Normal numbers with digit dependencies

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Uniform distribution sequences
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Expansion of a real number in an integer base

For a real number x, its expansion in an integer base $b \geqslant 2$ is a sequence of integers $a_1, a_2 \ldots$, where $0 \leqslant a_n < b$ for every n, such that

$$x - \lfloor x \rfloor = \sum_{n \geqslant 1} a_n b^{-n} = 0.a_1 a_2 a_3 \ldots$$

We require that $a_n < b - 1$ infinitely often to ensure that every number has a unique representation.

Borel normal numbers

Let integer $b \geqslant 2$. A real number x is simply normal to base b if every digit in $\{0, \ldots, b-1\}$ occurs in the base-b expansion of x with the same asymptotic frequency (that is, with frequency $1/b$).

A real number x is normal to base b if it is simply normal to all the bases $b, b^2, b^3, \ldots$.

A real number x is absolutely normal if it is normal to all integer bases.

Normality to base b is equivalent to normality to any base multiplicatively dependent to b (two integers are multiplicatively dependent if one is a rational power of the other).

Borel proved that almost all numbers, with respect to Lebesgue measure, are absolutely normal.

Examples and counterexamples of Borel normal numbers

$0.01010101010\ldots$ is simply normal to base 2 but not to 2^2 nor 2^3, etc.

Each number in Cantor middle third set is not simply normal to base 3.

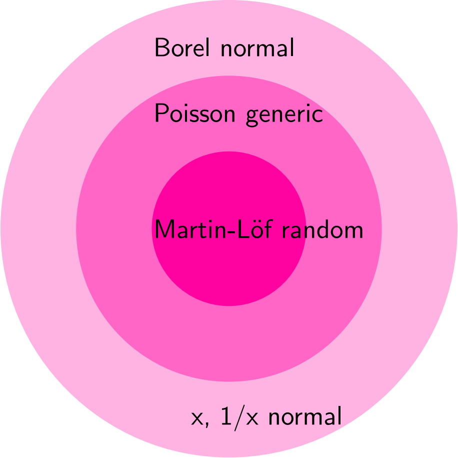
Champernowne's number in base 10

$$0.12345679101112131415161718192021$$

is normal to base 10, but is not known whether it is normal to bases multiplicatively independent to 10.

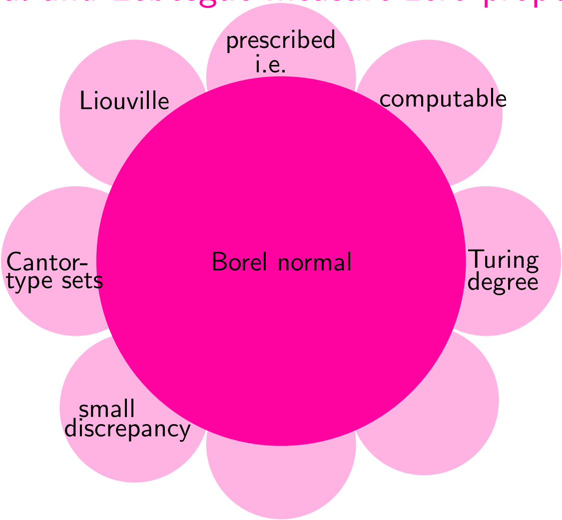
Stoneham number $\alpha_{2,3} = \sum_{n \geqslant 1} \frac{1}{3^n \ 2^{3^n}}$ is normal to base 2 but not simply normal to base 6 (Bailey and Borwein, 2012).

Borel normal numbers and other properties of full measure



Continued fraction normality, normality for other numeration systems such as Pisot bases.

Borel normal and Lebesgue measure zero properties



Turing(1937), Cassels (1959), Schmidt (1961/1962), Bugeaud (2002), Levin (1999)

Conjecture (Borel 1951)

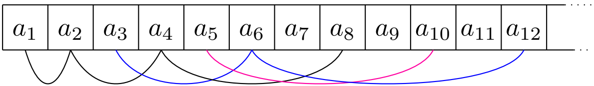
All algebraic irrational numbers are absolutely normal.

Toeplitz numbers (Jacobs and Keane 1969)

Let integer $b \geqslant 2$ and let $\mathcal{P}$ be a set of prime numbers.

The set of Toeplitz numbers $\mathcal{T}_{b,\mathcal{P}}$ is the set of all real numbers $x \in [0,1)$ such that if $x = \sum_{n \geqslant 1} a_n b^{-n}$ then $a_n = a_{np} \qquad (n \geqslant 1,\, p \in \mathcal{P})$.

