Fundamental Group of Quantum Field Theory

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- Want to introduce a notion of 'fundamental' group of QFT.
- Motivations? Fantasy?
- Need to have a 'definition' of QFT, a working definition at least.
- Need to construct such a group from such a definition of QFT.

1 Various constructions of fundamental group

Let X be a path connected topological space with some nice properties.

1.1 Definition

Let $p_1(X;x,y)$ be the set of homotopy types of paths from x to y, $x,y \in X$.

- $p_1(X;x,x)$ is a group denoted by $\pi_1(X,x)$;
- $\pi_1(X,x)$ and $\pi_1(X,y)$ are isomorphic, and $p_1(X;x,y)$ is the set of isomorphisms;
- $p_1(X;x,y)$ is a torsor of fundamental group.

1.2 Via fiber functors

Let $\mathbf{Cov}(X)$ be the category of covering spaces over X.

- Taking the fiber over x for each covering space defines a functor, called a fiber functor,

$$F_x : \mathbf{Cov}(X) \rightsquigarrow \mathbf{Set}.$$

- Let $\mathbf{Iso}(F_x, F_y)$ be the set of invertible natural transformaitions from F_x to F_y.
- There is an isomorphism $\mathbf{Iso}(F_x, F_y) \simeq p(X;x,y)$ and $\mathbf{Aut}(F_x) \simeq \pi_1(X;x)$.

1.3 Via the homotopy category *ho*$\mathbf{Top}_*$

Let *ho*$\mathbf{Top}_*$ be the homotopy category of based topological spaces, X_*, X'_* etc.

- Objects are based toplogical spaces;
- Morphisms are homotopy types of base point preserving continuous maps:
- $[X'_*, X_*]$: the set of morphisms from X'_* to X_*,

The based loop space $\Omega(X_*)$ defines a contravariant functor from *ho*$\mathbf{Top}_*$ to the category $\mathbf{Grp}$ of groups:

$$\mathscr{S}^{X_*} := [-, \Omega(X_*)] : ho\mathring{\mathbf{Top}}_* \rightsquigarrow \mathbf{Grp}$$

We have $\mathscr{S}^{X_*}(pt) \simeq \pi_1(X, x)$

$$\mathscr{S}^{X_*}(pt) = [pt, \Omega(X_*)] \simeq [S^1_*, X_*].$$

In other words $\mathscr{S}^{X_*}$ is a presheaf of groups over the homotopy category *ho*$\mathbf{Top}_*$ that is represented by the based loop space $\Omega(X_*)$.

1.4 Digression I

Let $ho\mathbf{ccdgC}(\mathbb{k})$ be the homotopy category of differential graded cocommuative coalgebras over $\mathbb{k}$, a field of characteristic zero such as $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

- Objects are ccdg-Coalgebras over $\mathbb{k}$;
- Morphisms are homotopy types of morphisms of ccdg-Coalgebra.
- $\mathrm{Hom}_{ho\mathbf{ccdgC}(\mathbb{k})}(C, C')$: the set of morphisms from C to C'.
- $\mathbb{k}$ is naturally a ccdg-Coalgebra, denoted by $\mathbb{k}^\vee$, with the degree concentrated to zero.

Let C_* be a ccdg-Coalgebra with a coagumentation.

- The Adams cobar construction provides us a ccdg-Hopf algebras $\Omega(C_*)$.
- We have a contravariant functor

$$\hat{\mathscr{S}}^{C_*} := \mathrm{Hom}_{ho\mathbf{ccdgC}(\mathbb{k})}\big(-, \Omega(C_*)\big) : ho\mathbf{ccd\mathring{g}C}(\mathbb{k}) \rightsquigarrow \mathbf{Grp}$$

- We call the group $\hat{\mathscr{S}}^{C_*}(\mathbb{k}^\vee) = \mathrm{Hom}_{ho\mathbf{ccdgC}(\mathbb{k})}\big(\mathbb{k}^\vee, \Omega(C_*)\big)$ the fundamental group of C_*.

In other words $\hat{\mathscr{S}}^{C_*}$ is a presheaf of groups over the homotopy category ccdg-Coalgebras that is represented by the cobar construct $\Omega(C_*)$ of C_*.

We shall use this construction for the would-be fundamental group of QFT.

1.5 Via the rational homotopy category $\mathbb{Q}ho\mathbf{Top}_*$

The rational homotopy theory, founded by Quillen and Sullivan, replace $ho\mathbf{Top}_*$ with something more manageable $\mathbb{Q}ho\mathbf{Top}_*$ after forgetting something like torsions.

