Primitive elements for the Hopf algebras of tableaux

~ o ~

Claudia Malvenuto
Sapienza Università di Roma

~ o ~

ESI "Vienna" 12-16 Oct, 2020

François Bergeron

Plan of the talk

~ ∘ ~

* Introduction
* Some notations
* Hopf algebras of permutations
* Primitive elements of $\mathbb{Z}S$
* Hopf algebras of tableaux
* Primitive elements of $\mathbb{Z}T$

* Introduction

- (1995) C.M. - $\boxed{\text{C. Reutenauer}}$ → in collaboration

  Hopf structures on permutations
  (inherited by concatenation/shuffle
  Hopf algebras on $T(V)$)

- (1995) S. Poirier - C. Reutenauer

Ann. Sci. Math. Québec **19** (1995), no. 1, 79–90.

ALGÈBRES DE HOPF DE TABLEAUX

STÉPHANE POIRIER ET CHRISTOPHE REUTENAUER

FRANÇOIS BERGERON

- M. Aguiar
F. Sottile
(2005)

Structure of the Malvenuto–Reutenauer Hopf algebra of permutations

Marcelo Aguiar and Frank Sottile[1]

Department of Mathematics, Texas A&M University, College Station, TX 77843, USA

Advances in Mathematics 191 (2005) 225–275

- M. Taskin
(2013)

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 141, Number 3, March 2013, Pages 837–856
S 0002-9939(2012)11415-7
Article electronically published on July 24, 2012

INNER TABLEAU TRANSLATION PROPERTY OF THE WEAK ORDER AND RELATED RESULTS

MÜGE TAŞKIN

* Some notations

- S_n : symmetric group on $\{1,2,\ldots,n\}$

- $\sigma \in S_n$: $\sigma(1)\sigma(2)\ldots\sigma(n)$

- $|\sigma|$ = # of letters of $\sigma = n$

- $\ell(\sigma)$ = length in Coxeter group

François Bergeron

- Inversion set for σ

$$\mathrm{Inv}(\sigma) := \{(j,i) : j>i\ ,\ \sigma^{-1}(j)<\sigma^{-1}(i)\}$$

Ex. $\sigma = 2\,5\,1\,7\,6\,4\,3$

$$\mathrm{Inv}(\sigma) = \{(2,1);(5,1);(5,4);(5,3);\\ (7,6);(7,4);(7,3);\\ (6,4);(6,3);(4,3)\}$$

$$\ell(\sigma) = |\mathrm{Inv}(\sigma)| = 10.$$

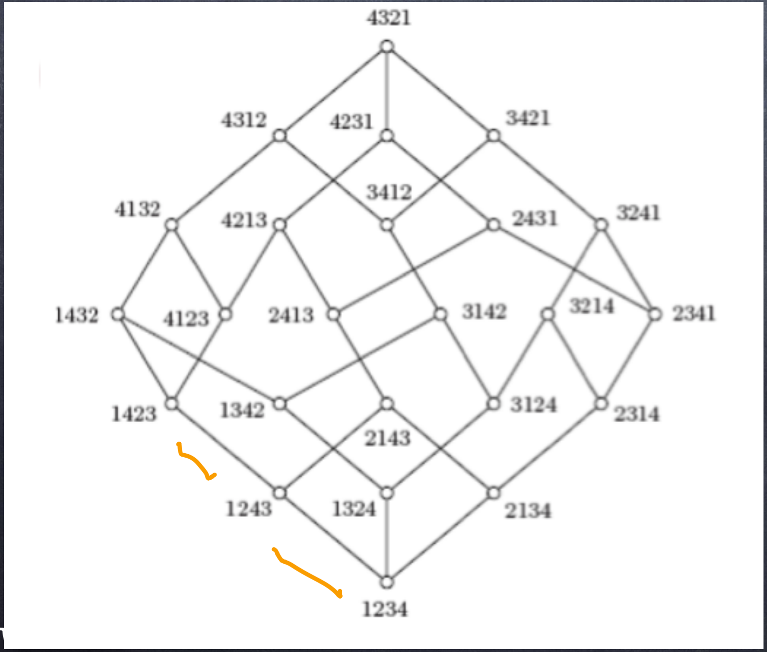
François Bergeron

- Right weak Bruhat order on S_n :

• reflexive
• transitive

closure of the relation

$$\boxed{u < v \iff \exists\, \tau : \; v = u \circ \tau}$$



- $u \le v \iff$

$$\mathrm{Inv}(u) \subseteq \mathrm{Inv}(v)$$

- Standardisation

v : a word on $\mathbb{N}^{>0}$ without repetition

$st(v) :=$ permutation
replace letters by
unique increasing
bijection from
$\mathrm{Aleph}(v) \longrightarrow \{1, 2, \dots |v|\}$

Ex.

