The structure group in regularity structures: avoiding trees

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Perspective on driven ODEs

Initial value problem for ODE with (rough) driver:
$\frac{du}{dt} = a(u)\xi, \quad u(0) = 0.$

Butcher/Rough-path point of view, all nonlinearities at once:
$u = u[a](t)$ (Gubinelli'10).

Inhomogeneous initial data, i. e. $\tilde{u}(0) = u_0$,
recovered via u-shift:
$\tilde{u} = u[a(\cdot + u_0)] + u_0$,
$\rightsquigarrow$ parameterization of solution manifold.

Re-centering, i. e. $u_1(1) = 0$,
recovered via suitable variable u-shift $\pi = \pi[a]$:
$u_1[a] = u[a(\cdot + \pi[a])] + \pi[a]$.

Perspective on driven PDEs

PDE with (rough) driver, e. g. gPAM:

$(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})u = a(u)\xi$ mod polynomials.

Parameterize solution manifold
by jets/space-time polynomials $p = p(x)$:

$u(x) = u[a,p](x)$ (Bruned&Chandra&Chevyrev&Hairer'19).

In this work: $u(x) = u[a,p](x)$ with p mod constants.

Guided by quasi-linear class:

$(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})u = a(u)\frac{\partial^2 u}{\partial x_1^2} + \xi$ mod polynomials.

Two actions on (a,p)-space

Set of $(a,p) \in \mathbb{R}[u] \times \mathbb{R}[x_1,x_2]/\mathbb{R}$.

$\mathbb{R}[x_1,x_2]/\mathbb{R} \cong \{\, p \in \mathbb{R}[x_1,x_2] \,|\, p(0) = 0 \,\}$.

Action of $\mathbb{R}^2 \ni y$ by **shift**: $\Big(a\big(\cdot + p(y)\big)\,,\, p(\cdot + y) - p(y)\Big)$.

Action of $\mathbb{R}[x_1,x_2] \ni q$ by **tilt**: $\Big(a\big(\cdot + q(0)\big)\,,\, p + q - q(0)\Big)$.

This work is on the **representation** of these actions.

Ignore algebra structure of $\mathbb{R}[u] \ni a$ (cf. Faà-di Bruno),

and vector space structure of $\mathbb{R}[x_1,x_2]/\mathbb{R} \ni p$ (but affine).

Lifting, variable tilt, and monoid structure

Lift maps $(a,p) \mapsto \Big(a\big(\cdot + p(y)\big)\,,\, p(\cdot + y) - p(y)\Big)$,

$$(a,p) \mapsto \Big(a\big(\cdot + q(0)\big)\,,\, p + q - q(0)\Big)$$

to endomorphisms of algebra of functions π on (a,p)-space: $(\Gamma_y^*\pi)[a,p] = \pi\Big[a\big(\cdot + p(y)\big)\,,\, p(\cdot + y) - p(y)\Big]$,

$$(\Gamma_q^*\pi)[a,p] = \pi\Big[a\big(\cdot + q(0)\big)\,,\, p + q - q(0)\Big].$$

Extend to variable tilt $\{\pi^{(n)} = \pi^{(n)}[a,p]\}_n$:

$$(\Gamma^*\pi)[a,p] = \pi\Big[a\big(\cdot + \pi^{(0)}[a,p]\big)\,,\, p + \textstyle\sum_{n\neq 0} \pi^{(n)}[a,p]x^n\Big].$$

Monoid: If $\left\{\begin{array}{c} \pi^{(n)} \mapsto \Gamma^* \\ \pi'^{(n)} \mapsto \Gamma'^* \end{array}\right\}$ then $\pi^{(n)} + \Gamma^*\pi'^{(n)} \mapsto \Gamma^*\Gamma'^*$

(cf. Bruned&Chevyrev&Friz&Preiss'19)

– need inverse for re-centering.