- For any X_* we have an augmented cdg-Algebra $A_{PL}(X_*)_*$ over $\mathbb{Q}$ by Sullivan.
- A $\Bbbk$-rational Sullivan model of X_* is any augmented cdg-Algebra over $\Bbbk$ quasi-isomorphic to $A_{PL}(X_*) \otimes_{\mathbb{Q}} \Bbbk$
- The cdg-Algebra of smooth differential forms on a smooth based and connected manifold is a $\mathbb{R}$-rational Sullivan model.
- A coaugmented ccdg-Coalgebra C_* over $\Bbbk$ is a $\Bbbk$-rational Quillen model of X_* if its dual is a $\Bbbk$-rational Sullivan model of X_*

If C_* is a $\mathbb{R}$-rational Quillent model of a smooth based and connected manifold X_*, the group

$$\hat{\mathscr{S}}^{C_*}(\Bbbk^\vee) = \mathrm{Hom}_{ho\mathbf{ccdgC}(\Bbbk)}\big(\Bbbk^\vee, \Omega(C_*)\big)$$

is isomorphic to the pro-unipotent fundamental group $\hat{\pi}_1(X_*)$, constructed by Chen via the iterated line integrals in the dual picture.

Let A_* be a augmented cdg-Algebra of differential forms on X_*, the bar construction gives us a cdg-Hopf algebra $B(A_*)$, which define a covariant functor

$$\hat{\mathscr{G}}_{A*} := \mathrm{Hom}_{ho\mathbf{cdgA}(\mathbb{R})}\big(B(A_*), -\big) : ho\mathbf{cdgA}(\mathbb{R}) \rightsquigarrow \mathbf{Grp}$$

such that $\hat{\mathscr{G}}_{A*}(\mathbb{R}) \simeq \hat{\pi}_1(X_*)$

1.6 Digression II

Return to the presheaf of groups over the homotopy category ccdg-Coalgebras

$$\hat{\mathscr{S}}^{C_*} := \mathrm{Hom}_{ho\mathbf{ccdgC}(\Bbbk)}\big(-, \Omega(C_*)\big) : ho\mathbf{cc\mathring{d}gC}(\Bbbk) \rightsquigarrow \mathbf{Grp},$$

that is represented by the cobar construction $\Omega(C_*)$ of a coaugmented ccdg-Colalgebra C_*.

We are looking for a sort of (dual of dg) Tannakian picture. [with JH Lee]

We consider linear representations of $\hat{\mathscr{S}}^{C_*}$.

- A linear representation is equivalent to a dg-module over the ccdg-Hopf algebra $\Omega(C_*)$ over $\Bbbk$.
- Linear reprentations of $\hat{\mathscr{S}}^{C_*}$ form a tensor dg-category, which is equivalent to a tensor dg-category $\underline{\mathbf{dgMod}}(\Omega(C_*))$ of dg-modules over $\Omega(C_*)$.
- We have the forgetful functor $\boldsymbol{\varpi} : \underline{\mathbf{dgMod}}(\boldsymbol{\Omega}(\mathbf{C}_*)) \rightsquigarrow \underline{\mathbf{Ch}}(\Bbbk)$, which is a tensor dg-functor, to the underlying tensor dg-category $\underline{\mathbf{Ch}}(\Bbbk)$ of chain complexes over $\Bbbk$.
- We have a presehaf of groups over the homotopy category ccdg-Coalgebras

  $$\mathrm{Z}_0\mathbf{Aut}^{\otimes}(-\otimes \boldsymbol{\varpi}) : ho\mathbf{cc\mathring{d}gC}(\Bbbk) \rightsquigarrow \mathbf{Grp}$$

  and an isomorphism $\hat{\mathscr{S}}^{C_*} \simeq \mathrm{Z}_0\mathbf{Aut}^{\otimes}(\boldsymbol{\varpi})$.

I claim that a (perturbative) QFT with a corresponds to a linear representations of $\hat{\mathscr{S}}^{C_*}$, where the relavant coaugmented ccdg-Colagebra C_* is that of homotopy equivariant $\hbar$-de Rham coalgebra on the space of off-shell quantum observables with an infinitesimal symmetry governed by a unital sL_∞-algebra.

2 On perturbative QFT

Let $\Bbbk$ be a field of characteristic zero. We regard the Planck constant $\hbar$ as a formal parameter. Let $\mathfrak{X}$ be a $\mathbb{Z}$-graded vector space over $\Bbbk$, whose grading is specified by the ghost number *gh*.