$st(5713) = 3412$

- $\sigma|_I =$ obtained by restriction to $I \subseteq \{1, \dots, n\}$

- $\sigma|I =$ obtained by erasing the letters <u>not</u> in I

Ex. $\sigma = \dot{2}\,\underline{5}\,\underline{1}\,7\,\dot{6}\,\underline{4}\,\dot{3}$ $\quad I = \{2, 3, 6\}$

$\sigma|_I = \underline{5}\,\underline{1}\,\underline{4}$ $\qquad \sigma|I = \dot{2}\,\dot{6}\,\dot{3}$

François Bergeron

$S = \bigcup_{n \geq 0} S_n$ Classical associative products on S

$u \in S_p$, $v \in S_q$, $\overline{v}$: add p to each letter of v

– right "shifted" concatenation $\square$

$u \,\square\, v := u\,\overline{v}$ **Ex.** $231 \,\square\, 12 = 23145 \in S_5$

– left "shifted" concatenation Δ

$v \,\Delta\, u := \overline{v}\, u$ **Ex.** $12 \,\Delta\, 231 = 45231$

Facts

- S is a free monoid with Δ

- Free generators: are the <u>indecomposable</u> permutations

- The weak order is compatible with Δ

$$u \leq u',\ v \leq v' \Longrightarrow v \,\Delta\, u \leq v' \,\Delta\, u'$$

- Δ and $\square$ are conjugate under $w \mapsto \widetilde{w}$

$$v \,\Delta\, u = \widetilde{\left(\widetilde{u} \,\square\, \widetilde{v}\right)}$$

François Bergeron

* Hopf algebras of permutations

$\mathbb{Z}S$: free module with basis S the permutations

– product $*$: destandardised concatenation

$\alpha \in S_p, \beta \in S_q \quad \alpha * \beta \in S_{p+q}$

$$\alpha * \beta = \sum_{uv \in S_{p+q}} uv, \qquad st(\alpha) = u, \; st(\beta) = v$$

Ex. $12 * 21 = 1221 + 1342 + 1432$

$2341 + 2431 + 3421 \in S_4$

François Bergeron

– coproduct δ: standardised unshuffling

$$\sigma \in S_n: \quad \delta(\sigma) = \sum_{i=1}^{n} \sigma|\{1,\ldots,i\} \otimes st(\sigma|\{i+1,\ldots,n\})$$

Ex.

$$\delta(3124) =$$

$$\begin{array}{rcl}
\varepsilon \otimes st(3124) & / & \varepsilon \otimes 3124 \\
1 \otimes st(324) & & 1 \otimes 213 \\
12 \otimes st(34) & & 12 \otimes 12 \\
312 \otimes st(4) & & 312 \otimes 1 \\
3124 \otimes \varepsilon & & 3124 \otimes \varepsilon
\end{array}$$

François Bergeron

- $(\mathbb{Z}S, *, \delta)$ is a graded Hopf algebra

- $(\mathbb{Z}S, *', \delta')$ is a graded Hopf algebra

$*'$: shifted shuffle

δ': standardised deconcatenation

- Duality between the two Hopf structures

$$\langle \sigma *' \alpha, \tau \rangle = \langle \sigma \otimes \alpha, \delta(\tau) \rangle$$

$$\langle \sigma * \alpha, \tau \rangle = \langle \sigma \otimes \alpha, \delta'(\tau) \rangle$$