Coordinates on (a,p)-space for matrix representation

Introduce coordinates on (a,p)-space
(arbitrarily fixing origin in (u,x)-space):

$$z_k[a] := \frac{1}{k!}\frac{d^k a}{du^k}(0), \; k \in \mathbb{N}_0, \qquad z_n[p] := \frac{1}{n!}\frac{d^n p}{dx^n}(0), \; n \in \mathbb{N}_0^2 - \{(0,0)\}.$$

Consider $\mathbb{R}[z_k, z_n]$ (= polynomial algebra)
$\subset$ {functions on (a,p) space}.
Difficulty: Γ_y^* maps $\mathbb{R}[z_k, z_n]$ only into $\mathbb{R}[[z_k, z_n]]$,
where $\mathbb{R}[[z_k, z_n]]$ (= formal power series algebra).

Task: Construct T^* (= dual of "abstract model space" T)
as linear subspace of $\mathbb{R}[[z_k, z_n]]$ such that $\Gamma_y^*, \Gamma_q^* \in \mathrm{End}(\mathsf{T}^*)$.

Formal power series $\mathbb{R}[[z_k, z_n]]$ natural for model Π

Formally: Π $=$ partial derivative $\prod_{k\geq 0, n\neq 0}(\frac{\partial}{\partial z_k})^{\beta(k)}(\frac{\partial}{\partial z_n})^{\beta(n)}$,
β multi-index, applied to general solution $u = u[a,p]$.

Set $\Pi_{e_n} = x^n$, solve inductively (mod polynomials)

$(\frac{\partial}{\partial x_2}-\frac{\partial^2}{\partial x_1^2})\Pi_0 = \xi$, $(\frac{\partial}{\partial x_2}-\frac{\partial^2}{\partial x_1^2})\Pi_{e_0} = \frac{\partial^2}{\partial x_1^2}\Pi_0$, $(\frac{\partial}{\partial x_2}-\frac{\partial^2}{\partial x_1^2})\Pi_{e_1} = \Pi_0\frac{\partial^2}{\partial x_1^2}\Pi_0$,
$(\frac{\partial}{\partial x_2}-\frac{\partial^2}{\partial x_1^2})\Pi_{2e_1} = \Pi_0\frac{\partial^2}{\partial x_1^2}\Pi_{e_1}+\Pi_{e_1}\frac{\partial^2}{\partial x_1^2}\Pi_0$, $(\frac{\partial}{\partial x_2}-\frac{\partial^2}{\partial x_1^2})\Pi_{e_1+e_{(1,0)}} = \Pi_{e_{(1,0)}}\frac{\partial^2}{\partial x_1^2}\Pi_0$.
$(\frac{\partial}{\partial x_2}-\frac{\partial^2}{\partial x_1^2})\Pi_\beta = \sum_{k\geq 0}\sum_{e_k+\beta_1+\ldots+\beta_{k+1}=\beta}\Pi_{\beta_1}\cdots\Pi_{\beta_k}\frac{\partial^2}{\partial x_1^2}\Pi_{\beta_{k+1}} + \delta_\beta^0\xi$.

Use algebra $\mathbb{R}[[z_k, z_n]]$ to write more compactly

$$(\frac{\partial}{\partial x_2}-\frac{\partial^2}{\partial x_1^2})\Pi = \sum_{k\geq 0} z_k \Pi^k \frac{\partial_1^2}{\partial x_1^2}\Pi + \xi 1.$$

Π_β "populated" only if $\beta \in \{e_n\}_{n\neq 0} \cup \{\beta \mid [\beta] \geq 0\}$,
where $[\beta] := \sum_{k\geq 0} k\beta(k) - \sum_{n\neq 0}\beta(n)$.

Abstract model space T^*

Set T^* direct product of index set $\{e_n\}_{n\neq 0} \cup \{\beta | [\beta] \geq 0\}$,

where $[\beta] := \sum_{k\geq 0} k\beta(k) - \sum_{n\neq 0} \beta(n)$;

note $[\beta] = (\text{homogeneity in } u) - (\text{homogeneity in } p)$.

Yields natural decomposition $\mathsf{T}^* = \bar{\mathsf{T}}^* \oplus \tilde{\mathsf{T}}^*$,

with $\bar{\mathsf{T}}^*$ dual of "polynomial sector" $\bar{\mathsf{T}}$ (mod constants).