A perturbative quantum field theory is governed by a morphism

$$\big(\mathfrak{X}[[\hbar]], 1_{\mathfrak{X}}, \boldsymbol{\ell}\big) \overset{\boldsymbol{\kappa}}{\dashrightarrow} \big(\Bbbk[[\hbar]], 1, 0\big)$$

of topologically-free unital sL_∞-algebras over $\Bbbk[[\hbar]]$:

- $\mathfrak{X}[[\hbar]]$ is the $\mathbb{Z}$-graded topologically-free $\Bbbk[[\hbar]]$-module of off-shell quantum observables with quantum symmetry encoded by the sL_∞-structure $\boldsymbol{\ell}$ whose unit $1_{\mathfrak{X}} \in \mathfrak{X}^0$ corresponds to a vacuum.
  - $\boldsymbol{\ell} : \overline{S}\big(\mathfrak{X}[[\hbar]]\big) \to \mathfrak{X}[[\hbar]]$ is a $\Bbbk[[\hbar]]$-linear map of $gh = 1$
  - $\mathbf{K} := \boldsymbol{\ell}_1 : \mathfrak{X}[[\hbar]] \to \mathfrak{X}[[\hbar]]$ is the quantum differential, making $\big(\mathfrak{X}[[\hbar]], 1_{\mathfrak{X}}, \mathbf{K}\big)$ a pointed cochain complex, i.e., $\mathbf{K} \circ \mathbf{K} = \mathbf{K}1_{\mathfrak{X}} = 0$, over $\Bbbk[[\hbar]]$.
- $\Bbbk[[\hbar]]$ with the multicative unit 1 is a unital sL_∞-algebra with the the zero sL_∞-structure 0;
- $\boldsymbol{\kappa} : \overline{S}\big(\mathfrak{X}[[\hbar]]\big) \to \Bbbk[[\hbar]]$ is a $\Bbbk[[\hbar]]$-linear map of $gh = 0$ is called *quantum cumulant functional*.
  - $\mathbf{c} := \boldsymbol{\kappa}_1 : \mathfrak{X}[[\hbar]] \to \Bbbk[[\hbar]]$ is the *quantum expectation*, which is a pointed cochain map

$$\big(\mathfrak{X}[[\hbar]], 1_{\mathfrak{X}}, \mathbf{K}\big) \xrightarrow{\ \mathbf{c}\ } \big(\Bbbk[[\hbar]], 1, 0\big)\,; \qquad \mathbf{c}(1_{\mathfrak{X}}) = 1, \quad \mathbf{c} \circ \mathbf{K} = 0.$$

  - $\boldsymbol{\kappa}_2 : S^2\mathfrak{X}[[\hbar]] \to \Bbbk[[\hbar]]$ is the *quantum covariance* and etc. etc.

The relation $\mathbf{c} \circ \mathbf{K} = 0$ says that $\mathbf{K}$ is an infinitesimal symmetry of quantum expectation $\mathbf{c}$.

A structure of sL_∞-algebra $\left(\mathfrak{X}[[\hbar]], \boldsymbol{\ell}\right)$ on $\mathfrak{X}[[\hbar]]$ is equivalent to a structure $\left(\overline{S}^{co}(\mathfrak{X}[[\hbar]]), \boldsymbol{\delta}_{\boldsymbol{\ell}}\right)$ ccdg-Coalgebra on the reduced symmetric coalgebra $\overline{S}^{co}(\mathfrak{X}[[\hbar]])$, such that an sL_∞-morphism is equivalent to a ccdg-Coalgbra map.

- From the sL_∞-structure $\boldsymbol{\ell} : \overline{S}(\mathfrak{X}[[\hbar]]) \to \mathfrak{X}[[\hbar]]$, we define $\boldsymbol{\delta}_{\boldsymbol{\ell}} : \overline{S}(\mathfrak{X}[[\hbar]]) \to \overline{S}(\mathfrak{X}[[\hbar]])$ such that, for all $n \geq 1$ and $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathfrak{X}[[\hbar]]$,

$$\boldsymbol{\delta}_{\boldsymbol{\ell}}(\mathbf{x}_1 \odot \ldots \odot \mathbf{x}_n) = \sum_{S_1 \sqcup S_2} \varepsilon(S_1 \sqcup S_2) J \mathbf{x}_{S_1} \odot \boldsymbol{\ell}(\mathbf{x}_{S_2}).$$

  Then we have $\boldsymbol{\delta}_{\boldsymbol{\ell}} \circ \boldsymbol{\delta}_{\boldsymbol{\ell}} = 0$.

- From the sL_∞-morphism $\boldsymbol{\kappa} : \overline{S}(\mathfrak{X}[[\hbar]]) \to \mathbb{k}[[\hbar]]$, we define $\boldsymbol{\beta}(\boldsymbol{\kappa}^{\hbar}) : \overline{S}(\mathfrak{X}[[\hbar]]) \to \overline{S}(\mathbb{k}[[\hbar]])$ such that, for all $n \geq 1$ and $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathfrak{X}[[\hbar]]$,

$$\boldsymbol{\beta}(\boldsymbol{\kappa})(\mathbf{x}_1 \odot \ldots \odot \mathbf{x}_n) := \sum_{r=1}^{n} \frac{1}{r!} \sum_{S_1 \sqcup \ldots \sqcup S_r = [n]} \varepsilon(\sqcup S) \boldsymbol{\kappa}(\mathbf{x}_{S_1}) \odot \ldots \odot \boldsymbol{\kappa}(\mathbf{x}_{S_r}).$$

  Then we have $\boldsymbol{\beta}(\boldsymbol{\kappa}) \circ \boldsymbol{\delta}_{\boldsymbol{\ell}} = 0$.