Conjugated via $\Theta(\sigma) = \sigma^{-1}$

François Bergeron

— Aguiar - Sottile : new linear basis $\{\mathcal{M}_\sigma : \sigma \in S\}$ for $\mathbb{Z}S$

$$\boxed{\sigma = \sum_{\sigma \leq w} \mathcal{M}_w}$$

They use

* left weak order $\leq$

* $(\mathbb{Z}S, *', \delta')$

Theorem $$\delta(\mathcal{M}_\sigma) = \sum_{\sigma = v \Delta u} \mathcal{M}_u \otimes \mathcal{M}_v$$

Lemma For $n = p + q$

$\sigma \in S_{p+q}$

$a = \sigma | \{1, \ldots, p\}$

$b = st(\sigma | \{i+1, \ldots, n\})$

$$\boxed{\begin{array}{c} \sigma \leq v \Delta u \\ \Longleftrightarrow \\ a \leq v \text{ and } b \leq u \end{array}}$$

Remark Proof in Aguiar - Sottile uses global descent

Def. $\sigma \in S_n$ has a global descent in $i \in \{1, 2, \dots, n-1\}$ if $\forall\, j \leq i,\ \forall\, k > i : \sigma(j) > \sigma(i)$

Ex.

$$7\,8\,4\,6\,5\,2\,1\,3 = 1\,2 \,\triangle\, 1\,3\,2 \,\triangle\, 2\,1\,3$$

$$\text{Global Descents} = \{2, 5\}$$

$$\underbrace{1\,2}_{S_2},\ \underbrace{1\,3\,2,\ 2\,1\,3}_{S_3}$$

indecomposable (no global descents)

FRANÇOIS BERGERON

* Primitive elements of KS

Corollary

The submodule of the primitive elements of $\mathbb{Z}S$

is spanned by the M_σ such that :

- σ has no global descents, equiv:
- σ is indecomposable for Δ, i.e the generators of the free monoïd (S, Δ)

François Bergeron

* Hopf algebras of tableaux

- T_n : standard Young tableaux with n cases, entries $\{1, 2, \dots, n\}$ $\qquad T = \bigcup_{n \geq 0} T_n$