Infinitesimal generators of u-shift, x^n-tilt, x_1-shift:

$$\frac{d}{dt}_{|t=0}\pi\Big[a(\cdot + t1), p\Big], \quad \frac{d}{dt}_{|t=0}\pi\Big[a, p + tx^n\Big] \text{ for } n \neq 0,$$

$$\frac{d}{dt}_{|t=0}\pi\Big[a\Big(\cdot + p(t,0)\Big), p(\cdot + (t,0)) - p(t,0)\Big].$$

... building blocks for Lie algebra L**.**

Towards a Lie algebra $\mathsf{L} \subset \mathsf{Der}(\mathbb{R}[[z_k, z_n]])$

Generator of u-shift: $D^{(0)} := \sum_{k\geq 0}(k+1)z_{k+1}\frac{\partial}{\partial z_k}$.

Generator of x^n-tilt: $D^{(n)} := \frac{\partial}{\partial z_n}$, $n \neq 0$.

Generator of x_1-shift: $\partial_1 := \sum_n (n_1+1) z_{n+(1,0)} D^{(n)}$,

(same for ∂_2).

Well-defined **derivations** on algebra $\mathbb{R}[[z_k, z_n]]$,
i. e. as elements of $\mathsf{Der}(\mathbb{R}[[z_k, z_n]])$.

Despite modding out constants,
commutators behave canonically:
$[D^{(n)}, D^{(m)}] = 0$, $[\partial_1, \partial_2] = 0$, $[D^{(n)}, \partial_1] = n_1 D^{(n-(1,0))}$
(same for ∂_2).

Almost a pre-Lie structure on L $\subset$ End(T*)

Extension to variable tilt: $z^\gamma D^{(n)} \in \mathrm{Der}(\mathbb{R}[[z_k, z_n]])$.
All $D \in \{\partial_i\}_{i=1,2} \cup \{z^\gamma D^{(n)}\}_{[\gamma]\geq 0,n}$ preserve T* and $\tilde{\mathrm{T}}$*.

Note $\mathrm{Der}(\mathbb{R}[[z_k, z_n]]) \cong$ {vector fields on flat $\mathbb{R}[[z_k, z_n]]$}.
Hence Lie algebra product $[\cdot,\cdot]$
arises from a pre-Lie algebra product $\triangleleft$ = covariant derivative.

For our derivations:
$D \triangleleft z^\gamma D^{(n)} = (Dz^\gamma) D^{(n)}, \quad z^\gamma D^{(n)} \triangleleft \partial_1 = n_1 z^\gamma D^{(n-(1,0))}$;
however, $\partial_1 \triangleleft \partial_1$ not of this form.

Gradations on pre-Lie algebra and definition of L

Two **gradations** on $\{\partial_i\} \cup \{z^\gamma D^{(n)}\}$ compatible with $\triangleleft$:

for $z^\gamma D^{(n)}$: $\left(1 + [\gamma], \sum_m |m|\gamma(m) - |n|\right)$,

for ∂_1: $(0, |(1,0)|)$, for ∂_2: $(0, |(0,1)|)$.

Fix parameter $\alpha > 0$, introduce **homogeneity**

$|\gamma| := \alpha(1 + [\gamma]) + \sum_{m\neq 0} |m|\gamma(m)$.

Define $\mathsf{L} := \mathrm{span}\Big(\{\partial_i\}_{i=1,2} \cup \{z^\gamma D^{(n)}\}_{[\gamma]\geq 0, |\gamma|>|n|}\Big)$, then

pre-Lie product $\triangleleft$ almost preserves L,

for $D \in \mathsf{L}$, $D^\dagger$ is strictly triangular w. r. t. $|\cdot|$.

From L to Hopf algebra U(L)

Since $[\partial_1, \partial_2] = 0$, L is Lie algebra since closed under

$[D, D'] = D \triangleleft D' - D' \triangleleft D$.

View L as direct sum indexed by

$\{(\gamma, n)\}_{[\gamma]\geq 0, |\gamma|>|n|} \cup \{1, 2\}$.

Consider universal enveloping algebra

U(L) := T(L)/$[\cdot, \cdot]$= Hopf algebra.

Lift representation ρ: U(L) $\rightarrow$ End(T*)

however no longer faithful (i. e. one-to-one).

