Truncated Rozansky–Witten models as extended TQFTs

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based on joint work with **Lóránt Szegedy** and with **Ilka Brunner**, **Daniel Roggenkamp**

https://carqueville.net/nils/RW.pdf

2d extended TQFTs

affine Rozansky–Witten models

2d extended TQFTs

affine Rozansky–Witten models

Theorem
Affine Rozansky–Witten models give truncated extended TQFTs

$$\mathrm{Bord}_{2,1,0} \longrightarrow (\text{algebraic non-semisimple 2-category } \mathcal{C})$$

Also: **$\mathbf{U(1)}$-equivariant** version, with **all defects**.

closed

TQFT

oriented closed

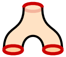
TQFT

Topological quantum field theory

A **2d oriented closed TQFT** is a symmetric monoidal functor

$$\mathrm{Bord}^{\mathrm{or}}_{2,1} \longrightarrow \mathrm{Vect}$$

$$S^1 \longmapsto V$$ (vector space)

 $\longmapsto \big(\mu\colon V \otimes V \longrightarrow V\big)$ (associative multiplication)

 $\longmapsto \big(\langle -,-\rangle\colon V \otimes V \longrightarrow \Bbbk\big)$ (nondegenerate μ-compatible pairing)

Theorem.
$\big\{$2d oriented closed TQFTs$\big\} \cong \big\{$commutative Frobenius algebras$\big\}$

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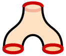 $\longmapsto \big(\mu\colon V \otimes V \longrightarrow V\big)$ (associative multiplication)

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Examples.

- $V = \Bbbk G$ for finite abelian group G and $\langle g, h\rangle = \delta_{g,h^{-1}}$
- $V = H^{\bullet}_{\mathrm{dR}}(X)$ for oriented closed manifold X and $\langle \alpha, \beta\rangle = \int_X \alpha \wedge \beta$
- $V = \mathrm{Jac}_W := \Bbbk[\underline{x}]/(\partial W)$ and $\langle f, g\rangle = \mathrm{Res}\big[\frac{f(x)g(x)\,\mathrm{d}x}{\partial_{x_1}W \dots \partial_{x_n}W}\big]$

Where to go from here?

A **2d oriented closed TQFT** is a symmetric monoidal functor

$$\mathrm{Bord}^{\mathrm{or}}_{2,1} \longrightarrow \mathrm{Vect}$$

Options:

- Increase "spacetime" dimension
- Promote source and target to higher categories
  $\Longrightarrow$ **extended** TQFTs
- Consider other tangential structures (framed, unoriented, spin, pin, string, . . .)
- Allow non-trivial stratifications of bordisms
  $\Longrightarrow$ **defect** TQFTs
- Consider targets other than (higher) vector spaces
- Study non-topological QFT. . .

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- ✓ Consider targets other than (higher) vector spaces
- – Study non-topological QFT...

(✓ in this talk)

extended
TQFT

framed extended

TQFT

Examples of symmetric monoidal 2-categories

$\mathrm{Bord}^{\mathrm{fr}}_{2,1,0}$

- objects: disjoint unions of 2-framed points $+, -$
- Hom categories: 2-framed bordisms of dimension 1 and 2

Alg (**state sum models**)

- objects: finite-dimensional $\Bbbk$-algebras
- Hom categories: finite-dimensional bimodules and bimodule maps

$\mathcal{V}\mathrm{ar}$ (**B-twisted sigma models**)

- objects: smooth projective varieties
- Hom categories: bounded derived categories of coherent sheaves

$\mathcal{LG}$ (**affine Landau–Ginzburg models**)

- objects: isolated singularities/potentials $W \in \mathbb{C}[x_1, \dots, x_n]$
- Hom categories: homotopy categories of matrix factorisations

$\mathrm{T}(\mathcal{RW}^{\mathrm{aff}})$ (**truncated affine Rozansky–Witten models**)

- objects: lists of variables $(x_1, \dots, x_n)$
- Hom categories: potentials and isom. classes of matrix factorisations

Pstragowski 2014, Căldăraru/Willerton 2007, Banks 2020, C/Murfet 2012, C/Montiel Montoya 2018, Kapustin/Rozansky 2009

3d graphical calculus

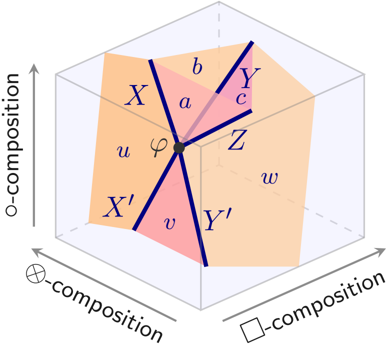
Fix symmetric monoidal 2-category with

monoidal product $\Box$

horizontal composition $\otimes$

vertical composition $\circ$

$\varphi \in \mathrm{Hom}\left(X' \otimes Y', X \otimes (Y \Box 1_a) \otimes (1_w \Box Z)\right)$



Willerton, Barrett/Meusburger/Schaumann 2012

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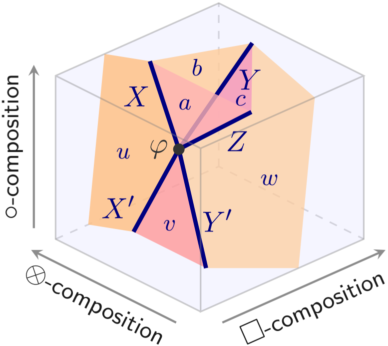
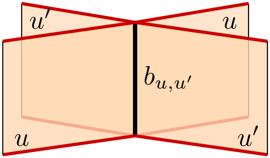
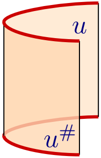


braiding:

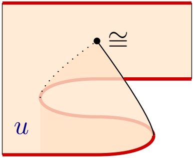


$$\widehat{=} \; b_{u,u'} \colon u \Box u' \longrightarrow u' \Box u$$

duals:



$$\widehat{=} \; \widetilde{\mathrm{ev}}_u \colon u \Box u^{\#} \longrightarrow \mathbb{1}$$



Willerton, Barrett/Meusburger/Schaumann 2012

Extended TQFT

Fix a symmetric monoidal 2-category $\mathcal{B}$. A **2d framed extended TQFT** valued in $\mathcal{B}$ is a symmetric monoidal 2-functor

$$\mathrm{Bord}^{\mathrm{fr}}_{2,1,0} \xrightarrow{\;\mathcal{Z}\;} \mathcal{B}$$

Theorem. (Framed **cobordism hypothesis** in 2d)
2d framed extended TQFTs are fully dualisable objects:

$$\begin{aligned} \mathrm{Fun}^{\mathrm{sym.\,mon.}}\Big(\mathrm{Bord}^{\mathrm{fr}}_{2,1,0}\,,\,\mathcal{B}\Big) &\xrightarrow{\;\cong\;} \big(\mathcal{B}^{\mathrm{fd}}\big)^{\times} \\ \mathcal{Z} &\longmapsto \mathcal{Z}(+) \end{aligned}$$

$$\begin{aligned} \mathcal{B}^{\mathrm{fd}} &:= \text{ full sub-2-category of fully dualisable objects} \\ \big(\mathcal{B}^{\mathrm{fd}}\big)^{\times} &:= \text{ maximal sub-2-groupoid of } \mathcal{B}^{\mathrm{fd}} \end{aligned}$$

Freed 1992, Baez/Dolan 1995, Lurie 2009, Schommer-Pries 2009, Pstragowski 2014

Cobordism hypothesis at work

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$$\begin{array}{ccc} \mathrm{Bord}^{\mathrm{fr}}_{2,1,0} & \xrightarrow{\mathcal{Z}} & \mathcal{B} \\ + & \longmapsto & u \in \mathcal{B}^{\mathrm{fd}} \end{array}$$

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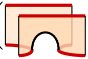
$$+ \longmapsto u \in \mathcal{B}^{\mathrm{fd}}$$

$$- \longmapsto u^{\#}$$

$$= \widetilde{\mathrm{ev}}_{+} \longmapsto \widetilde{\mathrm{ev}}_{u}$$

$$= {}^{\dagger}\widetilde{\mathrm{ev}}_{+} \longmapsto {}^{\dagger}\widetilde{\mathrm{ev}}_{u}$$

$$= \widetilde{\mathrm{ev}}_{+} \otimes {}^{\dagger}\widetilde{\mathrm{ev}}_{+} = S^1_1 \longmapsto \widetilde{\mathrm{ev}}_{u} \otimes {}^{\dagger}\widetilde{\mathrm{ev}}_{u}$$



$$\Big(= \mathrm{ev}_{\widetilde{\mathrm{ev}}_{+}} : {}^{\dagger}\widetilde{\mathrm{ev}}_{+} \otimes \widetilde{\mathrm{ev}}_{+} \longrightarrow 1_{+\sqcup -}\Big) \longmapsto \mathrm{ev}_{\widetilde{\mathrm{ev}}_{u}}$$

$$\Big(= \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_{+}} : \widetilde{\mathrm{ev}}_{+} \otimes \widetilde{\mathrm{ev}}^{\dagger}_{+} \longrightarrow 1_{\varnothing}\Big) \longmapsto \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_{u}}$$

2-framing on 1-manifold M is trivialisation $TM \oplus \mathbb{R} \cong \mathbb{R}^2$, described by immersion $\iota : M \hookrightarrow \mathbb{R}^2$ and trivialisation of normal bundle $\nu(\iota)$; normal vectors are blue.

Freed 1992, Baez/Dolan 1995, Lurie 2009, Schommer-Pries 2009, Pstragowski 2014

Extended TQFTs: semisimple example

$A \in \mathrm{Alg}$ fully dualisable $\iff$ A separable $\implies$ A semisimple

Corollary.
All extended TQFTs valued in Alg come from separable $\Bbbk$-algebras:

$$\begin{array}{rcl} \mathrm{Bord}^{\mathrm{fr}}_{2,1,0} & \longrightarrow & \mathrm{Alg} \\ + & \longmapsto & A \\ - & \longmapsto & A^{\mathrm{op}} \end{array}$$

$$= \widetilde{\mathrm{ev}}_+ \longmapsto {}_{\Bbbk}A_{A\otimes_{\Bbbk}A^{\mathrm{op}}}$$

$$= \mathrm{coev}_+ \longmapsto {}_{A\otimes_{\Bbbk}A^{\mathrm{op}}}A_{\Bbbk}$$

$$= \widetilde{\mathrm{ev}}_+ \otimes \mathrm{coev}_+ = S^1_0 \longmapsto A \otimes_{A\otimes_{\Bbbk}A^{\mathrm{op}}} A = \mathrm{HH}_0(A)$$

Every extended TQFT valued in Alg comes from semisimple data.