6			
5	9	11	
2	8	10	
1	3	4	7

$n = 11$

- $\sigma \in S_n \xrightarrow{RSK} \big(P(\sigma), Q(\sigma)\big)$

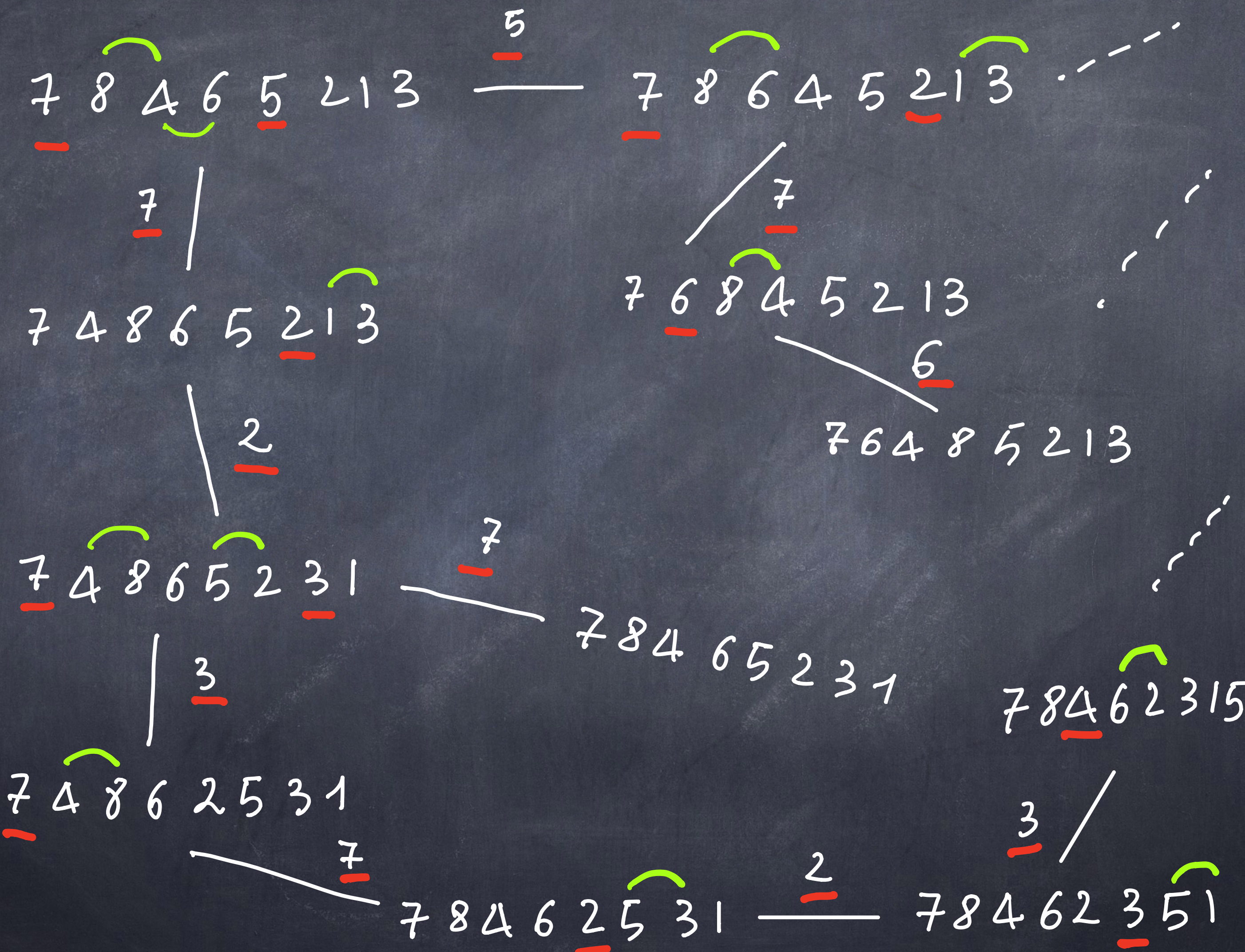
insertion recording

witness

- Knuth relations / plactic congruence $\begin{cases} \underline{j}ik \sim \underline{j}ki \\ ki\underline{j} \sim ik\underline{j} \end{cases}$ $\qquad i < \underline{j} < k$

François Bergeron

Ex.



François Bergeron

Theorem (Knuth)

$$\sigma \sim \tau \iff P(\sigma) = P(\tau)$$

$\mathbb{Z}\mathcal{T}$: free module with basis $\mathcal{T}$ the tableaux

I : module $\langle u - v : u \sim v \rangle$

Theorem (PR)

$$\mathbb{Z}S/I \cong \mathbb{Z}\mathcal{T}$$
$$\sigma \longmapsto P(\sigma)$$

$*$ product
δ coproduct

inherited by permutations

Obs. $(\mathbb{Z}\mathcal{T}, *, \delta)$ non-commutative not free associative algebra

FRANÇOIS BERGERON

- $(\mathbb{Z}\mathcal{T}, *', \delta')$ another Hopf structure:

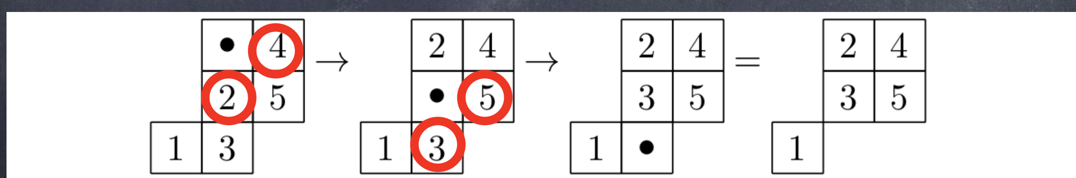
$t \in \mathcal{T}$: $\mathfrak{c}(t) = \sum_{\sigma \sim row(t)} \sigma$ — plactic class of t

Ex.