Corollary.

- Define $\boldsymbol{\ell}^{\hbar} : \overline{S}(\mathfrak{X}[[\hbar]]) \to \mathfrak{X}[[\hbar]]$ such that $\boldsymbol{\delta}_{\boldsymbol{\ell}^{\hbar}}(\mathbf{x}_1 \odot \ldots \odot \mathbf{x}_n) = (-\hbar)^{n-1} \boldsymbol{\delta}_{\boldsymbol{\ell}}(\mathbf{x}_1 \odot \ldots \odot \mathbf{x}_n)$. Then we also have $\boldsymbol{\delta}_{\boldsymbol{\ell}^{\hbar}} \circ \boldsymbol{\delta}_{\boldsymbol{\ell}^{\hbar}} = 0$.
- Define $\boldsymbol{\kappa}^{\hbar} : \overline{S}(\mathfrak{X}[[\hbar]]) \to \mathfrak{X}[[\hbar]]$ such that $\boldsymbol{\kappa}^{\hbar}(\mathbf{x}_1 \odot \ldots \odot \mathbf{x}_n) = (-\hbar)^{n-1} \boldsymbol{\kappa}(\mathbf{x}_1 \odot \ldots \odot \mathbf{x}_n)$. Then we also have $\boldsymbol{\beta}(\boldsymbol{\kappa}^{\hbar}) \circ \boldsymbol{\delta}_{\boldsymbol{\ell}^{\hbar}} = 0$.

The *quantum correlation functional* $\mu : S(\mathfrak{X}[[\hbar]]) \to \Bbbk[[\hbar]]$ is defined by

$$\mu := \boldsymbol{\pi}^{\Bbbk} \circ \boldsymbol{\beta}(\boldsymbol{\kappa}^{\hbar}),$$

whrere $\boldsymbol{\pi}^{\Bbbk}(\mathbf{a}_1 \odot \ldots \odot \mathbf{a}_n) = \mathbf{a}_1 \cdots \mathbf{a}_n$ for all $n \geq 1$ and $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \Bbbk[[\hbar]]$

For example, we have $\mu_1 = \boldsymbol{\kappa}_1 \equiv \mathbf{c}$ is the quantum expectation, and

$$\begin{aligned}
\mu_2(\mathbf{x}_1, \mathbf{x}_2) =& \boldsymbol{\kappa}_1(\mathbf{x}_1) \cdot \boldsymbol{\kappa}_1(\mathbf{x}_2) - \hbar \boldsymbol{\kappa}_2(\mathbf{x}_1, \mathbf{x}_2), \\
\mu_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) =& \boldsymbol{\kappa}_1(\mathbf{x}_1) \cdot \boldsymbol{\kappa}_1(\mathbf{x}_2) \cdot \boldsymbol{\kappa}_1(\mathbf{x}_3) - \hbar \boldsymbol{\kappa}_2(\mathbf{x}_1, \mathbf{x}_2) \cdot \boldsymbol{\kappa}_1(\mathbf{x}_3) - \hbar \boldsymbol{\kappa}_1(\mathbf{x}_1) \cdot \boldsymbol{\kappa}_2(\mathbf{x}_2, \mathbf{x}_3) \\
& - \hbar (-1)^{|\mathbf{x}_1||\mathbf{x}_2|} \boldsymbol{\kappa}_1(\mathbf{x}_2) \cdot \boldsymbol{\kappa}_2(\mathbf{x}_1, \mathbf{x}_3) + \hbar^2 \boldsymbol{\kappa}_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3).
\end{aligned}$$

From the condition that $\boldsymbol{\kappa}$ being an unital sL_∞-morphism, we have

- $\mu(1_{\mathfrak{X}}) = 1$ and $\mu(1_{\mathfrak{X}} \odot \mathbf{x}_1 \odot \ldots \odot \mathbf{x}_n) = \mu(\mathbf{x}_1 \odot \ldots \odot \mathbf{x}_n)$ for all $n \geq 1$;
- $\mu \circ \boldsymbol{\delta}_{\boldsymbol{\ell}^{\hbar}} = 0$, which says that $\boldsymbol{\delta}_{\boldsymbol{\ell}^{\hbar}}$ is an infinitesimal symmetry of the quantum correlation μ.

