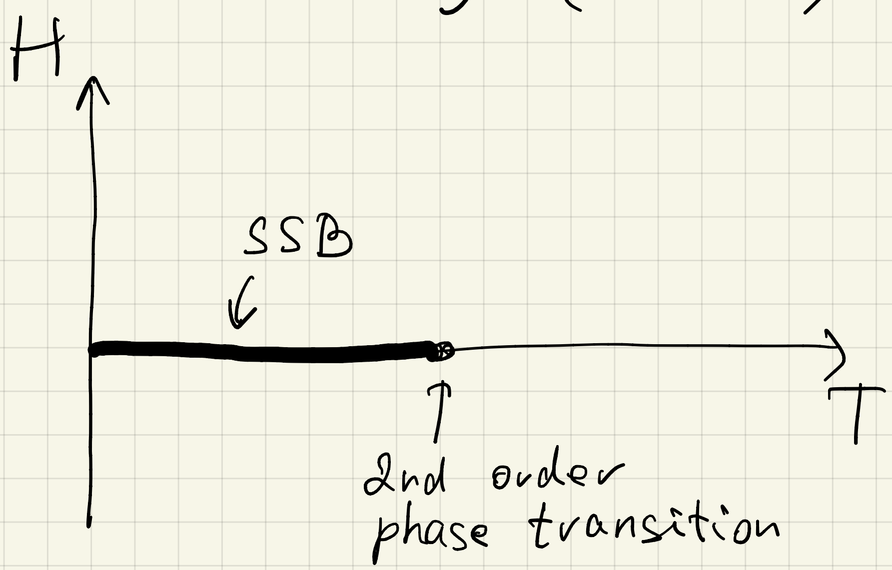
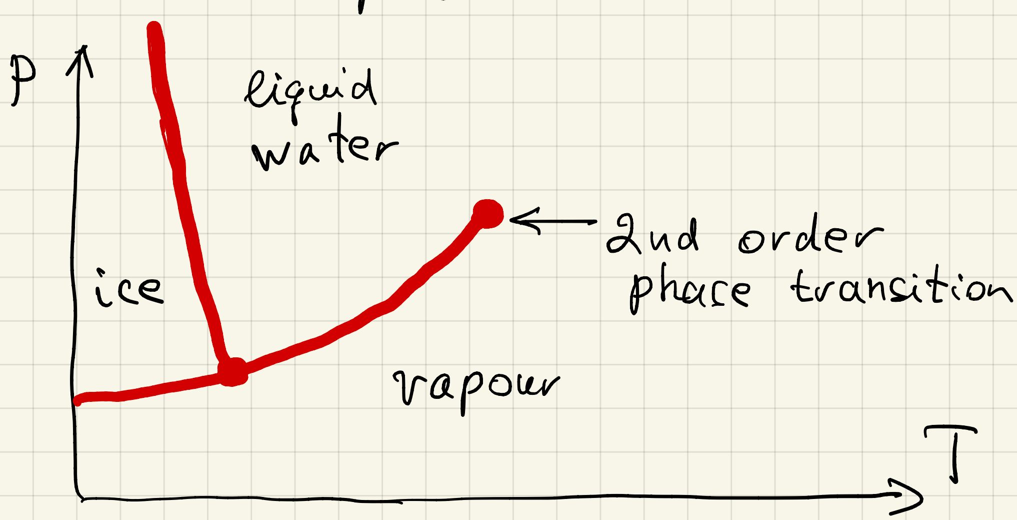
0. Gapped many-body systems and TQFT.

Many-body physics = physics of ∞-volume systems.

Key facts

- Sharp phase transitions as one varies parameters
- Possibility of spontaneous symmetry breaking (SSB)





Quantum phase transitions = transitions at $T=0$.

Phase transitions separate phases.

What are possible phases at $T=0$?

Quantum phases

- **gapless** (local excitations with arbitrarily small energy)
- **gapped**
  - Long-Range Entangled (**LRE**)
    - ???
    - Topologically ordered
    - Broken discrete symmetry
  - Short-Range Entangled (**SRE**)

Can one use QFT to describe all these phases?

Topologically ordered phases $\Longleftrightarrow$ non-trivial TQFT

SRE phases $\overset{?}{\Longleftrightarrow}$ "almost trivial TQFT".

"almost trivial" = invertible.

Invertible TQFT

- unique ground state on a closed spatial slice of any topology
- no non-trivial "anyons".
- partition function $Z_X \neq 0$ for any closed Euclidean space-time X.

Unitary invertible TQFTs have been completely classified (Freed, Hopkins).

Completely determined by Z_X for all X.

E.g. let $Z_X = \exp\left(i\pi \int_X w_2(TX)^2\right)$ for any 4d manifold X.

which SRE phase corresponds to which invertible TQFT?

Problems

- What is a "quantum phase" anyway?
- What is SRE anyway?
- How does one compute Z_X for any given SRE phase?