Incomplete pre-Lie structure yields canonical decomposition of $\mathsf{U}(\mathsf{L})$

Pre-Lie product $\triangleleft$ closed on $\tilde{\mathsf{L}} := \{z^\gamma D^{(n)}\}_{[\gamma]\geq 0, |\gamma|>|n|}$;
hence $\mathsf{U}(\tilde{\mathsf{L}}) \cong \mathsf{S}(\tilde{\mathsf{L}})$ (Oudom&Guin'08).

Derived algebra $[\mathsf{L},\mathsf{L}] \subset \tilde{\mathsf{L}}$ ideal;
hence $\otimes_m \mathsf{U}_m = \mathsf{U}(\mathsf{L}) \to \mathsf{U}(\mathsf{L}/\tilde{\mathsf{L}}) = \mathrm{span}\{\partial^m\}_m$.

Pre-Lie product $\triangleleft$: $\tilde{\mathsf{L}} \times \mathsf{L} \to \mathsf{L}$;
hence $\mathsf{U}_m \cong \mathsf{U}_0 = \mathsf{U}(\tilde{\mathsf{L}}) \cong \mathsf{S}(\tilde{\mathsf{L}})$.

Recall $\mathsf{L} =$ direct sum indexed by $\{(\gamma,n)\}_{[\gamma]\geq 0,|\gamma|<|n|} \cup \{1,2\}$.

Hence $\mathsf{U}(\mathsf{L})$

$=$ direct sum indexed by multi-indices J on $\{(\gamma,n)\}_{\dots} \cup \{1,2\}$.

Decomposition provides canonical pairings

Recall L $=$ direct sum indexed by $\{(\gamma, n)\}_{[\gamma]\geq 0, |\gamma|<|n|} \cup \{1, 2\}$,

think of L $=$ span$\{z^\gamma D^{(n)}, \partial_i\}$;

U(L)

$=$ direct sum indexed by multi-indices J on $\{(\gamma, n)\}_{\dots} \cup \{1, 2\}$,

think of U(L) $=$ span$\{z^{\gamma_1} \cdots z^{\gamma_k} \partial_1^{m_1} \partial_2^{m_2} D^{(n_k)} \cdots D^{(n_1)}\}$.

Get canonical pairing between U(L) and

T$^+$:= direct sum indexed by J's

$=$ polynomial algebra in variables indexed by $\{(\gamma, n)\}_{[\gamma]\geq 0, |\gamma|<|n|} \cup \{1, 2\}$.

Have canonical pairing between T* and

T := direct sum indexed by γ's.

Definition of Δ^+ and Δ by duality/pairing

Finiteness properties & gradedness imply:

(concatenation) product on U(L) yields
co-product $\Delta^+ \colon \mathsf{T}^+ \to \mathsf{T}^+ \otimes \mathsf{T}^+$,

representation $\rho \colon \mathsf{U}(\mathsf{L}) \to \mathsf{End}(\mathsf{T}^*)$ yields
co-module $\Delta \colon \mathsf{T} \to \mathsf{T}^+ \otimes \mathsf{T}$.

Injection $\gamma \mapsto J = e_{(\gamma,n)}$ yields $\mathcal{J}_n \colon \mathsf{T} \to \mathsf{T}^+$;
canonical definition of $x^m \in \mathsf{T}^+$.

Get intertwining axioms of Hairer'15:

$$\Delta^+ \mathcal{J}_n = (1 \otimes \mathcal{J}_n)\Delta + \sum_m \mathcal{J}_{m+n} \otimes \frac{x^m}{m!},$$

$$\Delta^+ x_1 = x_1 \otimes \{\} + \{\} \otimes x_1, \quad \text{same for } x_2.$$

