrak{F}_{\mathrm{red}}^*)$ on the Poisson manifold $\mathbb{M}_*^{\mathrm{red}} = \mathbb{M}_*/G$, where the Poisson subalgebras $\mathfrak{H}_{\mathrm{red}}^*$ and $\mathfrak{F}_{\mathrm{red}}^*$ of $C^\infty(\mathbb{M}_*^{\mathrm{red}}) = C^\infty(\mathbb{M}_*)^G$ arise from the restrictions of $\mathfrak{H}$ and $\mathfrak{F}_{\mathrm{red}} \simeq \Psi^*(C^\infty(\mathfrak{P}_- \times \mathfrak{P})^G)$ on $\mathbb{M}_* \subset \mathbb{M}$, respectively.*

Dynamical r-matrix formula for reduced Poisson brackets. Restrict to the dense, open, G-invariant submanifold $\pi_1^{-1}(G^{\rm reg}) = G^{\rm reg} \times \mathfrak{P} \subset \mathbb{M}$. Every G-orbit in this submanifold intersects $\mathbb{M}_0 := \{(Q,L) \in \mathbb{M} \mid Q \in G_0^{\rm reg}\}$. The intersection happens in orbits of the normalizer $\mathfrak{N} := N_G(G_0)$, and we obtain the identifications

$$(G^{\rm reg} \times \mathfrak{P})/G \simeq \mathbb{M}_0/\mathfrak{N} \quad \text{and} \quad C^\infty(G^{\rm reg} \times \mathfrak{P})^G \Longleftrightarrow C^\infty(\mathbb{M}_0)^{\mathfrak{N}}.$$

Let $\bar{\mathcal{F}}, \bar{\mathcal{H}} \in C^\infty(\mathbb{M}_0)^{\mathfrak{N}}$ be the restrictions of $\mathcal{F}, \mathcal{H} \in C^\infty(G^{\rm reg} \times \mathfrak{P})^G$. Then, we define their 'reduced Poisson bracket' by

$$\{\bar{\mathcal{F}}, \bar{\mathcal{H}}\}^{\rm red}_{\mathbb{M}_0}(Q,L) := \{\mathcal{F}, \mathcal{H}\}_{\mathbb{M}}(Q,L), \qquad \forall (Q,L) \in \mathbb{M}_0.$$

Its explicit form contains the dynamical r-matrix $\mathcal{R}(Q) \in \mathrm{End}(\mathcal{G}^{\mathbb{C}}_{\mathbb{R}})$:

$$\mathcal{R}(Q)(X) := \frac{1}{2}(\mathrm{Ad}_Q + \mathrm{id}) \circ (\mathrm{Ad}_Q - \mathrm{id})^{-1}_{|\mathcal{G}^{\mathbb{C}}_\perp}(X_\perp), \qquad \forall Q \in G_0^{\rm reg},\ X = (X_0 + X_\perp) \in \mathcal{G}^{\mathbb{C}}_{\mathbb{R}},$$

where $X_0 \in \mathcal{G}_0^{\mathbb{C}}$ and $X_\perp \in \mathcal{G}_\perp^{\mathbb{C}}$, in correspondence with $\mathcal{G}^{\mathbb{C}} = \mathcal{G}_0^{\mathbb{C}} + \mathcal{G}_\perp^{\mathbb{C}}$.

Theorem 5. *For $\bar{\mathcal{F}}, \bar{\mathcal{H}} \in C^\infty(\mathbb{M}_0)^{\mathfrak{N}}$, the definition implies the formula*

$$\{\bar{\mathcal{F}}, \bar{\mathcal{H}}\}^{\rm red}_{\mathbb{M}_0}(Q,L) = \langle \mathcal{D}_1\bar{\mathcal{F}}, \mathcal{D}_2\bar{\mathcal{H}}\rangle_{\mathbb{I}} - \langle \mathcal{D}_1\bar{\mathcal{H}}, \mathcal{D}_2\bar{\mathcal{F}}\rangle_{\mathbb{I}} + \langle \mathcal{R}(Q)\mathcal{D}_2\bar{\mathcal{H}}, \mathcal{D}_2\bar{\mathcal{F}}\rangle_{\mathbb{I}},$$