$t = \begin{array}{|c|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline \end{array}$ $row(t) = 312$ $\mathfrak{c}(t) = 312 + 132$

$312 \sim 132$

- Description of products + coproducts via "jeu de taquin"



backword slides

François Bergeron

Homomorphisms

- $\sigma \longmapsto P(\sigma)$ surjective Hopf morphism of $(\mathbb{Z}S, *, \delta)$ in $(\mathbb{Z}\mathcal{T}, *, \delta)$

- $(Sym, \cdot, \delta) \longrightarrow (\mathbb{Z}\mathcal{T}, *, \delta)$

$$\underset{\text{Schur function}}{s_\lambda} \longmapsto \sum_{sh(t)=\lambda} t$$

- $ev(t)$ Schützenberger evacuation of $t \in \mathcal{T}$
is an anti-automorphism of both Hopf algebras of tableaux

François Bergeron

* Primitive elements of $\mathbb{Z}\mathfrak{S}$

Weak order of tableaux:
- Melnikov 2004
- Duflo order
- M. Taskin 2013

$\leq_{weak}$

Def. A, U tableaux : $A \leq U \iff$

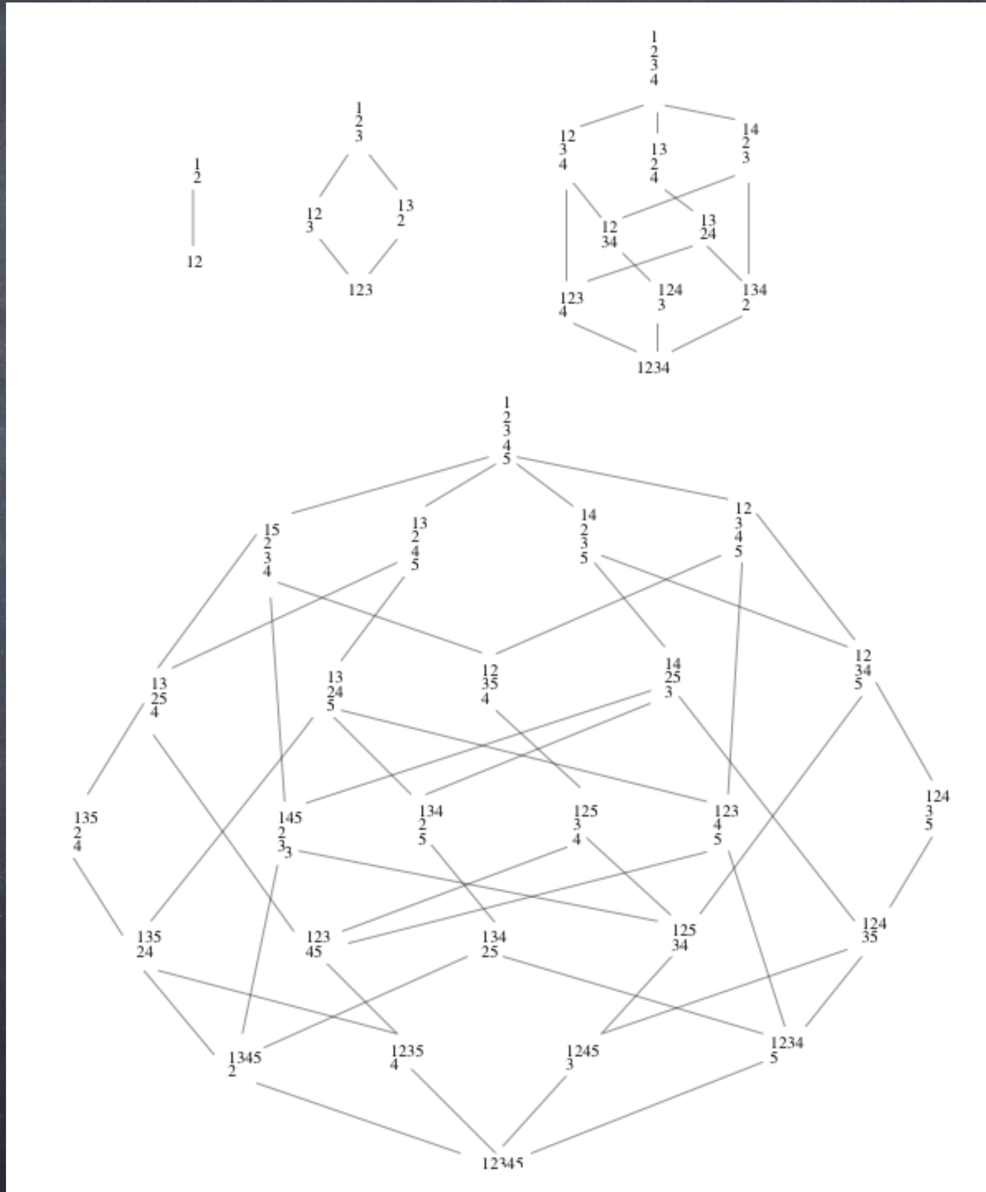
$\exists n, \exists \alpha_0, \ldots, \alpha_{n-1}, \beta_1, \ldots, \beta_n$ permutations :