Lurie 2009, Schommer-Pries 2009

Non-semisimple framed extended TQFTs

Theorem. Every $W \in \mathcal{LG}$ gives framed extended TQFT:

$$\mathrm{Bord}^{\mathrm{fr}}_{2,1,0} \longrightarrow \mathcal{LG}$$

$$+ \longmapsto W$$

 $= \widetilde{\mathrm{ev}}_+ \otimes {}^\dagger\widetilde{\mathrm{ev}}_+ = S^1_1 \longmapsto \mathrm{Jac}_W = \mathbb{C}[\underline{x}]/(\partial W)$

 $= 1_{\widetilde{\mathrm{ev}}_+} \otimes \mathrm{ev}_{\widetilde{\mathrm{ev}}_+} \otimes 1_{{}^\dagger\widetilde{\mathrm{ev}}_+} \longmapsto$ multiplication in Jac_W

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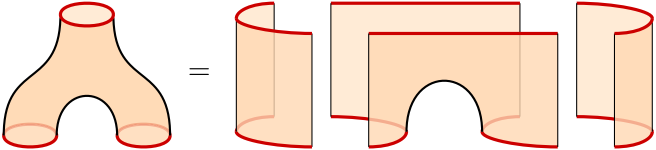

Carqueville/Montiel Montoya 2018

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Theorem. Every $X \in \mathcal{V}\mathrm{ar}$ gives framed extended TQFT

$$\mathrm{Bord}^{\mathrm{fr}}_{2,1,0} \longrightarrow \mathcal{V}\mathrm{ar}$$

Carqueville/Montiel Montoya 2018, Căldăraru/Willerton 2007, Banks 2020 oriented extended

TQFT

Oriented cobordism hypothesis

"Rotating frames" gives rise to SO_2-homotopy action on $\mathrm{Bord}^{\mathrm{fr}}_{2,1,0}$:

$$\Pi_{\leqslant 2}(\mathrm{SO}_2) \longrightarrow \mathrm{Aut}\Big(\mathrm{Bord}^{\mathrm{fr}}_{2,1,0}\Big)$$

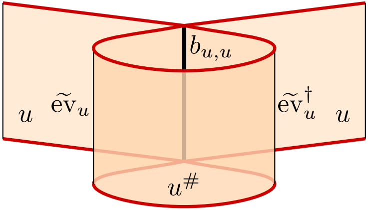
$$\pi_0(\mathrm{SO}_2) \cong \{*\} \ni * \longmapsto \mathrm{Id}$$

$$\pi_1(\mathrm{SO}_2) \cong \mathbb{Z} \ni -1 \longmapsto \big(S\colon \mathrm{Id} \longrightarrow \mathrm{Id}\big)\,, \quad S_+ =$$

For any $u \in \mathcal{B}^{\mathrm{fd}}$, have **Serre automorphism**

$$S_u := \big(1_u \,\square\, \widetilde{\mathrm{ev}}_u\big) \otimes \big(b_{u,u} \,\square\, 1_{u^\#}\big) \otimes \big(1_u \,\square\, \widetilde{\mathrm{ev}}_u^\dagger\big)$$

$=$



Lurie 2009, Douglas/Schommer-Pries/Snyder 2013

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\end{array}
$$

Theorem. (Oriented cobordism hypothesis in 2d)
2d oriented extended TQFTs are SO_2-homotopy fixed points:

$$
\mathrm{Fun}^{\mathrm{sym.\,mon.}}\Big(\mathrm{Bord}^{\mathrm{or}}_{2,1,0}\,,\,\mathcal{B}\Big) \xrightarrow{\;\cong\;} \Big[(\mathcal{B}^{\mathrm{fd}})^{\times}\Big]^{\mathrm{SO}_2}
$$

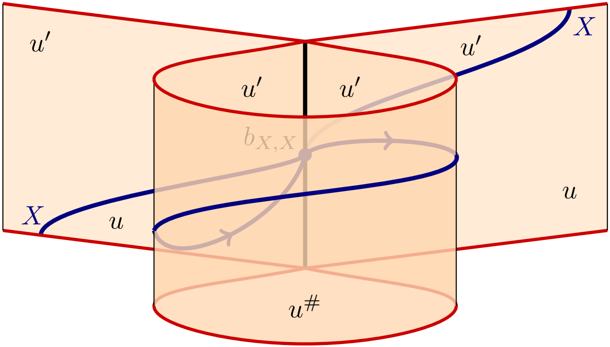
Such TQFTs $\mathcal{Z}$ are classified by objects $u := \mathcal{Z}(+) \in \mathcal{B}^{\mathrm{fd}}$ together with **trivialisation of Serre automorphism**, $S_u \xrightarrow{\cong} 1_u$.