where the derivatives $\mathcal{D}_1\bar{\mathcal{F}} \in \mathcal{B}_0$ and $\mathcal{D}_2\bar{\mathcal{F}} \in \mathcal{G}^{\mathbb{C}}_{\mathbb{R}}$ are taken at (Q,L). The Hamiltonian $\bar{\mathcal{H}}(Q,L) = \phi(L)$ with $\phi \in C^\infty(\mathfrak{P})^G$ induces the evolution equations

$$\dot{Q} = (\mathcal{D}\phi(L))_0 Q, \quad \dot{L} = [\mathcal{R}(Q)\mathcal{D}\phi(L), L] \qquad \textit{(up to residual gauge transformations).}$$

The formula defines a Poisson algebra structure on $C^\infty(\mathbb{M}_0)^{G_0}$ as well. For some purposes, it is advantageous to use, instead of $\mathbb{M}_0 = G_0^{\rm reg} \times \mathfrak{P}$, the equivalent model $\mathfrak{M}_0 := G_0^{\rm reg} \times B$. Then, the reduced Poisson bracket becomes

$$\{\bar{f}, \bar{h}\}^{\rm red}_{\mathfrak{M}_0}(Q,b) = \langle D_1\bar{f}, D_2\bar{h}\rangle_{\mathbb{I}} - \langle D_1\bar{h}, D_2\bar{f}\rangle_{\mathbb{I}}, + \langle \mathcal{R}(Q)(bD_2'\bar{h}b^{-1}), bD_2'\bar{f}b^{-1}\rangle_{\mathbb{I}}.$$

Here, the derivatives are evaluated at (Q,b), with $D_1\bar{f} \in \mathcal{B}_0$ and $D_2\bar{f}, D_2'\bar{f} \in \mathcal{G}$.

Canonically conjugate pairs and 'spin' variables. Let B_0 and B_+ be the subgroups of B associated with the subalgebras in $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_+$. Any $b \in B$ is uniquely decomposed as $b = e^p b_+$ with $p \in \mathcal{B}_0$, $b_+ \in B_+$. Then, we introduce new variables by means of the map

$$\zeta : \mathfrak{M}_0 = G_0^{\text{reg}} \times B \to G_0^{\text{reg}} \times \mathcal{B}_0 \times B_+$$

$$\zeta : (Q, e^p b_+) \mapsto (Q, p, \lambda) \quad \text{with} \quad \lambda := b_+^{-1} Q^{-1} b_+ Q.$$

The map ζ is a diffeomorphism. It is equivariant with respect to the G_0-actions for which $\eta_0 \in G_0$ sends (Q, b) to $(Q, \eta_0 b \eta_0^{-1})$ and (Q, p, λ) to $(Q, p, \eta_0 \lambda \eta_0^{-1})$. Consequently, ζ induces an isomorphism: $C^\infty(\mathfrak{M}_0)^{G_0} \Longleftrightarrow C^\infty(G_0^{\text{reg}} \times \mathcal{B}_0 \times B_+)^{G_0}$.

Any two functions $F, H \in C^\infty(G_0^{\text{reg}} \times \mathcal{B}_0 \times B_+)^{G_0}$ are related to unique $\bar{f}, \bar{h} \in C^\infty(\mathfrak{M}_0)^{G_0}$ by $F \circ \zeta = \bar{f}$, $H \circ \zeta = \bar{h}$. Thus, we can define $\{F, H\}_0^{\text{red}} \in C^\infty(G_0^{\text{reg}} \times \mathcal{B}_0 \times B_+)^{G_0}$ by

$$\{F, H\}_0^{\text{red}} \circ \zeta := \{\bar{f}, \bar{h}\}_{\mathfrak{M}_0}^{\text{red}}.$$

Theorem 6. *In terms of the new variables introduced via the map ζ, the reduced Poisson bracket acquires the following 'decoupled form':*

$$\{F, H\}_0^{\text{red}}(Q, p, \lambda) = \langle D_Q F, d_p H \rangle_{\mathbb{I}} - \langle D_Q H, d_p F \rangle_{\mathbb{I}} + \langle \lambda D'_\lambda F \lambda^{-1}, D_\lambda H \rangle_{\mathbb{I}},$$

where the derivatives of $F, H \in C^\infty(G_0^{\text{reg}} \times \mathcal{B}_0 \times B_+)^{G_0}$ are taken at (Q, p, λ).