- $A = P(\alpha_0)$,
- $\alpha_0 \leq \beta_1 \sim \alpha_1 \leq \beta_2 \sim \ldots \alpha_{n-1} \leq \beta_n$,
- $U = P(\beta_n)$

François Bergeron

The weak order on T_n

$n = 2, 3, 4, 5$



FRANÇOIS BERGERON

- **Lemma** Plactic equivalence $\sim$ is compatible with $\triangle$

  $$u \sim u' \implies v \triangle u \sim v \triangle u'$$

- Product $\triangle$ on tableaux

- $P(v \triangle u) = P(v) \triangle P(u)$:

  P is a homomorfism of the monoïds S and T.

- A simpler way to compute $\triangle$ on tableaux

Ex.

$$U = \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline\end{array}\ , \quad V = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

$$\begin{array}{cc} \overline{V} & \begin{array}{|c|c|}\hline 4 & 5 \\ \hline\end{array} \\ & \downarrow \\ U & \begin{array}{|c|c}\hline 3 & \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline\end{array} \end{array} \qquad V \triangle U \quad \begin{array}{|c|c}\hline 5 & \\ \hline 3 & \multicolumn{1}{|c|}{4} \\ \hline 1 & \multicolumn{1}{|c|}{2} \\ \hline\end{array}$$

- *Lemma* The weak order on tableaux is compatible with $\triangle$

$$U \leq U', \ V \leq V' \implies V \triangle U \leq V' \triangle U'$$

FRANÇOIS BERGERON

Recall for permutations

– *Lemme* For $n = p + q$

$\sigma \in S_n \quad v \in S_p \quad u \in S_q$

$a = \sigma | \{1, \dots, p\}$

$b = st(\sigma | \{i+1, \dots, n\})$

$$\boxed{\begin{array}{c} \sigma \leq v \,\Delta\, u \\ \Longleftrightarrow \\ a \leq v \text{ and } b \leq u \end{array}}$$

Holds for $\begin{array}{|c|c|c|c|} \hline 6 & & & \\ \hline 5 & 9 & 11 & \\ \hline 2 & 8 & 10 & \\ \hline 1 & 3 & 4 & 7 \\ \hline \end{array}$!!

– *Lemme* For $n = p + q$

$\Sigma \in T_n \quad v \in T_p \quad u \in T_q$

$A = \Sigma | \{1, \dots, p\}$

$B = st(\Sigma | \{i+1, \dots, n\})$

$$\boxed{\begin{array}{c} \Sigma \leq V \,\Delta\, U \\ \Longleftrightarrow \\ A \leq V \text{ and } B \leq U \end{array}}$$

François Bergeron

– Aguiar - Sottile method: define a new linear basis $\{M_\Sigma : \Sigma \in \mathcal{T}\}$ for $\mathbb{Z}\mathcal{T}$ via Möbius inversion in the poset of tableaux

$$\boxed{\Sigma = \sum_{\Sigma \leq W} M_W}$$

Theorem (C.M, C.R)

$$\delta(M_\Sigma) = \sum_{\Sigma = V \Delta U} M_U \otimes M_V$$

Corollary

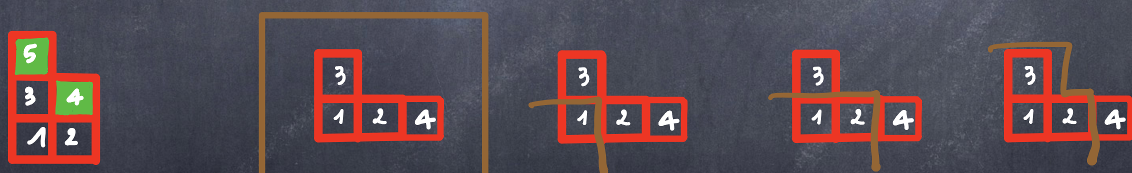
The submodule of the primitive elements of $\mathbb{Z}\mathcal{G}$

is spanned by the M_Σ such that:

Σ is indecomposable for Δ, i.e

the generators of the free monoïd $(\mathbb{Z}\mathcal{G}, \Delta)$

Ex.



decomposable

indecomposable

François Bergeron

* Final remarks

For S $\left\langle \begin{array}{l} \text{right shifted concatenation } \square \\ \text{left shifted concatenation } \triangle \end{array} \right.$

- $\square$ is compatible with $\sim$ $\Longrightarrow (\mathcal{T}, \square)$ is a monoid
- A simpler way to compute $\square$ on tableaux

Ex.

$$U = \begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array} \qquad V = \begin{array}{|c|c|} \hline 3 & \\ \hline 2 & \\ \hline 1 & 4 \\ \hline \end{array}$$

$$\underset{U}{\begin{array}{|c|c|} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}} \leftarrow \underset{\overline{V}}{\begin{array}{|c|c|} \hline 6 & \\ \hline 5 & \\ \hline 4 & 7 \\ \hline \end{array}} \qquad U \square V \quad \begin{array}{|c|c|c|c|} \hline 6 & & & \\ \hline 2 & 5 & & \\ \hline 1 & 3 & 4 & 7 \\ \hline \end{array}$$

Theorem (Loday-Ronco) 2002

$$u *' v = \sum_{u \,\square\, v \leq \sigma \leq v \,\Delta\, u} \sigma \qquad u, v \in S$$

$\sigma \in [u \,\square\, v, v \,\Delta\, u]$ interval of the weak Bruhat order of S

Theorem (Taskin) 2005

$$U *' V = \sum_{U \,\square\, V \leq t \leq V \,\Delta\, U} t \qquad U, V \in \mathcal{T}$$

t tableau in the interval of the Taskin-Duflo order of $\mathcal{T}$

François Bergeron

Theorem (A.-S.) In the linear basis $\{M_\sigma : \sigma \in S\}$ the structure constants are positive:

$$M_\sigma * M_\tau = \sum_\rho c_\sigma^\tau M_\rho \Rightarrow c_\sigma^\tau \geqslant 0$$

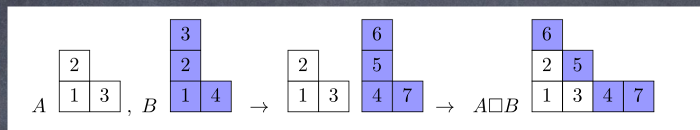
A counterexample (Franco Saliola UQAM)

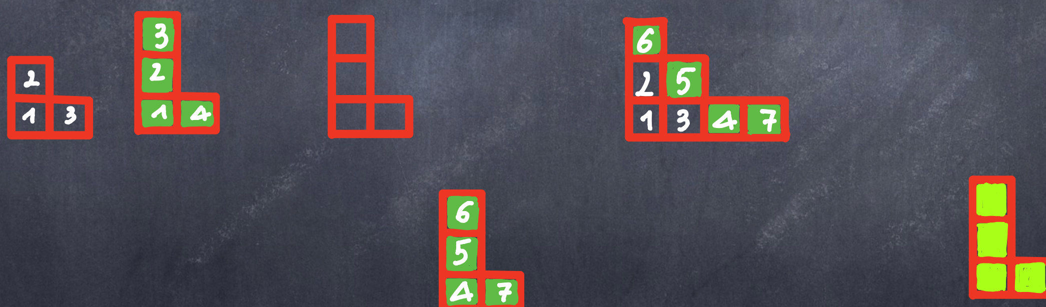
$$\mathcal{M}_{P(123)} * \mathcal{M}_{P(123)} = \mathcal{M}_{P(123456)}$$
$$-\mathcal{M}_{P(241356)} - \mathcal{M}_{P(251346)} - \mathcal{M}_{P(261345)} - \mathcal{M}_{P(351236)} - \mathcal{M}_{P(361245)} - \mathcal{M}_{P(461235)}$$
$$+\mathcal{M}_{P(256134)} + \mathcal{M}_{P(346125)} + \mathcal{M}_{P(356124)} + 2\mathcal{M}_{P(456123)}$$
$$+2\mathcal{M}_{P(362514)} - \mathcal{M}_{P(462513)} - M_{P(543126)}.$$