Definition of structure group $\mathsf{G} \subset \mathsf{End}(\mathsf{T})$

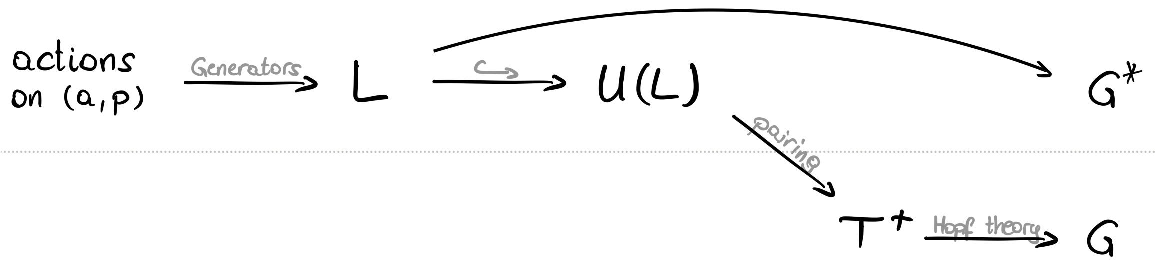
Define $\mathsf{G} \cong \mathsf{Alg}(\mathsf{T}^+, \mathbb{R}) = \{ f \in (\mathsf{T}^+)^* \mid f \text{ multiplicative} \}$.

Co-product $\Delta^+ \colon \mathsf{T}^+ \to \mathsf{T}^+ \otimes \mathsf{T}^+$ yields

product on G via $f \circ g = (f \otimes g)\Delta^+$.

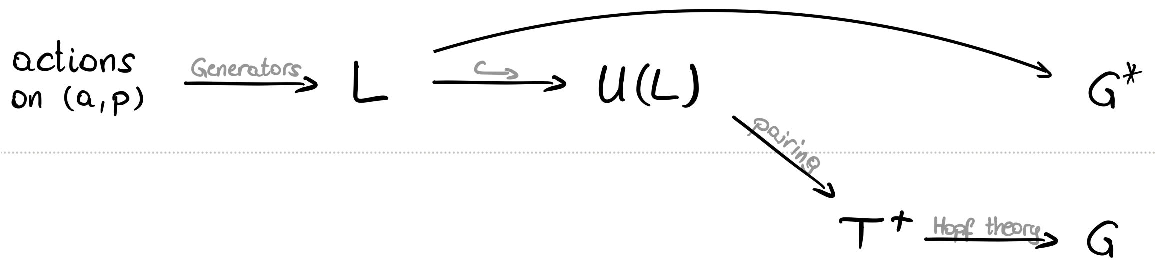
co-module $\Delta \colon \mathsf{T} \to \mathsf{T}^+ \otimes \mathsf{T}$ yields

representation $\mathsf{G} \subset \mathsf{End}(\mathsf{T})$ via $\Gamma_f = (f \otimes 1)\Delta$.



Initial tasks accomplished

For multiplicative $f \in (\mathsf{T}^+)^*$ set $\Gamma_f = (f \otimes 1)\Delta \in \mathrm{End}(\mathsf{T})$.



Shift by $y \in \mathbb{R}^2$: If $f^J = y_1^{J(1)} y_2^{J(2)}$ and $f^J = 0$ for $J(\gamma, n) \neq 0$
then $\Gamma_f^* \pi[a,p] = \pi\big[\, a(\cdot + p(y))\,,\, p(\cdot + y) - p(y) \,\big]$
for all $\pi \in \mathsf{T}^* \cap \mathbb{R}[z_k, z_n]$ and $(a,p) \in \mathbb{R}[u] \times \mathbb{R}[x_1, x_2]$.

Tilt by $\{\pi^{(n)}\}_n \subset \tilde{\mathsf{T}}^*$: If $f^J = \Pi_{(\gamma,n)} (\pi_\gamma^{(n)})^{J(\gamma,n)}$
and $f^J = 0$ for $J(i) \neq 0$
then $\Gamma_f^* \pi[a,p] = \pi\big[\, a(\cdot + \pi^{(0)}[a,p])\,,\, p + \Sigma_{n \neq 0}\, \pi^{(n)}[a,p] x^n \,\big]$.

gPAM: Construction of our model

$(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})\Pi_\beta = \sum_{k\geq 0}\sum_{e_k+\beta_1+\ldots+\beta_k=\beta}\Pi_{\beta_1}\cdots\Pi_{\beta_k}\xi,\ \ \beta \neq e_n.$

Examples:

$(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})\Pi_{e_0} = \xi,\ (\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})\Pi_{e_0+e_1} = \Pi_{e_0}\xi,$