For example $0.a_1 a_2 a_3 \ldots$ is a Toeplitz number for $\mathcal{P} = \{2\}$ if, for every $n \geqslant 1$, we have $a_n = a_{2n}$,



$a_1 = a_2 = a_4 = a_8 = \ldots$ (black)
$a_3 = a_6 = a_{12} = a_{24} = \ldots$ (blue)
$a_5 = a_{10} = a_{20} = a_{40} = \ldots$ (pink)
The variables with odd indices are independent.

Uniform measure on $\mathcal{T}_{b,\mathcal{P}}$

Let integer $b \geqslant 2$ and let $\mathcal{P}$ be a set of prime numbers.

Let $j_1, j_2, \ldots$ be the enumeration in increasing order of all positive integers that are not divisible by any of the primes in $\mathcal{P}$.

The Toeplitz transform $\tau_{b,\mathcal{P}} : [0,1) \to \mathcal{T}_{b,\mathcal{P}}$ is defined as

$$\tau_{b,\mathcal{P}}(0.b_1 b_2 b_3 \ldots) = 0.a_1 a_2 a_3 \ldots$$

such that when $n = j_k p_1^{e_1} \cdots p_r^{e_r} \quad (p_1, \cdots, p_r \in \mathcal{P})$,

$$a_n = b_k.$$

Let μ be the uniform probability measure on $\mathcal{T}_{b,\mathcal{P}}$, which is the push-forward of the Lebesgue measure λ by $\tau_{b,\mathcal{P}}$
For measurable $X \subseteq \mathcal{T}_{b,\mathcal{P}}$,

$$\mu(X) = \lambda(\tau_{b,\mathcal{P}}^{-1}(X)).$$

Theorem 1 (Aistleitner, Becher and Carton)

Let integer $b \geqslant 2$, $\mathcal{P} = \{2\}$ *and* μ *be the uniform probability measure on* $\mathcal{T}_{b,\mathcal{P}}$. *Then,* μ*-almost all elements of* $\mathcal{T}_{b,\mathcal{P}}$ *are absolutely normal.*

Proof of Theorem 1

For bases that are multiplicatively independent to b we adapt the work of Cassels 1959 and Schmidt 1961/1962 giving upper bounds for certain Riesz products.

Cassels worked on a Cantor-type set of real numbers whose ternary expansion avoids the digit 2 (hence not normal to base 3) and uniform measure supported on this Cantor-type set.

We deal with the set $\mathcal{T}_{b,\mathcal{P}}$ and the uniform measure μ.

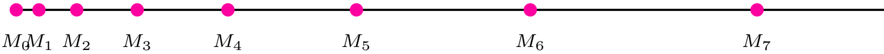
Proof of Theorem 1

Let positive intreger r be multiplicatively independent to b.
To prove μ-almost all $x \in \mathcal{T}_{b,\mathcal{P}}$ are normal to base r, by Weyl's criterion, show that the sequence of fractional parts of $x, rx, r^2x, \ldots$ is u.d. in $[0,1]$.
That is, we have to show that for every non-zero integer h,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} e(r^j hx) = 0.$$

where,as usual, $e(x) = e^{2\pi i x}$.

Let M_k for $k = 1, 2, 3\ldots$ be subexponential (such as $M_k = \sum_{j=1}^{k} \lceil e^{\sqrt{j}} \rceil$).
so that for every N there is a k such that $N - M_k$ is $o(N)$.



Show for every k large enough,

$$\frac{1}{M_k - M_{k-1}} \sum_{M_{k-1}\leqslant j < M_k} e(r^j hx) < \varepsilon_k$$

with $\varepsilon_k \to 0$, as $k \to \infty$

1 Fix $b, \mathcal{P}, r$. Fix a positive h. For each k define

$$Bad_k := \left\{ x \in \mathcal{T}_{b,\mathcal{P}} : \frac{1}{M_k - M_{k-1}} \sum_{M_{k-1} \leqslant j < M_k} e(r^j hx) > 1/k. \right\}$$

2 Show $\mu(Bad_k)$ is small enough to obtain a convergent series $\sum_k \mu(Bad_k)$.

By generalized Markov/Chebyshev inequality

$$\boxed{\mu(\{x \in X \ : \ |f(x)| > t\}) \ < \ 1/t^2 \int_X |f(x)|^2 \, d\mu(x)}$$

Our main lemma gives a suitable upper bound for

$$\int_0^1 \Big| \sum_{M_{k-1} \leqslant j < M_k} e(r^j hx) \Big|^2 d\mu(x)$$

3 Apply Borel Cantelli, obtain μ-almost all $x \in \mathcal{T}_{b,\mathcal{P}}$ are outside $\bigcup_k Bad_k$.

4 For any N there is k such that $N - M_k = o(N)$.Then, μ-almost all $x \in \mathcal{T}_{b,\mathcal{P}}$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} e(r^j hx) = 0.$$

5 Countably many h and r multiplicatively independent to b. $\square$

Proof of Theorem 1

Recall μ is the uniform probability measure on $\mathcal{T}_{b,\mathcal{P}}$.