Rough answers

- A quantum phase is an equivalence class of gapped lattice Hamiltonians.
- SRE phase = invertible phase (A. Kitaev)
- ???

a test case: Hall conductance at $T=0$

Expectations

- Defined for any gapped 2d system with a $U(1)$ symmetry
- a numerical invariant of a gapped phase.
- $\sigma_H \neq 0 \Rightarrow$ no gapped edge is possible
- For SRE systems $\sigma_H = n \cdot \frac{e^2}{h}$, $n \in \mathbb{Z}$
- For bosonic SRE systems $n \in 2\cdot\mathbb{Z}$.

Last two properties follow from TQFT:

$$Z_X(A) = \exp\left(\frac{in}{4\pi}\int_X A\,dA\right) =$$

↑ classical $U(1)$ gauge field

$$= \exp\left(\frac{in}{4\pi}\int_Y F\wedge F\right)$$

$$\partial Y = X$$

$Z_X(A)$ gauge-invariant $\Rightarrow n \in \mathbb{Z}$

$Z_X(A)$ independent of spin-structure $\Rightarrow n \in 2\cdot\mathbb{Z}$

1. Quantum lattice systems and quantum phases.

<u>Problem</u>: define and classify possible phases of lattice systems.

$\Lambda \subset \mathbb{R}^d$: countable uniformly discrete ("lattice").

$\mathcal{A} = \bigotimes_{p \in \Lambda} \mathcal{A}_p$: algebra of observables

$(\mathcal{A}_p \cong Mat(n_p, \mathbb{C}),\ n_p \in \mathbb{N})$

$\mathcal{A}_X = \bigotimes_{p \in X} \mathcal{A}_p \subset \mathcal{A}$: observables localized on $X \subset \Lambda$.

$H = \sum_{p \in \Lambda} H_p, \quad H_p \in \mathcal{A}_{B_p(r)}, \quad r > 0$

("Hamiltonian"). ($B_p(r)$: ball of radius r and center at p)

Want to study ground states of such Hamiltonians in infinite volume, up to some <u>equivalence</u>.

Remark The Hilbert space where H acts is not specified from the outset. It is constructed from the ground state via the GNS construction.

2. Gapped vs. gapless

Want to require an energy gap Δ between the (unique) ground state $|0\rangle$ and excited states.

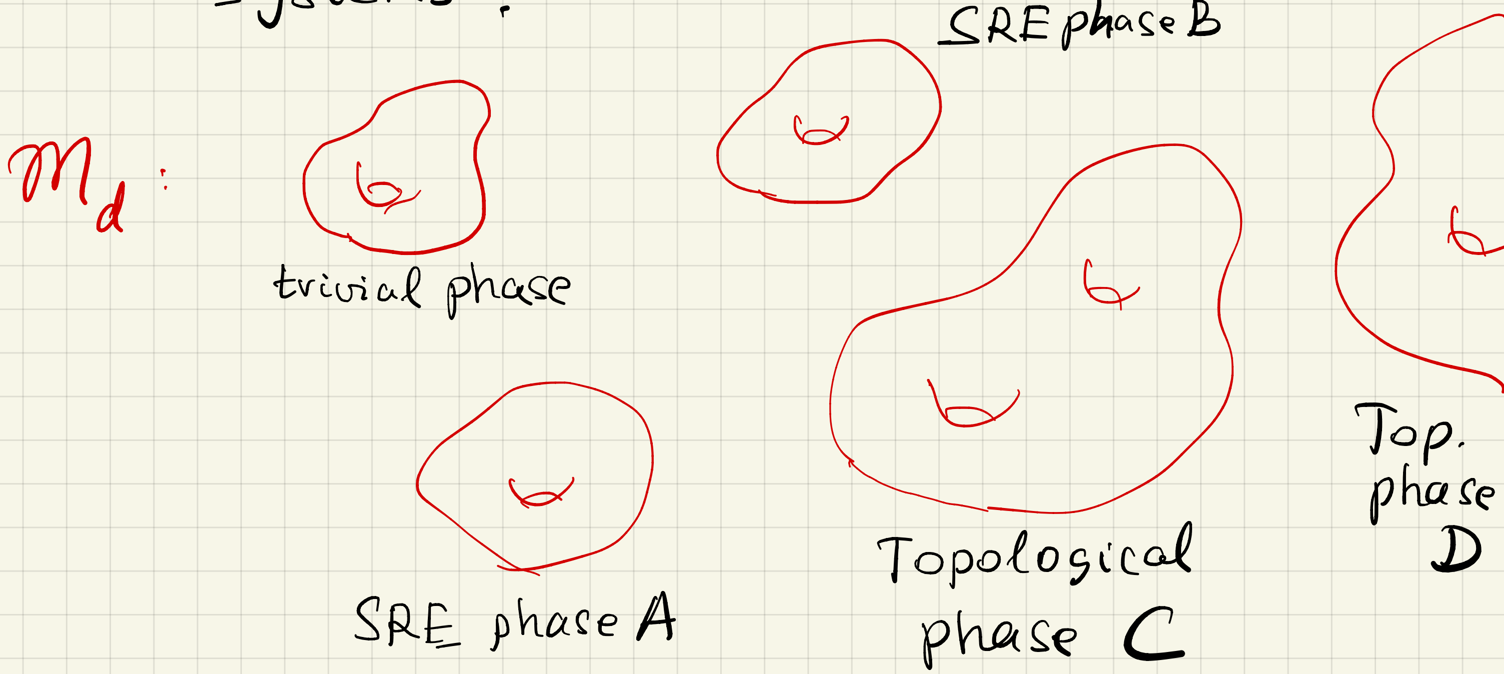
More precisely, there is a gap in the spectrum of the unbounded operator