Lurie 2009, Douglas/Schommer-Pries/Snyder 2013, Hesse/Schweigert/Valentino 2016, Hesse/Valentino 2017

Explicit description

Proposition. $\mathrm{Fun}^{\mathrm{sm}}(\mathrm{Bord}^{\mathrm{or}}_{2,1,0}\,\mathcal{B}) \cong \left[(\mathcal{B}^{\mathrm{fd}})^{\times}\right]^{\mathrm{SO}_2} \cong \mathcal{B}^{\circlearrowleft}$, where:

- Objects of $\mathcal{B}^{\circlearrowleft}$ are pairs (u,λ), with $u \in \mathcal{B}^{\mathrm{fd}}$ and $\lambda\colon S_u \xrightarrow{\cong} 1_u$.
- 1-cells $(u,\lambda) \longrightarrow (u',\lambda')$ are 1-cells $X\colon u \longrightarrow u'$ in $(\mathcal{B}^{\mathrm{fd}})^{\times}$ such that $(\lambda' \otimes 1_X) \circ S_X = 1_X \otimes \lambda$, where:

$S_X \; =$ 

- 2-cells $X \longrightarrow Y$ in $\mathcal{B}^{\circlearrowleft}$ are 2-cells $X \longrightarrow Y$ in $(\mathcal{B}^{\mathrm{fd}})^{\times}$.

Schommer-Pries 2009, Hesse/Schweigert/Valentino 2016, Hesse/Valentino 2017, Carqueville/Szegedy 2021

Oriented & spin extended TQFTs

Theorem. Every separable *symmetric Frobenius* algebra $A \in \mathrm{Alg}$ gives oriented extended TQFT $\mathrm{Bord}^{\mathrm{or}}_{2,1,0} \longrightarrow \mathrm{Alg}$.

Lurie 2009, Schommer-Pries 2009

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Theorem. Every separable *symmetric Frobenius* algebra $A \in \mathrm{Alg}$ gives oriented extended TQFT $\mathrm{Bord}^{\mathrm{or}}_{2,1,0} \longrightarrow \mathrm{Alg}$.

Lemma. $S_W \cong 1_W[n \bmod 2]$ for $W \equiv W(x_1, \dots, x_n) \in \mathcal{LG}$.

Theorem. Every $W(x_1, \dots, x_{2n}) \in \mathcal{LG}$ gives oriented extended TQFT

$$\mathrm{Bord}^{\mathrm{or}}_{2,1,0} \longrightarrow \mathcal{LG}$$

$$+ \longmapsto W$$

$$\widetilde{\mathrm{ev}}_+ \otimes {}^\dagger\widetilde{\mathrm{ev}}_+ =$$  $$\longmapsto \mathrm{Jac}_W$$

$$1_{\widetilde{\mathrm{ev}}_+} \otimes \mathrm{ev}_{\widetilde{\mathrm{ev}}_+} \otimes 1_{{}^\dagger\widetilde{\mathrm{ev}}_+} =$$  $$\longmapsto \text{multiplication}$$

$$\widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_+} =$$  $$\longmapsto \mathrm{Res}\Big[\frac{(-)\,\mathrm{d}x}{\partial_{x_1} W \dots \partial_{x_{2n}} W}\Big]$$

Theorem. *Every* $W \in \mathcal{LG}$ gives **spin** extended TQFT

$$\mathrm{Bord}^{\mathrm{spin}}_{2,1,0} \longrightarrow \mathcal{LG}$$

Lurie 2009, Schommer-Pries 2009, Carqueville/Montiel Montoya 2018, Carqueville/Szegedy 2021

truncated affine

Rozansky–Witten

models

Rozansky–Witten models

– rigorously constructed 3d TQFTs = Reshetikhin–Turaev models

Rozansky/Witten 1996, Kapranov 1997, Kontsevich 1997, Roberts/Willerton 2006, Kapustin/Rozansky/Saulina 2008, ...

Rozansky–Witten models

- rigorously constructed 3d TQFTs = Reshetikhin–Turaev models
- **RW models**: conjecturally 3d TQFTs from non-semisimple data
  - twisted 3d $\mathcal{N} = 4$ sigma model with holomorphic symplectic target
  - reduction on S^1 gives 2d B-model
  - "has local observables"
  - participate in 3d mirror symmetry
- Kapustin–Rozansky(–Saulina) propose defect 3-category $\mathcal{RW}$:
  - objects: holomorphic symplectic manifolds X
  - k-cells: "deformed Landau–Ginzburg models fibred over Lagrangians"
- **affine case** $X = T^*\mathbb{C}^n$
  - related to Chern–Simons theory for $\mathrm{psl}(1|1)$
  - related to free $\mathcal{N} = 4$ hypermultiplet
  - 3-category $\mathcal{RW}^{\mathrm{aff}}$ under explicit control

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Goal.
Construct RW models as extended TQFTs valued in $\mathcal{C} := \mathrm{T}(\mathcal{RW}^{\mathrm{aff}})$.

Rozansky/Witten 1996, Kapranov 1997, Kontsevich 1997, Roberts/Willerton 2006, Kapustin/Rozansky/Saulina 2008, ...

Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

There is a 2-category $\mathcal{C}$ with

- **objects** are lists of variables $\underline{x} := (x_1, x_2, \ldots, x_n)$, $n \in \mathbb{Z}_{\geqslant 0}$
- **1-cells** $\underline{x} \longrightarrow \underline{y}$ are pairs $(\underline{a}; W)$ with $W \in \mathbb{C}[\underline{a}, \underline{x}, \underline{y}]$:

- **horizontal composition**:

$$\big(\underline{b};\, V(\underline{b}, \underline{y}, \underline{z})\big) \circ \big(\underline{a};\, W(\underline{a}, \underline{x}, \underline{y})\big) = \big(\underline{a}, \underline{b}, \underline{y};\, V(\underline{b}, \underline{y}, \underline{z}) + W(\underline{a}, \underline{x}, \underline{y})\big)$$