$(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})\Pi_{e_1+e_{(2,0)}} = \Pi_{e_{(2,0)}}\xi = x_1^2\xi,$

$(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})\Pi_{e_0+e_1+e_2+e_{(2,0)}}$

$= \Pi_{e_0+e_2+e_{(2,0)}}\xi + 2\Pi_{e_0}\Pi_{e_1+e_{(2,0)}}\xi + 2\Pi_{e_0+e_1}\Pi_{e_{(2,0)}}\xi.$

Π populated only by β with $\sum_{k\geq 0}(k-1)\beta(k) - \sum_{n\neq 0}\beta(n) = -1.$

The model has redundancies

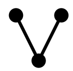
$(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})\Pi_{e_1+e_{(2,0)}} = \Pi_{e_{(2,0)}}\xi = x_1^2\xi = \Pi^2_{e_{(1,0)}}\xi = (\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2})\Pi_{e_2+2e_{(1,0)}}.$

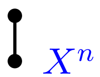
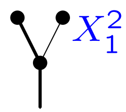
gPAM: Hairer's abstract model space T_H

Three symbols: $\bullet$, X_1, X_2.

Two operations: $\begin{cases} \text{Integration } \mathcal{I} \equiv \text{adding an edge,} \\ \text{multiplication} \equiv \text{adjoining on the root.} \end{cases}$

Examples:

$\mathcal{I}\bullet =$  , $\quad \bullet(\mathcal{I}\bullet)^2 =$  ,

$X^n \bullet \mathcal{I}\bullet =$  , $\quad \mathcal{I}\bullet(\mathcal{I}\bullet)\mathcal{I}X_1^2\bullet =$  .

Dictionary between β's and trees

Define $\phi, \phi^- : \mathsf{T} \to \mathsf{T}_H$ recursively via

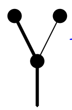
$$\phi^- z_{\beta=0} = 0, \qquad \phi z_\beta = \begin{cases} X^n & \text{for } \beta = e_n, \\ \mathcal{I}\phi^- z_\beta & \text{else,} \end{cases}$$

$$\phi^- z_\beta = \textstyle\sum_{k\geq 0} \sum_{e_k+\beta_1+\ldots+\beta_k=\beta} \phi z_{\beta_1} \cdots \phi z_{\beta_k} \bullet .$$

Examples:

$\phi^- z_{e_0} = \bullet$, $\quad \phi z_{e_0} =$ , $\quad \phi^- z_{e_1+e_{(2,0)}} = \bullet\, X_1^2$, $\quad \phi z_{e_1+e_{(2,0)}} =$  X_1^2 .

$\phi \oplus \phi^-$ is not onto:

$1 \notin \mathrm{im}\phi \oplus \mathrm{im}\phi^-$, $\quad \phi z_{e_0+e_1+e_2+e_{(2,0)}} = 2$  $X_1^2 + 2$  $X_1^2 + 2$  X_1^2 .

ϕ is not one-to-one:

$\phi z_\beta = 0$ for $\sum_{k\geq 0}(k-1)\beta(k) - \sum_{n\neq 0}\beta(n) \neq -1$,

$\phi z_{e_1+e_{(2,0)}} = \phi z_{e_2+2e_{(1,0)}} =$  X_1^2 .

A morphism between our and Hairer's structure

Recall $\phi, \phi^- : \mathsf{T} \to \mathsf{T}_H$ recursively defined via

$$\phi^- z_{\beta=0} = 0, \qquad \phi z_\beta = \begin{cases} X^n & \text{for } \beta = e_n, \\ \mathcal{I}\phi^- z_\beta & \text{else,} \end{cases}$$

$$\phi^- z_\beta = \textstyle\sum_{k\geq 0} \sum_{e_k+\beta_1+\ldots+\beta_k=\beta} \phi z_{\beta_1} \cdots \phi z_{\beta_k} \bullet .$$

Dictionary ϕ, ϕ^- extends to multiplicative $\Phi : \mathsf{T}^+ \to \mathsf{T}_H^+$ via intertwining $\Phi \mathcal{J}_n z_\beta = \mathcal{J}_n^H \phi^- z_\beta$.