Lemma

Let $r \geqslant 2$ be multiplicatively independent to b.
Then for all integers $h \geqslant 1$ there exist constants $c > 0$ and $\ell_0 > 0$, depending only on b, r and h such that for all positive integers ℓ, m satisfying $\ell \geqslant m + 1 + 2\log_r b \geqslant m_0$,

$$\int_0^1 \left| \sum_{j=\ell+1}^{\ell+m} e(r^j h x) \right|^2 d\mu(x) \leqslant m^{2-c}.$$

Proof of Theorem 1

Using Euler's formula $e^{ix} = \cos x + i \sin x$, the Riesz-like product appears.

Lemma (adapted from Schmidt's Hilfssatz 5, 1961)

Let r and b be multiplicatively independent. There is $c > 0$, depending only on r and b, such that for all positive integers K and L with $L \geqslant b^K$,

$$\sum_{n=0}^{N-1} \prod_{\substack{k=K+1 \\ k \text{ odd}}}^{\infty} \left(\frac{1}{b} + \frac{b-1}{b} \left| \cos\left(\pi r^n L b^{-k}\right) \right| \right) \leqslant 2N^{1-c}.$$

Lemma (Schmidt's Hilfssatz 5, 1961)

Let r and b be multiplicatively independent. There is $c_* > 0$, depending only on r and b, such that for all positive integers K and L with $L \geqslant b^K$,

$$\sum_{n=0}^{N-1} \prod_{k=K+1}^{\infty} |\cos(\pi r^n L b^{-k})| \leqslant 2N^{1-c_*}.$$

Schmidt's proof uses $|\cos(\pi x)|$ is periodic, $|\cos(\pi x)| \leqslant 1$ and $|\cos(\pi/b^2)| < 1$.

All these properties also hold for $\dfrac{1}{b} + \dfrac{b-1}{b} |\cos(\pi x)|$.

To finish the proof of Theorem 1 we must show that μ-almost all numbers in $\mathcal{T}_{b,\mathcal{P}}$ are normal to base b. This is proved in Theorem 2. $\square$

Theorem 2 (Aistleitner, Becher and Carton)

Let integer $b \geqslant 2$, let $\mathcal{P}$ be a finite set of primes and let μ be the uniform probability measure on $\mathcal{T}_{b,\mathcal{P}}$. Then, μ-almost all elements of $\mathcal{T}_{b,\mathcal{P}}$ are normal to base b.

For $\mathcal{P} = \{2\}$ was obtained by Alexander Shen (2016), and by Lingmin Liao and Michal Rams (2021).

Proof of Theorem 2

Recall $\tau_{b,\mathcal{P}} : [0,1) \to \mathcal{T}_{b,\mathcal{P}}$ is defined as $\tau_{b,\mathcal{P}}(0.b_1 b_2 \ldots) = 0.a_1 a_2 \ldots$ such that when $n = j_k p_1^{e_1} \cdots p_r^{e_r}$, $(p_1, \ldots, p_r \in \mathcal{P})$, $a_n = b_k$.

Definition

Let $\mathcal{P} = \{p_1, \ldots, p_r\}$ be a set of r primes. The equivalence relation $\sim$ on the set of positive integers is defined as follows: $n \sim n'$ whenever there are exponents $e_1, \ldots e_r, e'_1, \ldots e'_r$ and a positive integer k such that

j is not divisible by any $p \in \mathcal{P}$,

$n = j \cdot p_1^{e_1} \ldots p_r^{e_r}$ and $n' = j \cdot p_1^{e'_1} \ldots p_r^{e'_r}$.

For example, for $\mathcal{P} = \{2, 3\}$, $2 \sim 3$, $3 \sim 36$, $36 \not\sim 5$.

Lemma (follows from Tijdeman 1973)

There is n_0 such that if $n' \sim n$ and $n' > n > n_0$, then $n' - n > 2\sqrt{n}$.

The Toeplitz transform $\tau_{b,\mathcal{P}}$ induces a function $\delta: \mathbb{N} \mapsto \mathbb{N}$

$$\tau_{b,\mathcal{P}}(0.b_1 b_2 b_3 \cdots) = 0.a_1 a_2 a_3 \cdots = 0.b_{\delta(1)} b_{\delta(2)} b_{\delta(3)} \cdots$$

where $\delta(n) = \delta(n')$ exactly when $n \sim n'$. By the previous lemma,

$$\delta(n), \delta(n+1), \ldots, \delta(n + 2\lfloor\sqrt{n}\rfloor)$$

are pairwise different.