We *demand* that the *quantum corelation functional* $\mu : S(\mathfrak{X})[[\hbar]] \to \Bbbk[[\hbar]]$ factors through the *quantum expectation* $\mathbf{c} : \mathfrak{X}[[\hbar]] \to \Bbbk[[\hbar]]$ via a cochain map $\boldsymbol{\pi} : S(\mathfrak{X})[[\hbar]] \to \mathfrak{X}[[\hbar]]$, called *quantum corelator*:

$$\begin{array}{ccc} S(\mathfrak{X}[[\hbar]]) & \xrightarrow{\mu} & \Bbbk[[\hbar]] \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle \mathbf{c}} & \\ \mathfrak{X}[[\hbar]] & & \end{array}, \qquad \mu = \mathbf{c}\circ\boldsymbol{\pi}, \qquad \boldsymbol{\pi}\circ\boldsymbol{\delta}_{\hbar\boldsymbol{\ell}} = \mathbf{K}\circ\boldsymbol{\pi}. \tag{1}$$

Note that $\boldsymbol{\pi}_1 = \mathbb{I}_{\mathfrak{X}[[\hbar]]}$ Combining with the definition $\mu := \boldsymbol{\pi}^{\Bbbk}\circ\boldsymbol{\beta}(\boldsymbol{\kappa}^{\hbar})$, we have $\boldsymbol{\pi}^{\Bbbk}\circ\boldsymbol{\beta}(\boldsymbol{\kappa}^{\hbar}) = \mathbf{c}\circ\boldsymbol{\pi}$.

If we fix a quantum correlator $\boldsymbol{\pi}$, we have

- the sL_∞-structure $\boldsymbol{\ell}$ is determined by the quantum differential $\mathbf{K}$ by the formula $\boldsymbol{\pi}\circ\boldsymbol{\delta}_{\hbar\boldsymbol{\ell}} = \mathbf{K}\circ\boldsymbol{\pi}$. For example, we have $\boldsymbol{\ell}_1 = \mathbf{K}$ and

$$\begin{aligned} -\hbar\,\boldsymbol{\ell}_2(\mathbf{x}_1,\mathbf{x}_2) &= \mathbf{K}\boldsymbol{\pi}_2(\mathbf{x}_1,\mathbf{x}_2) - \boldsymbol{\pi}_2(\mathbf{K}\mathbf{x}_1,\mathbf{x}_2) - \boldsymbol{\pi}_2(J\mathbf{x}_1,\mathbf{K}\mathbf{x}_2),\\ \hbar^2\,\boldsymbol{\ell}_3(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) &= \mathbf{K}\boldsymbol{\pi}_3(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) - \boldsymbol{\pi}_3(\mathbf{K}\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3) - \boldsymbol{\pi}_3(J\mathbf{x}_1,\mathbf{K}\mathbf{x}_2,\mathbf{x}_3) - \boldsymbol{\pi}_3(J\mathbf{x}_1,J\mathbf{x}_2,\mathbf{K}\mathbf{x}_3)\\ &\quad - \hbar\boldsymbol{\pi}_2(\boldsymbol{\ell}_2(\mathbf{x}_1,\mathbf{x}_2),\mathbf{x}_3) - \hbar\boldsymbol{\pi}_2(J\mathbf{x}_1,\boldsymbol{\ell}_2(\mathbf{x}_2,\mathbf{x}_3)) - \hbar(-1)^{|\mathbf{x}_1||\mathbf{x}_2|}\boldsymbol{\pi}_2(J\mathbf{x}_2,\boldsymbol{\ell}_2(\mathbf{x}_1,\mathbf{x}_3)). \end{aligned}$$

- the quantum cumulant functional $\kappa : S(\mathfrak{X})[[\hbar]] \to \Bbbk[[\hbar]]$ is determined by the quantum expectation $\mathbf{c}$ by the formula $\boldsymbol{\pi}^{\Bbbk}\circ\boldsymbol{\beta}(\boldsymbol{\kappa}^{\hbar}) = \mathbf{c}\circ\boldsymbol{\pi}$. For example, we have $\kappa_1 = \mathbf{c}$ and

$$\begin{aligned} -\hbar\boldsymbol{\kappa}_2(\mathbf{x}_1,\mathbf{x}_2) &= \mathbf{c}(\boldsymbol{\pi}_2(\mathbf{x}_1,\mathbf{x}_2)) - \mathbf{c}(\mathbf{x}_1)\cdot\mathbf{c}(\mathbf{x}_2),\\ \hbar^2\boldsymbol{\kappa}_3(\mathbf{x}_1,\mathbf{x}_2) &= \mathbf{c}(\boldsymbol{\pi}_3(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)) - \mathbf{c}(\mathbf{x}_1)\cdot\mathbf{c}(\mathbf{x}_2)\cdot\mathbf{c}(\mathbf{x}_3) + \hbar\boldsymbol{\kappa}_2(\mathbf{x}_1,\mathbf{x}_2)\cdot\mathbf{c}(\mathbf{x}_3) + \hbar\mathbf{c}(\mathbf{x}_1)\cdot\boldsymbol{\kappa}_2(\mathbf{x}_2,\mathbf{x}_3)\\ &\quad + \hbar(-1)^{|\mathbf{x}_1||\mathbf{x}_2|}\mathbf{c}(\mathbf{x}_2)\cdot\boldsymbol{\kappa}_2(\mathbf{x}_1,\mathbf{x}_3). \end{aligned}$$