- $\mathbf{1}_{\underline{x}} := \big(\underline{a};\, \underline{a} \cdot (\underline{x}' - \underline{x})\big)$, where $\underline{a} \cdot (\underline{x}' - \underline{x}) := \sum_{i=1}^{n} a_i \cdot (x_i' - x_i)$

Kapustin/Rozansky 2009, Oblomkov/Rozansky 2018, Brunner/Carqueville/Roggenkamp 2022

Matrix factorisations (will give 2-cells)

– **Matrix factorisation** of $f \in \mathbb{C}[\underline{x}]$ is (X, d_X), where
  - $X = X^0 \oplus X^1$ is free $\mathbb{Z}_2$-graded $\mathbb{C}[\underline{x}]$-module
  - $d_X \colon X \longrightarrow X$ is odd $\mathbb{C}[\underline{x}]$-linear module map with $d_X^2 = f \cdot 1_X$

Example: $f = y^4 - x^3, \quad X = \mathbb{C}[x,y]^2 \oplus \mathbb{C}[x,y]^2,$

$$d_X = \begin{pmatrix} 0 & 0 & -y^2 & -x \\ 0 & 0 & x^2 & y^2 \\ -y^2 & -x & 0 & 0 \\ x^2 & y^2 & 0 & 0 \end{pmatrix}$$

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- For $p_i, q_i \in \mathbb{C}[\underline{x}]$, have **Koszul matrix factorisation** of $f = \sum_i p_i \cdot q_i$:

$$[\underline{p}, \underline{q}] := \big(K(\underline{p}, \underline{q}), d_{K(\underline{p},\underline{q})}\big) , \quad K(\underline{p}, \underline{q}) = \bigwedge \Big(\textstyle\bigoplus_{i=1}^k \mathbb{C}[\underline{x}] \cdot \theta_i \Big)$$

$$d_{K(\underline{p},\underline{q})} = \textstyle\sum_{i=1}^k \big(p_i \cdot \theta_i + q_i \cdot \theta_i^*\big)$$

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$$[\underline{p}, \underline{q}] := \left(K(\underline{p}, \underline{q}), d_{K(\underline{p},\underline{q})}\right), \quad K(\underline{p}, \underline{q}) = \bigwedge \left(\bigoplus_{i=1}^k \mathbb{C}[\underline{x}] \cdot \theta_i\right)$$

$$d_{K(\underline{p},\underline{q})} = \textstyle\sum_{i=1}^k (p_i \cdot \theta_i + q_i \cdot \theta_i^*)$$

- With $\partial_{[i]}^{x',x} f := \frac{f(x_1,\dots,x_{i-1},x_i',\dots x_n') - f(x_1,\dots,x_i,x_{i+1}',\dots x_n')}{x_i' - x_i}$ have

$$I_f := \left[\underline{\partial^{x',x} f}, \underline{x}' - \underline{x}\right]$$

Matrix factorisations (will give 2-cells)

- **Matrix factorisation** of $f \in \mathbb{C}[\underline{x}]$ is (X, d_X), where
  - $X = X^0 \oplus X^1$ is free $\mathbb{Z}_2$-graded $\mathbb{C}[\underline{x}]$-module
  - $d_X \colon X \longrightarrow X$ is odd $\mathbb{C}[\underline{x}]$-linear module map with $d_X^2 = f \cdot 1_X$
- For $p_i, q_i \in \mathbb{C}[\underline{x}]$, have **Koszul matrix factorisation** of $f = \sum_i p_i \cdot q_i$:

$$\big[\underline{p}, \underline{q}\big] := \big(K(\underline{p}, \underline{q}), d_{K(\underline{p},\underline{q})}\big), \quad K(\underline{p}, \underline{q}) = \bigwedge \Big(\bigoplus_{i=1}^k \mathbb{C}[\underline{x}] \cdot \theta_i\Big)$$

$$d_{K(\underline{p},\underline{q})} = \sum_{i=1}^k \big(p_i \cdot \theta_i + q_i \cdot \theta_i^*\big)$$

- With $\partial_{[i]}^{x',x} f := \frac{f(x_1,\dots,x_{i-1},x'_i,\dots x'_n) - f(x_1,\dots,x_i,x'_{i+1},\dots x'_n)}{x'_i - x_i}$ have

$$I_f := \Big[\underline{\partial^{x',x} f}, \underline{x}' - \underline{x}\Big]$$

- **homotopy category of matrix factorisations** has as morphisms even cohomology classes of differential

$$\mathrm{Hom}_{\mathbb{C}[\underline{x}]}(X, X') \longrightarrow \mathrm{Hom}_{\mathbb{C}[\underline{x}]}(X, X')$$

$$\zeta \longmapsto d_{X'} \circ \zeta - (-1)^{|\zeta|} \zeta \circ d_X$$

- $\mathrm{hmf}(\mathbb{C}[\underline{x}], f)^\omega :=$ idempotent completion of finite-rank objects

Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

There is a 2-category $\mathcal{C}$ with

- **objects** are lists of variables $\underline{x} := (x_1, x_2, \dots, x_n)$, $n \in \mathbb{Z}_{\geqslant 0}$
- **1-cells** $\underline{x} \longrightarrow \underline{y}$ are pairs $(\underline{a}; W)$ with $W \in \mathbb{C}[\underline{a}, \underline{x}, \underline{y}]$:

- **horizontal composition**:

$$\big(\underline{b};\, V(\underline{b}, \underline{y}, \underline{z})\big) \circ \big(\underline{a};\, W(\underline{a}, \underline{x}, \underline{y})\big) = \big(\underline{a}, \underline{b}, \underline{y};\, V(\underline{b}, \underline{y}, \underline{z}) + W(\underline{a}, \underline{x}, \underline{y})\big)$$

- $\mathbf{1}_{\underline{x}} = \big(\underline{a};\, \underline{a} \cdot (\underline{x}' - \underline{x})\big)$

Kapustin/Rozansky 2009, Oblomkov/Rozansky 2018, Brunner/Carqueville/Roggenkamp 2022

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- $\mathbf{1}_{\underline{x}} = (\underline{a};\, \underline{a} \cdot (\underline{x}' - \underline{x}))$

- Let $(\underline{a}; W), (\underline{b}; V) \colon \underline{x} \longrightarrow \underline{y}$. A **2-cell** $(\underline{a}; W) \longrightarrow (\underline{b}; V)$ is an isomorphism class X of objects in $\operatorname{hmf}(\mathbb{C}[\underline{a}, \underline{b}, \underline{x}, \underline{y}], V - W)^{\omega}$.

- $\mathbf{1}_{(\underline{a};W)} := I_W = [\underline{\partial^{x',x} W}, \underline{x}' - \underline{x}]$

Kapustin/Rozansky 2009, Oblomkov/Rozansky 2018, Brunner/Carqueville/Roggenkamp 2022

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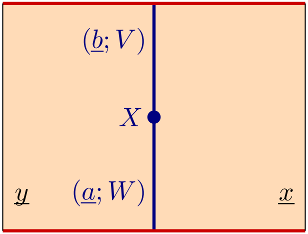

Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

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Horizontal and **vertical composition** via $\big(X \otimes Y,\, d_{X \otimes Y}\big) =$

$$\Big(\big((X^0 \otimes Y^0) \oplus (X^1 \otimes Y^1)\big) \oplus \big((X^0 \otimes Y^1) \oplus (X^1 \otimes Y^0)\big),\, d_X \otimes 1 + 1 \otimes d_Y\Big)$$

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Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

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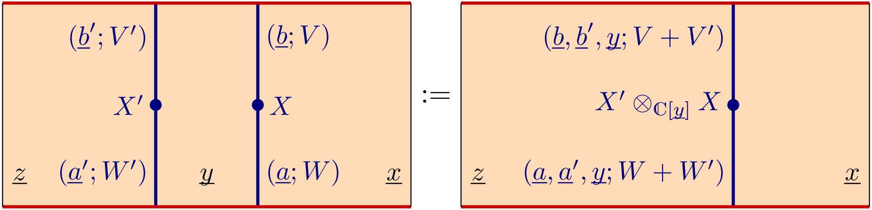

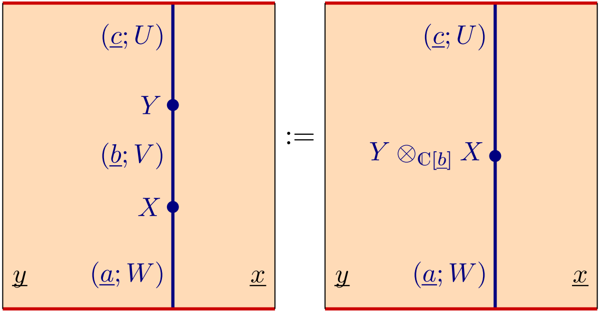

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Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

Monoidal product $\Box\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$,

$$(x_1, \dots, x_n) \,\Box\, (y_1, \dots, y_m) := (x_1, \dots, x_n, y_1, \dots, y_m)$$

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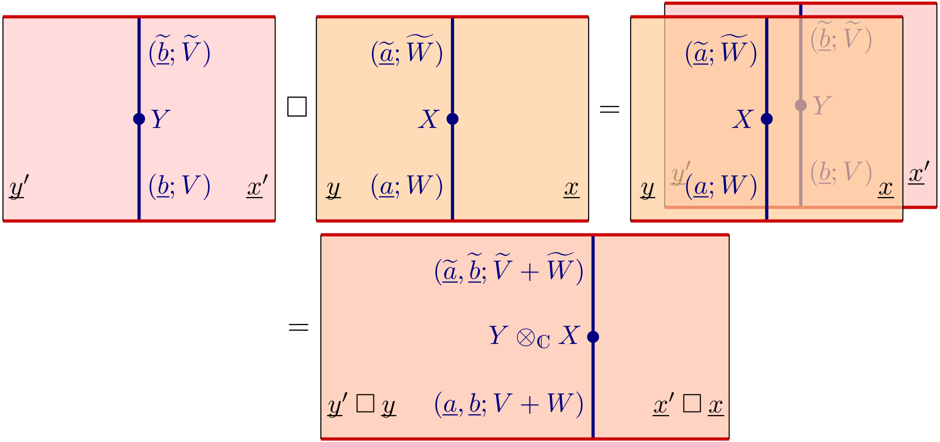

Monoidal unit $= \varnothing$

(structure 2-cells explicit and unsurprising)

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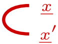
Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

Theorem. $\mathcal{C}$ is symmetric monoidal 2-category.

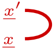
Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

Theorem. $\mathcal{C}$ is symmetric monoidal 2-category.