Using the identification $(\mathcal{B}_+)^* \simeq \mathcal{G}_\perp$, the derivatives $D_\lambda F, D'_\lambda F \in \mathcal{G}_\perp$ are defined by

$$\langle X_+, D_\lambda F(Q, p, \lambda) \rangle_{\mathbb{I}} + \langle X'_+, D'_\lambda F(Q, p, \lambda) \rangle_{\mathbb{I}} = \left. \frac{d}{dt} \right|_{t=0} F(Q, p, e^{tX_+} \lambda e^{tX'_+}), \ \forall X_+, X'_+ \in \mathcal{B}_+.$$

Comparison with the reduction of T^*G. The 'linear analogue' of the Poisson algebra $(C^\infty(G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+)^{G_0}, \{-,-\}_0^{\mathrm{red}})$,

$$\{F,H\}_0^{\mathrm{red}}(Q,p,\lambda) = \langle D_Q F, d_p H\rangle_{\mathbb{I}} - \langle D_Q H, d_p F\rangle_{\mathbb{I}} + \langle \lambda D'_\lambda F \lambda^{-1}, D_\lambda H\rangle_{\mathbb{I}},$$

is given by $(C^\infty(G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times \mathcal{B}_+)^{G_0}, \{-,-\}_{\mathrm{lin}})$ with

$$\{f,h\}_{\mathrm{lin}}(Q,p,X) := \langle D_Q f, d_p h\rangle_{\mathbb{I}} - \langle D_Q h, d_p f\rangle_{\mathbb{I}} + \langle X, [d_X f, d_X h]\rangle_{\mathbb{I}},$$

where the derivatives are taken at (Q,p,X), and $d_X f \in \mathcal{G}_\perp \simeq (\mathcal{B}_+)^*$ denotes the differential of f with respect to its third variable. An interpretation of these brackets comes by observing that $B \simeq G^*$ and $\mathcal{B} \simeq \mathcal{G}^*$, and the reductions of $(B, \{-,-\}_B)$ and $(\mathcal{G}^*, \{-,-\}_{\mathcal{G}^*})$ with respect to the Hamiltonian actions of G_0, at the zero value of the $\mathcal{G}_0^*$-valued moment map, give precisely the third term of the respective Poisson brackets, i.e., they represent $G^*//G_0$ and $\mathcal{G}^*//G_0$, respectively. [Beware, previously (on page 7) we used the model $\mathcal{G}^* \simeq \mathcal{G}$. Thus, $\xi \in \mathcal{G}_\perp$ used before is now replaced by $X \in \mathcal{B}_+$.]

The Poisson algebra $(C^\infty(G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times \mathcal{B}_+)^{G_0}, \{-,-\}_{\mathrm{lin}})$ arises from the Poisson reduction of the cotangent bundle T^*G by the obvious conjugation action, whereby the kinetic energy of the bi-invariant Riemannian metric of G reduces to the **spin Sutherland Hamiltonian**:

$$H_{\mathrm{spin-Suth}}(e^{\mathrm{i}q}, p, X) = \frac{1}{2}\langle p,p\rangle + \frac{1}{8}\sum_{\alpha\in\mathfrak{R}^+} \frac{1}{|\alpha|^2}\frac{|X_\alpha|^2}{\sin^2(\alpha(q)/2)} \quad \text{with} \quad X = \sum_{\alpha\in\mathfrak{R}^+} X_\alpha E_\alpha \in \mathcal{B}_+.$$

Proposition 7. *For any real $\epsilon > 0$, let us define the G_0-equivariant diffeomorphism*

$$\mu_\epsilon : G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times \mathcal{B}_+ \to G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+, \quad \mu_\epsilon : (Q,p,X) \mapsto (Q, \epsilon p, \exp(\epsilon X)).$$

Then, $\{-,-\}_{\mathrm{lin}}$ is the 'scaling limit' of $\{-,-\}_0^{\mathrm{red}}$ according to the formula

$$\{f,h\}_{\mathrm{lin}} = \lim_{\epsilon\to 0} \epsilon \{f\circ\mu_\epsilon^{-1}, h\circ\mu_\epsilon^{-1}\}_0^{\mathrm{red}} \circ \mu_\epsilon.$$