For each n, consider $a_n(x)$ as a random variable on space $([0,1), \mathcal{B}(0,1), \lambda)$. Since $a_n(x) = b_{\delta(n)}(x)$ for all n, two random variables a_n and $a_{n'}$ are independent, with respect to λ and μ, if and only if, $\delta(n) \neq \delta(n')$.

Thus, $b_{\delta(n)}, b_{\delta(n+1)}, \ldots, b_{\delta(n+2\lfloor\sqrt{n}\rfloor)}$ are mutually independent.

At every position n the number of independent variables is $2\lfloor\sqrt{n}\rfloor$ which exceeds the minimal required to ensure normality, established in Theorem 3. $\square$

Theorem 3 (Aistleitner, Becher and Carton)

Let integer $b \geqslant 2$. Let $X_1, X_2, \ldots$ be a sequence of random variables from a given probability space $(\Omega, \mathcal{F}, P)$ into $\{0, .., b-1\}$.

Assume that for every n, X_n is uniformly distributed on $\{0, .., b-1\}$. Suppose there is a function $g: \mathbb{N} \mapsto \mathbb{R}$ unbounded and monotonically increasing such that for all sufficiently large n the random variables

$$X_n,\ X_{n+1},\ \ldots,\ X_{n+\lceil g(n)\log\log n\rceil}$$

are mutually independent. Then, P-almost surely $x = 0.X_1X_2\ldots$ is normal to base b.

Furthermore, $\lceil g(n)\log\log n\rceil$ is sharp.

Example of a simply normal number to base b in $\mathcal{T}_{b,\mathcal{P}}$

Let $\mathbb{P}$ be the set of primes and let $\mathcal{P} \subset \mathbb{P}$. Define the completely additive arithmetical funtion $\Omega_{\mathcal{P}} : \mathbb{N} \to \mathbb{N}$, $\Omega_{\mathcal{P}}(n)$ is the sum of the exponents in the factorization of n of those prime factors that are *not* in $\mathcal{P}$.

For example, for $\mathcal{P} = \{2, 3\}$,
$\Omega_{\mathcal{P}}(2) = \Omega_{\mathcal{P}}(3) = \Omega_{\mathcal{P}}(6) = \Omega_{\mathcal{P}}(8) = 0$; $\Omega_{\mathcal{P}}(5) = \Omega_{\mathcal{P}}(10) = 1$; $\Omega_{\mathcal{P}}(2^4 \cdot 3^6 \cdot 5^2 \cdot 7) = 3$

Given $\mathcal{P} \subset \mathbb{P}$ and integer $b \geqslant 2$, the number

$$x := \sum_{n \geqslant 1} t_n b^{-n}$$

where

$$t_n := (\Omega_{\mathcal{P}}(n) \mod b).$$

Clearly, $x \in \mathcal{T}_{b,\mathcal{P}}$.

Theorem 4 (Becher, Marchionna and Tenenbaum 2023)

Let integer $b \geqslant 2$ and $\mathcal{P} \subset \mathbb{P}$. The number x is simply normal to base b if, and only if, $\sum_{p \in (\mathbb{P} \setminus \mathcal{P})} 1/p = \infty$. *Moreover, defining for* $k = 0, \ldots, (b-1)$

$$\varepsilon_{N,k} := \left| \frac{1}{N} |\{n : 1 \leqslant n \leqslant N, (\Omega_{\mathcal{P}}(n) \mod b) = k\}| - \frac{1}{b} \right|$$

we have

$$\varepsilon_{N,k} \ll \frac{1}{b} \mathrm{e}^{-E(N)/180b^2}, \text{ where } E(N) := \sum_{p \leqslant N,\, p \in (\mathbb{P} \setminus \mathcal{P})} 1/p \quad (N \geqslant 1)$$

Proof uses Tenenbaum's quantitative version of effective mean-value estimates for arithmetic functions (quantitative versions of Wirsing's estimates).

Since $E(N)$ is asymptotyically equal to $\log\log N + M$, where M is Meissel-Mertens constant,

$$\varepsilon_{N,k} \ll \frac{1}{2} e^{-(\log\log N + M)/720} = \frac{1}{2} e^{-M/720} (\log N)^{-1/720}.$$

This decays extremely slowly, like a very small negative power of $\log N$. (Champernowne constant has discrepancy exactly in the order $(\log N)^{-1}$)

Example of a normal number in $\mathcal{T}_{b,\mathcal{P}}$ for singleton $\mathcal{P}$

Theorem 5 (Becher, Carton and Heiber 2018)

We construct a number in $\mathcal{T}_{b,\mathcal{P}}$ for $b = 2$ and $\mathcal{P} = \{2\}$, normal to base 2.

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