Finally, we impose the following $\hbar$-condition to a quantum correlator $\boldsymbol{\pi}: S(\mathfrak{X})[[\hbar]] \to \mathfrak{X}[[\hbar]]$:

- There is a family $\mathbf{m}_2^{\mathrm{O}}, \mathbf{m}_3^{\mathrm{O}}, \ldots$ of $\Bbbk[[\hbar]]$-linear maps

$$\mathbf{m}_{k+2}^{\mathrm{O}}: S^k\mathfrak{X}[[\hbar]] \otimes \mathfrak{X}[[\hbar]] \otimes \mathfrak{X}[[\hbar]] \to \mathfrak{X}[[\hbar]] \qquad k = 0, 1, 2, \ldots$$

  such that

$$\boldsymbol{\pi}_{n+2}(\boldsymbol{x}_1 \odot \ldots \odot \boldsymbol{x}_n \odot \boldsymbol{y} \otimes \boldsymbol{z}) = \sum_{S_1 \sqcup S_2 = [n]} (-\hbar)^{n-|S_1|} \varepsilon(S_1 \sqcup S_2)\ \boldsymbol{\pi}_{|S_1|+1}\big(\boldsymbol{x}_{S_1} \odot \boldsymbol{m}_{|S_2|+2}^{\mathrm{O}}\big(\boldsymbol{x}_{S_2} \otimes y \otimes z\big)\big)$$

- For example we have $\mathbf{m}_2^{\mathrm{O}} = \boldsymbol{\pi}_2$ and

$$\begin{aligned} \mathbf{m}_2^{\mathrm{O}} &= \boldsymbol{\pi}_2, \\ -\hbar \mathbf{m}_3^{\mathrm{O}} &= \boldsymbol{\pi}_3 - \mathbf{m}_2^{\mathrm{O}} \circ \big(\mathbb{I} \otimes \mathbf{m}_2^{\mathrm{O}}\big) \end{aligned}$$

  etc.

Finally we call the resulting tuple $\mathfrak{X}_{QFTA} = \big(\mathfrak{X}[[\hbar]], 1_{\mathfrak{X}}, \boldsymbol{\ell}, \boldsymbol{m}_2^{\mathrm{O}}, \boldsymbol{m}_3^{\mathrm{O}}, \ldots\big)$ a strucuture of (pertubative) QFT algebra on $\mathfrak{X}$ over $\Bbbk$.

Some examples.

- A QFT algebra is called binary if $\boldsymbol{m}_k^0 = 0$ for all $k \geq 3$.
- A binary QFT algebra $\big(\mathfrak{X}[[\hbar]], 1_{\mathfrak{X}}, \boldsymbol{\ell}, \boldsymbol{m}_2^0\big)$ is a BV-QFT algebra if $\boldsymbol{m}_2^0$ does not depends on $\hbar$ and $\boldsymbol{\ell}_k = 0$ for all $k \geq 3$. For example:

$$K = -\hbar\Delta_{BV} + (S, -)_{BV}, \qquad \boldsymbol{\ell}_2 = (-,-)_{BV}$$

  where Δ_{BV} is the BV operator and $(-,-)_{BV}$ is the BV-bracket, and, S is a quantum master action $-\hbar\Delta + \frac{1}{2}(S,S) = 0$.
- $\Bbbk_{QFTA} = \big(\Bbbk[[\hbar]], 1, 0, \cdot\big)$ is a QFT algebra.

We have the natural notions of morphisms of QFT algebras and homotopy of morphisms, so that we can form the (homotopy)category (ho)**QFTA**$(\Bbbk)$ QFT algebras over $\Bbbk$.

A quantum expectation $\mathbf{c} : \mathfrak{X}[[\hbar]] \to \Bbbk[[\hbar]]$ is a morphism of QFT algebra from $\mathfrak{X}_{QFTA}$ to $\Bbbk_{QFTA}$. A QFT is such an arrow $\mathfrak{X}_{QFTA} \xrightarrow{\ \mathbf{c}\ } \Bbbk_{QFTA}$.

There is an ample room as well as computational needs to extend the notion of QFT-algebra to that of homotopy QFT-algebra.

Let $C^{co}\big(\mathfrak{X}[[\hbar]]\big) = S^{co}\big(\mathfrak{X}[[\hbar]]\big) \otimes \Lambda^{co}\big(\mathfrak{X}[[\hbar]]\big)$ be the tensor product of the symmetric and exterior coalgebras over $\Bbbk[[\hbar]]$ generated by $\mathfrak{X}[[\hbar]]$.

