ESI, Vienna 26/8/2022

MORITA INVARIANCE of QUASI POISSON STRUCTURES

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based on

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IMRN (2022), 9, 6627-6704

symplectic derived algebraic geometry
(PTVV 2013)

showed clear connections
with different areas
that are developed with the tools
of differential geometry

TQFT : AKSZ construction

Poisson geometry: } Lie groupoids
Graded geometry } graded symplectic and Poisson structures

despite very different tools

need of a dictionary

SOME PARTS of THE PICTURE ARE ALREADY CLEAR

Differentiable stacks are Lie groupoids up to Morita equivalence

[Behrend, Xu; 2011]

1- symplectic differentiable stacks are quasi symplectic groupoids

[Xu; 2004]

What are Poisson differentiable stacks?
A natural candidate is quasi Poisson groupoids
[Iglesias Ponte, Laurent-Gengoux, Xu; 2012]

What is Morita equivalence for quasi Poisson groupoids?

On the DAG side

- Calaque, Pantev, Toen, Vaquie, Vezzosi 2017
- Melani, Safronov

Look at Les Diablerets lectures by P. Safronov on YouTube

A BRIEF REMINDER of MORITA EQUIVALENCE

$Z \rightrightarrows X \qquad \Gamma \rightrightarrows M$ Lie groupoids

A Lie groupoid morphism

$$\begin{tikzcd} Z \arrow[r, "\phi"] \arrow[d, shift left] \arrow[d, shift right] & \Gamma \arrow[d, shift left] \arrow[d, shift right] \\ X \arrow[r, "\varphi"'] & M \end{tikzcd}$$

is a Morita morphism if

i) φ is a surjective submersion

ii)
$$\begin{tikzcd} Z \arrow[r, "t\times s"] \arrow[d, "\phi"'] & X\times X \arrow[d, "\varphi\times\varphi"] \\ \Gamma \arrow[r, "t\times s"] & M\times M \end{tikzcd}$$

is cartesian in the category of smooth manifolds i.e. Z is isomorphic to the pull-back groupoid

$$\Gamma[X] = X \times_M \Gamma \times_M X \rightrightarrows X$$

$\Gamma_1 \rightrightarrows M_1$ and $\Gamma_2 \rightrightarrows M_2$ are **Morita equivalent** if there exists $Z \rightrightarrows X$ and two Morita morphisms

$$\begin{tikzcd} & Z \arrow[dl, "\Phi_1"'] \arrow[dr, "\Phi_2"] & \\ \Gamma_1 & & \Gamma_2 \end{tikzcd}$$

We look at the class $[M/\Gamma]$ of Lie groupoids that are Morita equivalent to $\Gamma \rightrightarrows M$ as a differentiable stack.

All geometrical structures should be expressed as Morita invariant structures

SYMPLECTIC STRUCTURES ON $[M/\Gamma]$

A quasi symplectic groupoid is the triple $(\Gamma \rightrightarrows M, \omega, H)$ [Xu, 2004; Bursztyn, Crainic, Weinstein, Zhu, 2004]

$$\omega \in \Omega^2\Gamma$$
$$H \in \Omega^3 M$$

i) $d\omega = \partial^* H$

$\partial^*\omega = 0$

$dH = 0$

ii) $\ker\omega_m \cap \ker ds|_m \cap \ker dt|_m = 0$

$\forall\, m \in M \hookrightarrow \Gamma$

THEOREM [Xu, 2004]

Let
$$\begin{array}{ccc} \Gamma(X) & \xrightarrow{\phi} & \Gamma \\ \downarrow\downarrow & & \downarrow\downarrow \\ X & \xrightarrow[\varphi]{} & M \end{array}$$
φ surjective submersion

then
$(\Gamma[X] \rightrightarrows X, \phi^*\omega, \varphi^* H)$ is a quasi symplectic groupoid

QUASI POISSON GROUPOIDS

$$(\Gamma \rightrightarrows M, \Pi, \Lambda)$$

$$\Pi \in \Gamma(\wedge^2 T\Gamma)$$

$$\Lambda \in \Gamma(\wedge^3 A) \qquad A \Rightarrow M = \mathrm{Lie}(\Gamma \rightrightarrows M)$$

is a quasi Poisson groupoid if

$$[\Pi, \Pi] = \overrightarrow{\Lambda} - \overleftarrow{\Lambda}$$

$$[\Pi, \overrightarrow{\Lambda}] = 0$$

Two quasi Poisson structures $(\Gamma, \Pi_1, \Lambda_1)$ and $(\Gamma, \Pi_2, \Lambda_2)$ are twist equivalent if there exists $T \in \Gamma(\wedge^2 A)$ s.t.

$$\Pi_2 = \Pi_1 + \overrightarrow{T} - \overleftarrow{T}$$

$$\Lambda_2 = \Lambda_1 + \delta_{\Pi_1}(T) - \frac{1}{2}[T, T]$$

EXAMPLES

- Poisson Groupoids $\Lambda = 0$
  - Poisson-Lie groups $(G \rightrightarrows *, \pi)$
  - symplectic groupoids $(\Gamma \rightrightarrows M, \pi = \Omega^{-1})$
- Quasi Poisson groups $(G \rightrightarrows *, \pi, \Lambda)$

every quasitriangular Poisson Lie group is twist equivalent to a quasi Poisson group with $\pi = 0$