Interpretation as spin RS model: Consider the new variable $\lambda = b_+^{-1} Q^{-1} b_+ Q$ using

$$\lambda = e^{\sigma}, \quad b_+ = e^{\beta}, \quad \sigma = \sum_{\alpha>0} \sigma_\alpha E_\alpha, \quad \beta = \sum_{\alpha>0} \beta_\alpha E_\alpha, \quad Q = e^{\mathrm{i}q}.$$

The Baker-Campbell-Hausdorff formula gives

$$\exp(-\beta + Q^{-1}\beta Q + \frac{1}{2}[Q^{-1}\beta Q, \beta] + \cdots) = \exp(\sigma).$$

As a consequence, β_α can be expressed in terms of σ and $e^{\mathrm{i}q}$:

$$\beta_\alpha = \frac{\sigma_\alpha}{e^{-\mathrm{i}\alpha(q)} - 1} + \sum_{k\geq 2} \sum_{\varphi_1,\dots,\varphi_k} f_{\varphi_1,\dots,\varphi_k}(e^{\mathrm{i}q}) \sigma_{\varphi_1} \dots \sigma_{\varphi_k},$$

where $\alpha = \varphi_1 + \cdots + \varphi_k$ and $f_{\varphi_1,\dots,\varphi_k}$ depends rationally on $e^{\mathrm{i}q}$. This gives a construction of the inverse of the map $\zeta : (Q, e^p b_+) \to (Q, p, \lambda)$.

Take any finite dimensional irreducible representation $\rho : G^{\mathbb{C}} \to \mathrm{SL}(V)$. Introduce an inner product on V so that the dagger, $K^\dagger = \Theta(K^{-1})$, becomes the usual adjoint. Then, the (normalized) character $\phi^\rho(L) = \mathrm{tr}_\rho(L) := c_\rho \mathrm{tr}\rho(L)$ gives an element of $C^\infty(\mathfrak{P})^G$. (Here, c_ρ is a constant, so that $\mathrm{tr}_\rho(XY) := c_\rho \mathrm{tr}(\rho(X)\rho(Y)) = \langle X, Y \rangle$, $\forall X, Y \in \mathcal{G}^{\mathbb{C}}$.)

Using the 'decoupled variables' (Q, p, σ), $H^\rho := \mathrm{tr}_\rho(e^p b_+ b_+^\dagger e^p)$ can be expanded as

$$H^\rho(e^{\mathrm{i}q}, p, \sigma) = c_\rho \mathrm{tr} \left(e^{2p} \left(\mathbf{1}_\rho + \frac{1}{4} \sum_{\alpha>0} \frac{|\sigma_\alpha|^2 \rho(E_\alpha)\rho(E_{-\alpha})}{\sin^2(\alpha(q)/2)} + \mathrm{o}_2(\sigma, \sigma^*) \right) \right).$$

I call this a spin Ruijsenaars–Schneider (RS) type Hamiltonian.

By expanding e^{2p},

$$H^\rho(e^{\mathrm{i}q},p,\sigma)=c_\rho\mathrm{dim}_\rho+2\mathrm{tr}_\rho(p^2)+\frac{1}{2}\sum_{\alpha>0}\frac{1}{|\alpha|^2}\frac{|\sigma_\alpha|^2}{\sin^2(\alpha(q)/2)}+\mathrm{o}_2(\sigma,\sigma^*,p).$$

Leading term of $\frac{1}{4}(H^\rho-c_\rho\mathrm{dim}_\rho)$ matches the Hamiltonian $H_{\mathrm{spin-Suth}}(e^{\mathrm{i}q},p,X)$. In other words, with the 'scaling map' μ_ϵ, we have

$$H_{\mathrm{spin-Suth}}=\lim_{\epsilon\to 0}\frac{1}{4\epsilon^2}(H^\rho\circ\mu_\epsilon-c_\rho\mathrm{dim}_\rho).$$

The Poisson brackets of the functions of the 'spin variables' X and σ follow from

$$\{X^i,X^j\}_{\mathcal{G}^*}(X)=\langle[Y^i,Y^j],X\rangle_{\mathbb{I}},\quad \{\sigma^i,\sigma^j\}_B(e^\sigma)=\langle[Y^i,Y^j],\sigma\rangle_{\mathbb{I}}+\mathrm{o}(\sigma),$$

where $X^i=\langle X,Y^i\rangle_{\mathbb{I}}$ for a basis $\{Y^i\}$ of $\mathcal{G}_\perp\subset\mathcal{G}=\mathcal{G}_0+\mathcal{G}_\perp$, and similarly for σ. Proposition 7 is a consequence of the latter expansion.