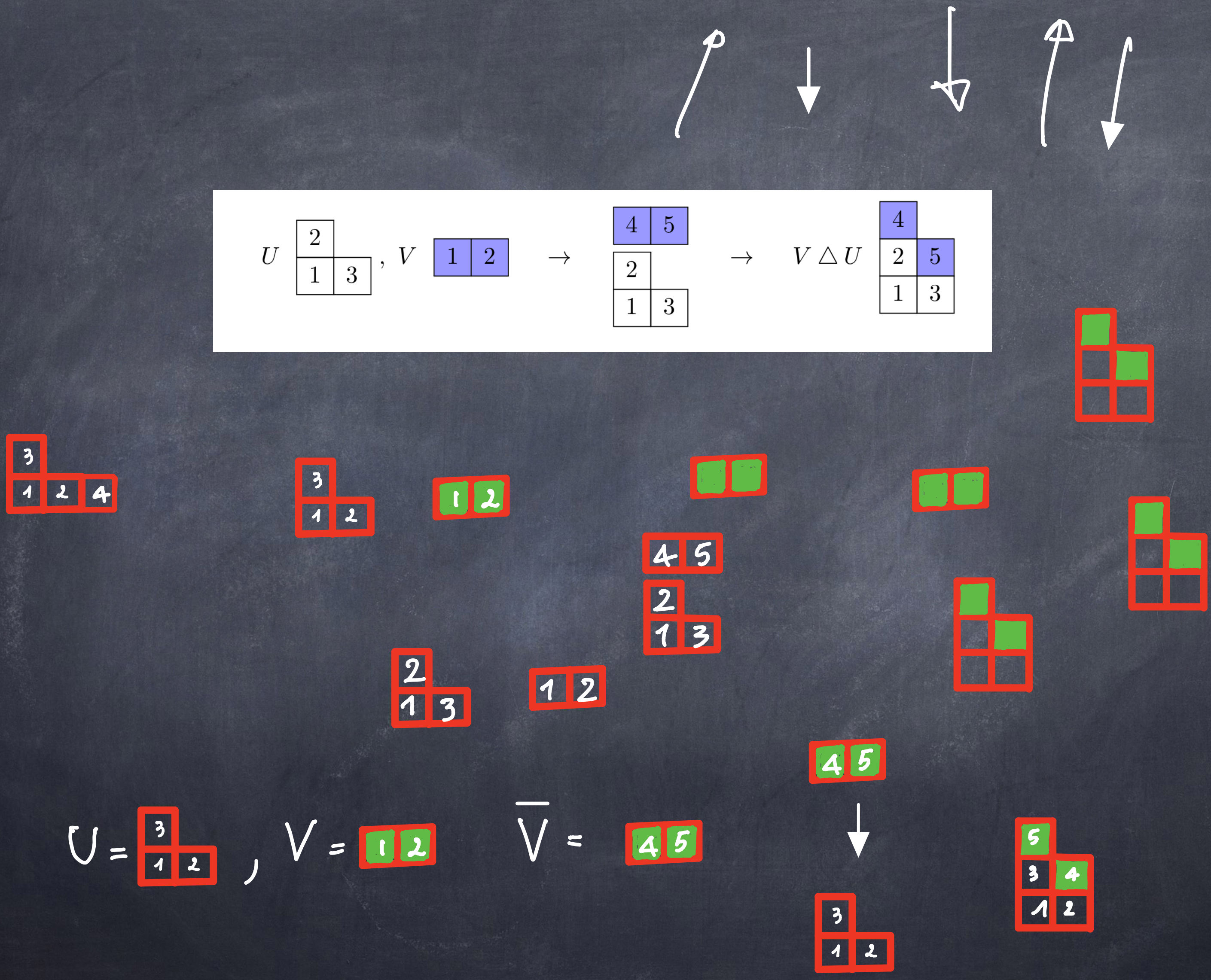
Grazie
per
l'attenzione

François Bergeron

François Bergeron


A
, B
→
→
A□B





François Bergeron