- AMM Groupoid

$G \ltimes G \rightrightarrows G$ action groupoid (wrt adjoint)

$\mathfrak{g} = \mathrm{Lie}\, G$ quadratic, $\Lambda \in \Lambda^3 \mathfrak{g}$ Cartan 3-form

$$\pi = \dots$$

[Iglesias-Ponte, Laurent-Gengoux, P. Xu, 2012]

MORITA EQUIVALENCE OF QUASI-POISSON
(naive attempt)

Let $(\Gamma \rightrightarrows M, \pi, \Lambda)$ be q.P.

and
$$\begin{tikzcd} \Gamma[X] \arrow[r, "\Phi"] \arrow[d, shift left] \arrow[d, shift right] & \Gamma \arrow[d, shift left] \arrow[d, shift right] \\ X \arrow[r] & M \end{tikzcd}$$
be a Morita morphism

We want to transport the qP structure on $\Gamma[X]$

- choose a connection on φ
$$\lambda_\nabla : \varphi^* TM \longrightarrow TX$$

it fixes
$$\lambda_\nabla : \varphi^* A \longrightarrow A[X] \equiv \mathrm{Lie}\,\Gamma[X]$$
$$\lambda_\nabla : \Phi^* T\Gamma \longrightarrow T\Gamma[X]$$

- define
$$\pi_X = \lambda_\nabla(\pi) \quad \Lambda_X = \lambda_\nabla(\Lambda)$$

but $[\pi_X, \pi_X] \neq \overrightarrow{\Lambda_X} - \overleftarrow{\Lambda_X}$

$\mathbb{Z}$- GRADED 2-LIE ALGEBRAS

$$\mathfrak{u} \xrightarrow{d} \mathfrak{g}$$

- $\mathfrak{u}, \mathfrak{g}$ graded Lie Algebras
- $d : \mathfrak{u} \longrightarrow \mathfrak{g}$ gLA morphism
- $\delta : \mathfrak{g} \longrightarrow \mathrm{Der}\,\mathfrak{u}$ gLA morphism

s.t.

a) $[\pi, da] = d\,\delta_\pi(a)$

b) $[a, b] = \delta_{da}(b)$

$a, b \in \mathfrak{u} \quad \pi \in \mathfrak{g}$

Associated dgLA

$$\nu(\mathfrak{u} \xrightarrow{d} \mathfrak{g}) \equiv \mathfrak{u}[1] \oplus \mathfrak{g}$$

$$[a_1 + \pi_1, a_2 + \pi_2] = \delta_{\pi_1}(a_2) - (-1)^{\pi_2 + 1} \delta_{\pi_2}(a_1) + [\pi_1, \pi_2]$$

$$D(a + \pi) = 0 + da$$

Maurer cartan set

$$MC(\mathfrak{u} \xrightarrow{d} \mathfrak{g}) \equiv MC(\mathcal{V}(\mathfrak{u} \xrightarrow{d} \mathfrak{g}))$$

$$\Lambda \oplus \pi \in \mathcal{V}^2 = \mathfrak{u}^1 \oplus \mathfrak{g}^2$$

if

$$d\Lambda + \frac{1}{2}[\pi,\pi] = 0$$

$$\delta_\pi \Lambda = 0$$

Gauge transformations

$$T \in \mathfrak{u}^1 \hookrightarrow \mathcal{V}^0 = \mathfrak{u}^1 \oplus \mathfrak{g}^0$$

$$e^T(\Lambda \oplus \pi) =$$

$$\left(\Lambda - \delta_\pi T - \frac{1}{2}[T,T]\right) \oplus (\pi + dT)$$

THE $\mathbb{Z}$-GRADED 2-LIE ALGEBRA of MULTIPLICATIVE POLYVECTOR FIELDS

$T^*[1]\Gamma \rightrightarrows A^*[1]$ is a graded groupoid

Multiplicative polyvector fields are groupoid 1-cocycles of $T^*[1]\Gamma \rightrightarrows A^*[1]$

- trivial cocycles $\partial^* a = \overrightarrow{a} - \overleftarrow{a}$ $a \in \Gamma(\wedge A)$
- degree 1 $v \in \Gamma(T\Gamma)$ s.t. $v: T^*\Gamma \to \mathbb{R}$ is a groupoid morphism
- degree 2 $\pi \in \Gamma(\wedge^2 T\Gamma)$ s.t $\pi: \begin{array}{ccc} T^*\Gamma & \to & T\Gamma \\ \downarrow\downarrow & & \downarrow\downarrow \\ A^* & \to & TM \end{array}$ is a groupoid morphism

$$\mathcal{T}_{mult}(\Gamma) \equiv$$

$$Z^1(T^*[1]\Gamma) \subset C^\infty(T^*[1]\Gamma) = \Gamma(\wedge T\Gamma)$$

Let $\Sigma(A)$ be the gLA of sections of $\wedge A$

$\mathcal{T}(\Gamma)$ be the gLA of sections of $T\Gamma$

PROPOSITION

[Iglesias-Ponte, Laurent-Gengoux, P.Xu; 2012]