The elements of $C^\infty(\mathfrak{P})^G$ yield G-invariant functions of 'Lax matrix' $L(e^{\mathrm{i}q},p,\sigma):=e^pb_+b_+^\dagger e^p$, where $b_+=b_+(e^{\mathrm{i}q},\sigma)$ expresses the inverse of our map ζ. In any representation,

$$L(e^{\mathrm{i}q},p,\sigma)=\mathbf{1}+2p+\sum_{\alpha>0}\left(\frac{\sigma_\alpha}{e^{-\mathrm{i}\alpha(q)}-1}E_\alpha+\frac{\sigma_\alpha^*}{e^{\mathrm{i}\alpha(q)}-1}E_{-\alpha}\right)+\mathrm{o}(\sigma,\sigma^*,p).$$

This matches the standard, $\mathcal{G}$-valued, spin Sutherland Lax matrix.

In conclusion, our models are 'deformations' of the spin Sutherland models, which can be recovered in the 'scaling limit'.

Explicit formulas for $G^{\mathbb{C}} = \mathrm{SL}(n,\mathbb{C})$: Now parametrize $b_+ \in B$ by its matrix elements. We have $b = e^p b_+$, and can find b_+ from the relation

$$Q^{-1} b_+ Q = b_+ \lambda,$$

where $Q = \mathrm{diag}(Q_1, \ldots, Q_n) \in G_0^{\mathrm{reg}}$, $\lambda \in B_+$ is the constrained 'spin' variable and b_+ is an upper triangular matrix with unit diagonal.

Introducing $\mathcal{I}_{a,a+j} := \frac{1}{Q_{a+j}Q_a^{-1} - 1}$, we have $(b_+)_{a,a+1} = \mathcal{I}_{a,a+1}\lambda_{a,a+1}$, and, for $k = 2, \ldots, n-a$, the matrix element $(b_+)_{a,a+k}$ equals

$$\mathcal{I}_{a,a+k}\lambda_{a,a+k} + \sum_{\substack{m=2,\ldots,k \\ (i_1,\ldots,i_m)\in\mathbb{N}^m \\ i_1+\cdots+i_m=k}} \prod_{\alpha=1}^{m} \mathcal{I}_{a,a+i_1+\cdots+i_\alpha}\lambda_{a+i_1+\cdots+i_{\alpha-1},a+i_1+\cdots+i_\alpha}.$$

Then $H = \mathrm{tr}(bb^\dagger)$ gives

$$H(e^{\mathrm{i}q}, p, \lambda) = \sum_{a=1}^{n} e^{2p_a} + \frac{1}{4}\sum_{a=1}^{n-1} e^{2p_a} \sum_{k=1}^{n-a} \frac{|\lambda_{a,a+k}|^2}{\sin^2((q_{a+k}-q_a)/2)} + \mathrm{o}_2(\lambda, \lambda^\dagger).$$

Next, we explain that restricting λ to a minimal dressing orbit of $\mathrm{SU}(n)$ results in the standard (spinless) real, trigonometric Ruijsenaars–Schneider model.

Taking $G = \mathrm{SU}(n)$, let us go back to

$$\{F, H\}_0^{\mathrm{red}}(Q, p, \lambda) = \langle D_Q F, d_p H\rangle_{\mathbb{I}} - \langle D_Q H, d_p F\rangle_{\mathbb{I}} + \langle \lambda D'_\lambda F \lambda^{-1}, D_\lambda H\rangle_{\mathbb{I}},$$

and restrict λ to a minimal dressing orbit. This is the orbit $\mathcal{O}(y) \subset B(n)$ through

$$\Delta(y) := \exp\left(\mathrm{diag}((n-1)y/2, -y/2, \cdots, -y/2)\right), \quad \text{for some} \quad y \in \mathbb{R}^*.$$