- bigraded by (gh,fm) where an element in $C_k\big(\mathfrak{X}[[\hbar]]\big) := \Lambda^k\mathfrak{X}[[\hbar]] \otimes S\big(\mathfrak{X}[[\hbar]]\big)$ is assigned to $fm = -k$.
- Koszul differential $\partial : C_k\big(\mathfrak{X}[[\hbar]]\big) \to C_{k-1}\big(\mathfrak{X}[[\hbar]]\big)$ of $(gh,fm) = (0,1)$ such that $\big(C^{co}\big(\mathfrak{X}[[\hbar]]\big), -\hbar\partial\big)$ is a ccdg-Coalgebra.

From an sL_∞-structure $\big(\mathfrak{X}[[\hbar]], \boldsymbol{\ell}\big)$ of on $\mathfrak{X}[[\hbar]]$, we have a ccdg-Coalgebra $\big(S^{co}\big(\mathfrak{X}[[\hbar]]\big), \boldsymbol{\delta}_{\boldsymbol{\ell}}\big)$ over $\Bbbk[[\hbar]]$.

Then the differential $\boldsymbol{\delta}_{\boldsymbol{\ell}}$ on $S\big(\mathfrak{X}[[\hbar]]\big)$ has a unique extension to a differential $\boldsymbol{\mathfrak{D}}_{\boldsymbol{\ell}}$ on $C\big(\mathfrak{X}[[\hbar]]\big)$ with $(gh,fm) = (1,0)$ such that

- $\big(C\big(\mathfrak{X}[[\hbar]]\big), \boldsymbol{\mathfrak{D}}_{\boldsymbol{\ell}}\big)$ is a dg-comodule over the ccdg-Coalgebra $\big(S^{co}\big(\mathfrak{X}[[\hbar]]\big), \boldsymbol{\delta}_{\boldsymbol{\ell}}\big)$;
- $\big(C^{co}\big(\mathfrak{X}[[\hbar]]\big), -\hbar\partial + \boldsymbol{\mathfrak{D}}_{\boldsymbol{\ell}}\big)$ is a ccdg-Coalgebra over $\Bbbk[[\hbar]]$.

We use the notation

$$C_{\hbar dR}\big(\mathfrak{X}[[\hbar]], \boldsymbol{\ell}\big) = \big(C^{co}\big(\mathfrak{X}[[\hbar]]\big), -\hbar\partial + \boldsymbol{\mathfrak{D}}_{\boldsymbol{\ell}}\big)$$

call it the *homotopy equivariant $\hbar$-de Rham coalgebra* cogenerated by the sL_∞-algebra $\big(\mathfrak{X}[[\hbar]], \boldsymbol{\ell}\big)$.

Restoring a unit $1_{\mathfrak{X}}$, we are led to completed and coaugmented homotopy equivariant $\hbar$-de Rham Coalgebra $\widehat{C}^*_{\hbar dR}\big(\mathfrak{X}[[\hbar]], 1_{\mathfrak{X}}, \boldsymbol{\ell}\big)$, whose coaugmentation is induced from $1_{\mathfrak{X}}$.

A cotwisting coefficient system over $C_{\hbar dR}\big(\mathfrak{X}[[\hbar]],\boldsymbol{\ell}\big)$ is a tuple $\big(\mathscr{V}[[\hbar]],\boldsymbol{\omega}\big)$ where

- $\mathscr{V}[[\hbar]]$ is a cochain complex over $\Bbbk[[\hbar]]$ and
- $\boldsymbol{\omega} : C\big(\mathfrak{X}[[\hbar]]\big)\otimes\mathscr{V}[[\hbar]] \to \mathscr{V}[[\hbar]]$ is a cotwisting matrix of the total degree $gh+fm = 1$, satisfying the following integrability condition $\mathscr{R}(\boldsymbol{\omega})=0$, where

$$\mathscr{R}(\boldsymbol{\omega}) := d_{\mathscr{V}[[\hbar]]}\circ\boldsymbol{\omega}+\boldsymbol{\omega}\circ\Big(-\hbar\partial\otimes\mathbb{I}+\mathfrak{D}_{\boldsymbol{\ell}}\otimes\mathbb{I}+\mathbb{I}\otimes d_{\mathscr{V}[[\hbar]]}\Big)+\boldsymbol{\omega}\circ(\mathbb{I}\otimes\boldsymbol{\omega})\circ(\triangle\otimes\mathbb{I})$$

See the diagram

$$C\big(\mathfrak{X}[[\hbar]]\big)\otimes\mathscr{V}[[\hbar]] \xrightarrow{\triangle\otimes\mathbb{I}} C\big(\mathfrak{X}[[\hbar]]\big)\otimes C\big(\mathfrak{X}[[\hbar]]\big)\otimes\mathscr{V}[[\hbar]] \xrightarrow{\mathbb{I}\otimes\boldsymbol{\omega}} C\big(\mathfrak{X}[[\hbar]]\big)\otimes\mathscr{V}[[\hbar]] \xrightarrow{\boldsymbol{\omega}} \mathscr{V}[[\hbar]]$$