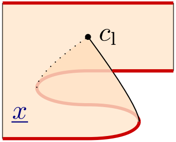
Lemma. Every $\underline{x} \in \mathcal{C}$ has **dual** $\underline{x}^{\#} := \underline{x}$ with



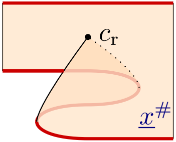
$$= \widetilde{\mathrm{ev}}_{\underline{x}} := \big(\underline{a};\, \underline{a} \cdot (\underline{x}' - \underline{x})\big) \colon \underline{x} \,\Box\, \underline{x}^{\#} = (\underline{x}, \underline{x}') \longrightarrow \varnothing$$



$$= \widetilde{\mathrm{coev}}_{\underline{x}} := \big(\underline{a};\, \underline{a} \cdot (\underline{x} - \underline{x}')\big) \colon \varnothing \longrightarrow \underline{x}^{\#} \,\Box\, \underline{x} = (\underline{x}', \underline{x})$$



$$= c_{\mathrm{l}} \colon \big(\widetilde{\mathrm{ev}}_{\underline{x}} \,\Box\, 1_{\underline{x}}\big) \circ \big(1_{\underline{x}} \,\Box\, \widetilde{\mathrm{coev}}_{\underline{x}}\big) \xrightarrow{\cong} 1_{\underline{x}}$$



$$= c_{\mathrm{r}} \colon \big(1_{\underline{x}^{\#}} \,\Box\, \widetilde{\mathrm{ev}}_{\underline{x}}\big) \circ \big(\widetilde{\mathrm{coev}}_{\underline{x}} \,\Box\, 1_{\underline{x}^{\#}}\big) \xrightarrow{\cong} 1_{\underline{x}^{\#}}$$

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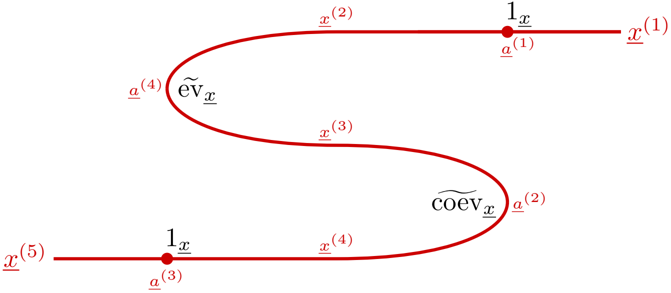
Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

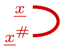
Proof.

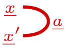


$$
\begin{aligned}
&= \underline{a}^{(1)} \cdot \big(\underline{x}^{(2)} - \underline{x}^{(1)}\big) + \underline{a}^{(2)} \cdot \big(\underline{x}^{(4)} - \underline{x}^{(3)}\big) + \underline{a}^{(3)} \cdot \big(\underline{x}^{(5)} - \underline{x}^{(4)}\big) \\
&\qquad\qquad + \underline{a}^{(4)} \cdot \big(\underline{x}^{(3)} - \underline{x}^{(2)}\big) \\
&\cong \underline{a}^{(2)} \cdot \big(\underline{x}^{(5)} - \underline{x}^{(3)}\big) + \underline{a}^{(4)} \cdot \big(\underline{x}^{(3)} - \underline{x}^{(1)}\big) \\
&= \underline{x}^{(3)} \cdot \big(\underline{a}^{(4)} - \underline{a}^{(2)}\big) + \underline{a}^{(2)} \cdot \underline{x}^{(5)} - \underline{a}^{(4)} \cdot \underline{x}^{(1)} \\
&\cong \underline{a}^{(2)} \cdot \big(\underline{x}^{(5)} - \underline{x}^{(1)}\big) \cong 1_{\underline{x}}
\end{aligned}
$$

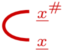
Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

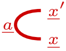
Theorem. Every $\underline{x} \in \mathcal{C}$ is fully dualisable:



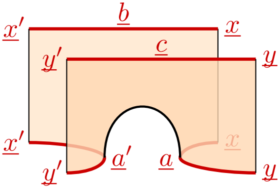
$\equiv$


$$= \mathrm{coev}_{\underline{x}} = {}^{\dagger}\widetilde{\mathrm{ev}}_{\underline{x}} = \widetilde{\mathrm{ev}}^{\dagger}_{\underline{x}} := (\underline{a};\, \underline{a} \cdot (\underline{x} - \underline{x}'))$$



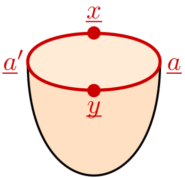
$\equiv$


$$= \mathrm{ev}_{\underline{x}} = {}^{\dagger}\widetilde{\mathrm{coev}}_{\underline{x}} = \widetilde{\mathrm{coev}}^{\dagger}_{\underline{x}} := (\underline{a};\, \underline{a} \cdot (\underline{x}' - \underline{x}))$$



$$= \mathrm{ev}_{\widetilde{\mathrm{ev}}_{\underline{x}}} = \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{coev}}_{\underline{x}}}$$

$$:= [\underline{c} - \underline{a},\, \underline{y} - \underline{y}'] \otimes [\underline{b} - \underline{a}',\, \underline{x}' - \underline{x}] \otimes [\underline{a}' - \underline{a},\, \underline{y}' - \underline{x}]$$



$$= \mathrm{coev}_{\widetilde{\mathrm{ev}}_{\underline{x}}} = \widetilde{\mathrm{coev}}_{\widetilde{\mathrm{coev}}_{\underline{x}}} := [\underline{a}' - \underline{a},\, \underline{x} - \underline{y}]$$