Proposition:

$$(\Phi \otimes \phi^-)\Delta = \Delta_H \phi^-, \quad (\Phi \otimes \Phi)\Delta^+ = \Delta_H^+ \Phi.$$

Relation between $\triangleleft$ and tree grafting

(BCCH'19, Bruned&Manchon'20, Bailleul&Bruned'21)

Branched rough paths $\subset$ gPAM:

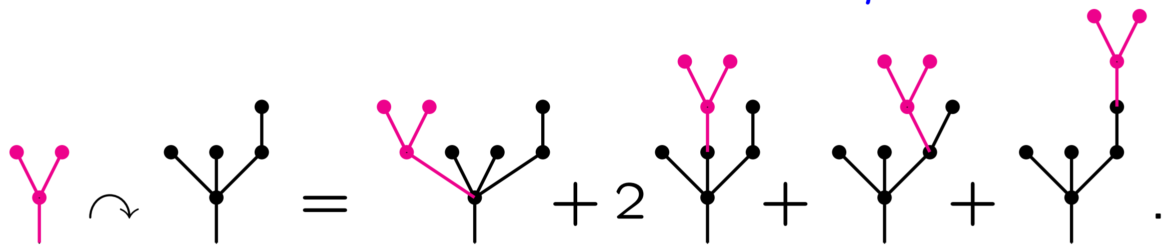
ϕ: $\mathsf{T}^1 \to \mathsf{T}_B$ one-to-one on $\mathsf{T}^1 := \mathrm{span}\{z_\beta | \beta(n) = 0\}$.

Then $\phi^\dagger$: $\mathcal{L}^1 \to \mathsf{L}$ is pre Lie-algebra morphism:

$\phi^\dagger(z_\tau * z_{\tau'}) = \phi^\dagger z_\tau \triangleleft \phi^\dagger z_{\tau'}$.

Connes-Kreimer: pre-Lie $*$ on Grossman-Larson's $\mathcal{L}^1$

comes from grafting: $\sigma(\tau) z_\tau * \sigma(\tau') z_{\tau'} =$ "$\sigma(\tau \curvearrowright \tau') z_{\tau \curvearrowright \tau'}$"

Example: 

Form of counter-term for quasi-linear class

Seek $(\frac{\partial}{\partial x_2}-\frac{\partial^2}{\partial x_1^2})u+h=a(u)\frac{\partial^2}{\partial x_1^2}+\xi$ with
h local, i. e. independent of p, only on $u(x)$,
h deterministic, thus not explicitly dependent on x,
h covariant, i. e. $h[a](u+v)=h[a(\cdot+v)](u)$.

Parameterization by deterministic $c\in\mathsf{T}^*$,
with $D^{(n)}c=0$ for $n\neq 0$, such that

$$h[a](u)=c[a(\cdot+u)]=\sum_\beta c_\beta\prod_{k\geq 0}(\frac{1}{k!}\frac{d^ka}{du^k}(u))^{\beta(k)}$$
$$=\sum_\beta c_\beta(a(u))\prod_{k\geq 1}(\frac{1}{k!}\frac{d^ka}{du^k}(u))^{\beta(k)}.$$

Requires truncation, $|\cdot|$ not coercive on e_0.

Structure group G compatible with renormalization

Annealed stochastic estimates on model (Π_x, Γ_{xy})

For (regularized) $C^{\alpha-2}$-noise, BPHZ-choice of renormalization and with (irrational) $\alpha > \frac{1}{4}$

$$\mathbb{E}^{\frac{1}{p}}|\Pi^-_{x\beta\lambda}(x)|^p \lesssim \lambda^{|\beta|-2},$$
$$\mathbb{E}^{\frac{1}{p}}|\Pi_{x\beta}(y)|^p \lesssim |x-y|^{|\beta|},$$
$$\mathbb{E}^{\frac{1}{p}}|\Gamma^{\gamma}_{xy\beta}|^p \lesssim |x-y|^{|\beta|-|\gamma|}.$$

in progress with P. Linares, M. Tempelmayr, P. Tsatsoulis.

BPHZ-choice (expectation) fits spectral gap (variance),

Malliavin calculus instead of Feynman diagrams

– without passing via trees.