It turns out that

$$\mathcal{O}(y) \cap B(n)_+ = \{T\nu(y)T^{-1} \mid T \in G_0\},$$

with the matrix $\nu(y) \in B(n)_+$ given by $\nu(y)_{jk} = (1 - e^{-y})\exp((k-j)y/2)$, $\forall j < k$. Therefore the G_0-reduced orbit now consist of a single point, and the reduced Poisson (symplectic) structure is encoded by

$$\{F, H\}_0^{\mathrm{red}}(Q, p) = \langle D_Q F, d_p H\rangle_{\mathbb{I}} - \langle D_Q H, d_p F\rangle_{\mathbb{I}}.$$

For fixed $\lambda = \nu(y)$ and Q, the equation $b_+^{-1}Q^{-1}b_+Q = \nu(y)$ determines b_+. We find

$$(b_+)_{kl} = Q_k \bar{Q}_l \prod_{m=1}^{l-k} \frac{e^{\frac{y}{2}}\bar{Q}_k - e^{-\frac{y}{2}}\bar{Q}_{k+m-1}}{\bar{Q}_k - \bar{Q}_{k+m}}, \quad 1 \le k < l \le n, \quad \bar{Q}_k = Q_k^{-1} = e^{-\mathrm{i}q_k}.$$

Then, after the canonical transformation $(q, p) \to (q, \theta)$ with

$$\theta_k = p_k - \frac{1}{4}\sum_{m<k} \ln\left[1 + \frac{\sinh^2(y/2)}{\sin^2((q_k - q_m)/2)}\right] + \frac{1}{4}\sum_{m>k} \ln\left[1 + \frac{\sinh^2(y/2)}{\sin^2((q_k - q_m)/2)}\right],$$

we obtain the trigonometric Ruijsenaars–Schneider Hamiltonian from $b = e^p b_+$:

$$H_{\mathrm{RS}}(q, \theta) := \sum_{k=1}^n \cosh(2\theta_k) \prod_{m \ne k} \left[1 + \frac{\sinh^2(y/2)}{\sin^2((q_k - q_m)/2)}\right]^{\frac{1}{2}} = \frac{1}{2}\mathrm{tr}(bb^\dagger) + (bb^\dagger)^{-1}).$$

The symplectic leaf is $T^*G_0^{\mathrm{reg}}/S_n$ and (q, θ) parametrizes $T^*G_0^{\mathrm{reg}}$, which motivated the transformation.

The dual system in a nutshell

We have the following 3 models of the Heisenberg double

$$G^{\mathbb{C}}_{\mathbb{R}} \simeq G \times B \simeq G \times \mathfrak{P}.$$

To study the 'dual master system', the first model, $M = G^{\mathbb{C}}_{\mathbb{R}}$, is convenient.

Recall that $K \in G^{\mathbb{C}}_{\mathbb{R}}$ admits the Iwasawa decompositions

$$K = g_L b_R^{-1} = b_L g_R^{-1} \quad \text{with} \quad g_L, g_R \in G,\, b_L, b_R \in B,$$

which yield the 'Iwasawa maps' $\Xi_L, \Xi_R : G^{\mathbb{C}}_{\mathbb{R}} \to G$ and $\Lambda_L, \Lambda_R : G^{\mathbb{C}}_{\mathbb{R}} \to B$,

$$\Xi_L(K) := g_L, \quad \Xi_R(K) := g_R, \quad \Lambda_L(K) := b_L, \quad \Lambda_R(K) := b_R.$$

The Abelian Poisson algebra of the 'dual system' is $\tilde{\mathfrak{H}} := \Xi_R^*(C^\infty(G)^G)$. To describe the integral curve of $\Xi_R^*(\chi) \in \tilde{\mathfrak{H}}$ through $K(0) \in G^{\mathbb{C}}_{\mathbb{R}}$, we need the decomposition

$$\exp(\mathrm{i}t\nabla\chi(g_R(0))) = \beta(t)^{-1}\gamma(t) \quad \text{with} \quad \beta(t) \in B,\ \gamma(t) = G.$$