1) $\mathcal{T}_{mult}(\Gamma) \subset \mathcal{T}(\Gamma)$ is a sub gLA

2) $d : \Sigma(A) \longrightarrow \mathcal{T}_{mult}(\Gamma)$

$$da \equiv \overrightarrow{a} - \overleftarrow{a}$$

is a dgLA morphism

3) for each $\pi \in \mathcal{T}_{mult}(\Gamma)$ and $a \in \Sigma(A)$ there exists a unique $\delta_\pi(a) \in \Sigma(A)$ s.t.

$$\overrightarrow{\delta_\pi(a)} = [\pi, \vec{a}]$$

4) $\delta : \mathcal{T}_{mult}(\Gamma) \longrightarrow \mathrm{Der}(\Sigma(A))$ and

$$d\,\delta_\pi(a) = [\pi, da] \qquad a, b \in \Sigma$$

$$\delta_{d(a)}(b) = [a, b] \qquad \pi \in \mathcal{T}_{mult}$$

COROLLARY

- $(\Gamma \rightrightarrows M, \pi, \Lambda)$ is quasi Poisson $\Leftrightarrow$ $\Lambda \oplus \pi \in MC(\mathcal{V})$
- twist equivalence is gauge equivalence

THEOREM

Let $\Gamma_1 \rightrightarrows M_1$ and $\Gamma_2 \rightrightarrows M_2$ be Morita equivalent groupoids then the $\mathbb{Z}$-graded 2-Lie algebras $\Sigma(A_1) \to \tau_{mult}(\Gamma_1)$ and $\Sigma(A_2) \longrightarrow \tau_{mult}(\Gamma_2)$ are homotopically equivalent

[this generalizes Berwick-Evans/Lerman result on the (non graded) 2-Lie algebra of vector fields]

we think the homotopy equivalence class of $\Sigma(A) \xrightarrow{d} \tau_{mult}(\Gamma)$ as the space of polyvector fields on $[M/\Gamma]$

Back to Morita equivalence of q Poisson

$$\begin{tikzcd} T[X] \arrow[r, "\Phi"] \arrow[d, shift left] \arrow[d, shift right] & \Gamma \arrow[d, shift left] \arrow[d, shift right] \\ X \arrow[r, "\varphi"'] & M \end{tikzcd}$$

Morita morphism

choose a connection over φ

$$\lambda_\nabla : \varphi^* TM \longrightarrow TX$$

that fixes

$$\lambda_\nabla : \varphi^* A \longrightarrow A[X] \qquad \lambda_\nabla : \Phi^* T\Gamma \longrightarrow TT[X]$$

Let $\Psi_1 = \lambda_\nabla$

Now we can complete with

$$\Psi_2 : \Lambda^2 T_{mult}(\Gamma) \longrightarrow \Sigma(A[X])$$

so that

$$\Psi = (\Psi_1, \Psi_2) : \begin{pmatrix} \Sigma(A) \\ \downarrow \\ T_{mult}(\Gamma) \end{pmatrix} \longrightarrow \begin{pmatrix} \Sigma(A[X]) \\ \downarrow \\ T_{mult}(T[X]) \end{pmatrix}$$

is a morphism of graded 2-Lie algebras

$$\Psi(\Lambda, \pi) = \left(\lambda_\nabla(\Lambda) + \Psi_2(\pi,\pi),\ \lambda_\nabla(\pi)\right)$$

the term we were missing

DEFINITION

Let $(Z \rightrightarrows X, \Pi_X, \Lambda_X)$ and $(T \rightrightarrows M, \Pi, \Lambda)$ be q Poisson groupoids. By Morita morphism of q Poisson groupoid we mean a Morita morphism of Lie gpd

$$\begin{array}{ccc} Z & \xrightarrow{\phi} & \Gamma \\ \downdownarrows & & \downdownarrows \\ X & \xrightarrow[\varphi]{} & M \end{array}$$

and $T \in \Sigma^1(A[x])$ s.t.

$e^{T} \cdot (\Lambda_X, \Pi_X)$ projects to (Λ, π)

DEFINITION

Two q Poisson groupoids $(T_1 \rightrightarrows M_1, \Lambda_1, \Pi_1)$ and $(T_2 \rightrightarrows M_2, \Lambda_2, \Pi_2)$ are Morita equivalent if there exists a third q Poisson groupoid $(Z \rightrightarrows X, \Lambda_Z, \Pi_Z)$ and two Morita morphisms of q Poisson groupoids

$$\begin{array}{ccccc} & & (Z, \Lambda, \pi) & & \\ & \swarrow & & \searrow & \\ (T_1, \Lambda_1, \pi_1) & & & & (T_2, \Lambda_2, \Pi_2) \end{array}$$