$$\hat{H} \text{ "=" } \pi_{GNS}(H), \qquad \hat{H}|0\rangle = 0.$$

Want to classify such H up to a suitable equivalence. Very hard.

Can we at least construct some "topological invariants"?

3. The "space" of gapped lattice systems.



$\pi_0(\mathcal{M}_d)$ = set of gapped phases in dimension d.

$$\mathcal{M}_d \rightsquigarrow \mathcal{M}_d^B \quad \text{(bosonic)}$$

$$\mathcal{M}_d \rightsquigarrow \mathcal{M}_d^F \quad \text{(fermionic)}$$

$$\pi_0(\mathcal{M}_1^F) \overset{?}{=} \mathbb{Z}_2 \quad \text{(Kitaev chain)}$$

$$\pi_0(\mathcal{M}_1^B) \overset{?}{=} 0$$

$$\pi_0(\mathcal{M}_2^{B/F}) \overset{?}{=} \mathbb{Q} \quad \text{(chiral central charge)}$$

Kitaev argued that spaces $\mathcal{M}_d^{SRE}$ are all related:

$$\Omega \mathcal{M}_d^{SRE} \sim \mathcal{M}_{d-1}^{SRE} \quad \left(\Rightarrow \Omega^k \mathcal{M}_d^{SRE} \sim \mathcal{M}_{d-k}^{SRE}\right)$$

That is, they form a "loop spectrum".

"Practical" consequence:

$$\pi_n\left(\mathcal{M}_d^{SRE}\right) = \pi_n\left(\Omega^k \mathcal{M}_{d+k}^{SRE}\right) = \pi_{n+k}\left(\mathcal{M}_{d+k}^{SRE}\right)$$

$$\forall k > 0.$$

But we are still very far from proving this.

4. Topological invariants of families of gapped lattice systems.

Choose a connected component of $\mathfrak{M}_d$.

Say, the one in a trivial phase.

What are its homotopy groups?

What are its (co)homology groups?

What is its homotopy type?

How does one detect the nontrivial topology of the space of systems in a particular gapped phase?

<u>Let's look at the case $d=0$</u> first.

5. Berry connection, Berry curvature, Berry class.

Consider a Hamiltonian $H(\lambda)$ depending on parameters $\lambda \in M$.

The Hilbert space V is fixed (is a trivial bundle over M).

Ground state is unique $\forall \lambda \in M \Rightarrow$ get a line bundle $\mathcal{B}$ over M.

- Canonical connection ∇ on $\mathcal{B}$
- $\frac{1}{i}\nabla^2 = F \in \Omega^2(M)$ is a closed form with periods $\in 2\pi\mathbb{Z}$
- $\left[\frac{F}{2\pi}\right] = c_1(\mathcal{B})$.

If $\left[\frac{F}{2\pi}\right] \neq 0$, the family of Hamiltonians is topologically non-trivial.

$[\frac{F}{2\pi}]$ is a <u>complete</u> invariant of a family.

Space of $d=0$ Hamiltonians with a unique ground state $\sim$ Space of rank-1 projectors $\sim U(V)/U(1)$ (V: ∞d Hilbert space)

$$U(V)/U(1) = PU(V) \text{ is a } K(\mathbb{Z},2):$$

$$\pi_k(PU(V)) = \begin{cases} \mathbb{Z}, & k=2 \\ 0, & k\neq 2 \end{cases}$$

$$\Rightarrow \quad \pi(M, PU(V)) = H^2(M,\mathbb{Z}).$$

(Equivalently, a line bundle of ground states is determined (topologically) by $[\frac{F}{2\pi}]$.)

Note: $\mathcal{M}_0^{SRE} \sim K(\mathbb{Z},2)$ implies (assuming Kitaev)

$$\pi_{d+2}(\mathcal{M}_d^{SRE}) = \mathbb{Z}, \quad \pi_k(\mathcal{M}_d^{SRE}) = 0 \text{ if } k=d, d+1, \text{ or } k>d+2$$

In particular, $\mathcal{M}_1^{triv} \sim K(\mathbb{Z},3)$.

Can we check all this directly?