For the class function $\chi \in (C^\infty(G)^G)$, we use the $\mathcal{G}$-valued derivative $\nabla\chi$ defined by $\langle X, \nabla\chi(g)\rangle := \frac{d}{dt}\big|_{t=0} \chi(e^{tX}g), \quad \forall g \in G,\ X \in \mathcal{G}$. Then, the integral curve is

$$K(t) = K(0)\beta(t)^{-1} \longleftrightarrow b_R(t) = \beta(t)b_R(0),\ b_L(t) = b_L(0)\beta(t)^{-1},\ g_L(t) = g_L(0),$$

and $g_R(t) = \gamma(t)g_R(0)\gamma(t)^{-1}$. Since $L(t) = b_R(t)b_R(t)^\dagger = \beta(t)L(0)\beta(t)^\dagger$, we also have the integral curve in terms of the model $G \times \mathfrak{P}$.

In this case, we have the **map of constants of motion**

$$\tilde{\Psi} : G_{\mathbb{R}}^{\mathbb{C}} \to G_{\mathbb{R}}^{\mathbb{C}} \quad \text{defined by} \quad \tilde{\Psi}(K) := b_L b_R g_L^{-1} \equiv b_L g_R b_L^{-1}.$$

This is equivariant with respect to the conjugation action of G on the target space $G_{\mathbb{R}}^{\mathbb{C}}$ and the action of G on the Heisenberg double that is induced by the Poisson-Lie moment map $\Lambda = \Lambda_L \Lambda_R$. The $\tilde{\Psi}$-pullback of the ring of invariants

$$C^\infty(G_{\mathbb{R}}^{\mathbb{C}})^G := \{F \in C^\infty(G_{\mathbb{R}}^{\mathbb{C}}) \mid F(\eta K \eta^{-1}) \ \forall \eta \in G,\ K \in G_{\mathbb{R}}^{\mathbb{C}}\}$$

yield constants of motion that descend to the reduced phase space. These guarantee the degenerate integrability of the dual master system and its Poisson reduction.

Let me finish by mentioning the example of dual Ruijsenaars–Schneider system, given by the 'main Hamiltonian'

$$\tilde{H}_{\text{RS}} := \sum_{k=1}^{n} \cos(2\hat{\theta}_k) \prod_{m \neq k} \left[1 - \frac{\sinh^2(y/2)}{\sinh^2((\hat{q}_k - \hat{q}_m)/2)} \right]^{\frac{1}{2}}.$$

To interpret this, we consider $G = SU(n)$ and pick the same symplectic leaf as before, which belongs to the specific moment map value $\nu(y)$.

In fact, [LF-Klimcik 2011], $\tilde{H}_{\text{RS}}$ descends from the class function $\chi(g) := \frac{1}{2}\Re(\text{tr}(g))$. The 'dual position variables $\hat{q}_k$ arise from the eigenvalues of $L = b_R b_R^\dagger$. This formula of the reduced Hamiltonian is valid on a dense open subset. It was shown by Ruijsenaars in 1995 that $\tilde{H}_{\text{RS}}$ is Liouville integrable on its complete(d) phase space, and this result received a natural interpretation in the reduction approach.

This exemplifies the so-called Ruijsenaars duality (or action-position duality) between two integrable many-body systems.

Conclusion and open questions

1. I constructed 'Poisson–Lie deformations' of trigonometric spin Sutherland models.

2. I proved their degenerate integrability after restriction on the honest Poisson manifold $\mathbb{M}_*^{\text{red}} \subset \mathbb{M}^{\text{red}}$ as well as on the maximal symplectic leaves of the open dense subset $\mathbb{M}_{**}^{\text{red}} \subset \mathbb{M}_*^{\text{red}}$.

3. For lack of time, I did not present it, but recently I also proved integrability on arbitrary symplectic leaves of $\mathbb{M}_{**}^{\text{red}}$ (by a different method). What about integrability on arbitrary symplectic leaves of $\mathbb{M}_*/G$ and of the full reduced phase phase $\mathbb{M}/G$?

4. I also studied the 'dual systems' but details of their reductions are still to be explored.

5. Quantization by quantum Hamiltonian reduction?

6. An old open question: Can one derive the spinless (real, repulsive) hyperbolic RS model by Hamiltonian reduction of a **real** master integrable system?

7. Elliptic generalization?