Equivalently, we have the cotwisted cofree comodule $\big(C\big(\mathfrak{X}[[\hbar]]\big)\otimes\mathscr{V}[[\hbar]],{}^{\hbar}\boldsymbol{\nabla}^{\omega}\big)$ over $C_{\hbar dR}\big(\mathfrak{X}[[\hbar]],\boldsymbol{\ell}\big)$ with the cotwisted differential

$${}^{\hbar}\boldsymbol{\nabla}^{\omega} = -\hbar\partial\otimes\mathbb{I}+\mathfrak{D}_{\boldsymbol{\ell}}\otimes\mathbb{I}+\mathbb{I}\otimes d_{\mathscr{V}[[\hbar]]}+(\mathbb{I}\otimes\boldsymbol{\omega})\circ(\triangle\otimes\mathbb{I})$$

satisfying ${}^{\hbar}\boldsymbol{\nabla}^{\omega}\circ{}^{\hbar}\boldsymbol{\nabla}^{\omega}=0$.

We say such a cotwising coefficient system over $C_{\hbar dR}\big(\mathfrak{X}[[\hbar]],\boldsymbol{\ell}\big)$ is tangential if $\mathscr{V}[[\hbar]]=\mathfrak{X}[[\hbar]]$.

A homotopy QFT algebra is a tangential cotwisting coefficient system $\left(\mathfrak{X}[[\hbar]], \boldsymbol{\omega}\right)$ over the completed and coaugmented homotopy equivariant $\hbar$-de Rham coalgebra $\widehat{C}^*_{\hbar dR}\left(\mathfrak{X}[[\hbar]], 1_{\mathfrak{X}}, \boldsymbol{\ell}\right)$ cogenerated by a sL_∞-algebra $\left(\mathfrak{X}[\hbar]], \boldsymbol{\ell}\right)$ over $\Bbbk[[\hbar]]$.

Note that the total degree of $\boldsymbol{\omega} : \widehat{C}\left(\mathfrak{X}[[\hbar]]\right)\hat{\otimes}\mathfrak{X}[[\hbar]] \to \mathfrak{X}[[\hbar]]$ is $gh + fm = 1$. Decomposing $\widehat{\boldsymbol{\omega}}$ into $\boldsymbol{\omega}_k : C_k\left(\mathfrak{X}[[\hbar]]\right) \otimes \mathfrak{X}[[\hbar]] \to \mathfrak{X}[[\hbar]]$ of degree $(gh, fm) = (1-k, k)$, we obtain a family $\boldsymbol{\omega}_0, \boldsymbol{\omega}_1, \widehat{\boldsymbol{\omega}}_2, \ldots$.

Define $\boldsymbol{m}^{1-k}_{n+k+1} : S^n\mathfrak{X}[[\hbar]] \otimes \Lambda^n\mathfrak{X}[[\hbar]] \otimes \mathfrak{X}[[\hbar]] \to \mathfrak{X}[[\hbar]]$ of $gh = 1-k$ such that

$$\boldsymbol{m}^{1-k}_{n+k+1} = \widehat{\omega}_k\left(\boldsymbol{x}_1 \odot \ldots \boldsymbol{x}_n \otimes \boldsymbol{x}_{n+1} \wedge \ldots \wedge \boldsymbol{x}_{n+k} \otimes \boldsymbol{x}_{n+k+1}\right)$$

We obtain the following set of multi-linear operations on $\mathfrak{X}[[\hbar]]$, indexed by the ghost number and arity:

	$gh\backslash arity$	1	2	3	4	$\cdots$
$\boldsymbol{\ell}$:	$+1$	K	$\boldsymbol{\ell}_2$	$\boldsymbol{\ell}_3$	$\boldsymbol{\ell}_4$	$\cdots$
$\boldsymbol{\omega}_1$:	0		$\boldsymbol{m}^0_2$	$\boldsymbol{m}^0_3$	$\boldsymbol{m}^0_4$	$\cdots$
$\boldsymbol{\omega}_2$:	-1			$\boldsymbol{m}^{-1}_3$	$\boldsymbol{m}^{-1}_4$	$\cdots$
$\boldsymbol{\omega}_3$:	-2				$\boldsymbol{m}^{-2}_4$	$\cdots$
$\vdots$	$\vdots$					$\ddots$

satisfying the set of relations summarized by the integrability ${}^{\hbar}\boldsymbol{\nabla}^{\omega} \circ {}^{\hbar}\boldsymbol{\nabla}^{\omega} = 0$.

We can form the (homotopy)category $(ho)\mathbf{QFTA}_\infty(\Bbbk)$ of homotopy QFT-algebras. Many nice things happen there...

But we have also opened a door to something unknown.