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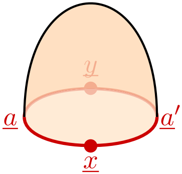
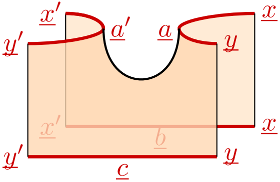
Truncated affine Rozansky–Witten 2-category $\mathcal{C}$



$$= \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_{\underline{x}}} = \mathrm{ev}_{\widetilde{\mathrm{coev}}_{\underline{x}}} := \left[\underline{y} - \underline{x},\, \underline{a}' - \underline{a}\right]$$

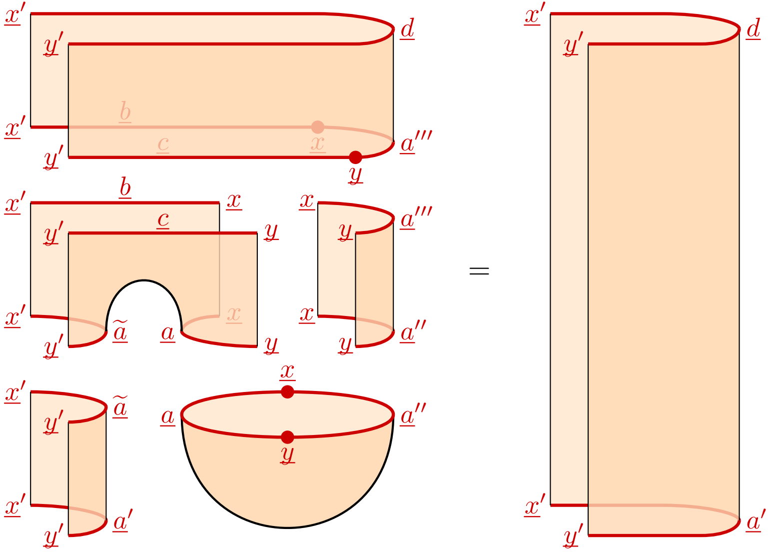


$$= \widetilde{\mathrm{coev}}_{\widetilde{\mathrm{ev}}_{\underline{x}}} = \mathrm{coev}_{\widetilde{\mathrm{coev}}_{\underline{x}}}$$

$$:= \left[\underline{c} - \underline{a},\, \underline{y}' - \underline{y}\right] \otimes \left[\underline{b} - \underline{a},\, \underline{x} - \underline{x}'\right] \otimes \left[\underline{y}' - \underline{x}',\, \underline{a} - \underline{a}'\right]$$

Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

Proof. Explicit computation of Zorro moves, e. g.



Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

Lemma. For all $\underline{x} \in \mathcal{C}$, there are precisely two isomorphisms

$$S_{\underline{x}} \xrightarrow{\cong} 1_{\underline{x}}$$

represented by the matrix factorisations $I_{1_{\underline{x}}}$ and $I_{1_{\underline{x}}}[1]$.

Truncated affine Rozansky–Witten 2-category $\mathcal{C}$

Lemma. For all $\underline{x} \in \mathcal{C}$, there are precisely two isomorphisms

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represented by the matrix factorisations $I_{1_{\underline{x}}}$ and $I_{1_{\underline{x}}}[1]$.

Proof.

$$\begin{aligned} S_{\underline{x}} &= (1_{\underline{x}} \,\square\, \widetilde{\mathrm{ev}}_{\underline{x}}) \circ (b_{\underline{x},\underline{x}} \,\square\, 1_{\underline{x}^\#}) \circ (1_{\underline{x}} \,\square\, \widetilde{\mathrm{ev}}^\dagger_{\underline{x}}) \\ &= \Big(\underline{a}^{(1)}, \dots, \underline{a}^{(7)}, \underline{x}^{(2)}, \dots, \underline{x}^{(7)} \,;\, \sum_{i=1}^{7} \underline{a}^{(i)} \cdot \big(\underline{x}^{(i+1)} - \underline{x}^{(i)}\big)\Big) \\ &= \Big(\underline{a}^{(1)};\ \underline{a}^{(1)} \cdot \big(\underline{x}^{(2)} - \underline{x}^{(1)}\big)\Big) \circ \Big(\underline{a}^{(2)};\ \underline{a}^{(2)} \cdot \big(\underline{x}^{(3)} - \underline{x}^{(2)}\big)\Big) \\ &\qquad \circ \dots \circ \Big(\underline{a}^{(7)};\ \underline{a}^{(7)} \cdot \big(\underline{x}^{(8)} - \underline{x}^{(7)}\big)\Big) \ = \ (1_{\underline{x}})^7 \end{aligned}$$

and

$$\begin{aligned} \mathrm{hmf}\left(\mathbb{C}[\underline{a},\underline{b},\underline{x},\underline{y}], (\underline{a}-\underline{b})\cdot(\underline{x}-\underline{y})\right)^\omega &\cong \mathrm{hmf}\left(\mathbb{C}[\underline{a},\underline{b},\underline{x},\underline{y}], \underline{b}\cdot\underline{y}\right)^\omega \\ &\cong \mathrm{hmf}\left(\mathbb{C}[\underline{a},\underline{x}], 0\right)^\omega \\ &\cong \mathrm{mod}^{\mathbb{Z}_2}(\mathbb{C}[\underline{a},\underline{x}]) \end{aligned}$$

Truncated affine Rozansky–Witten models

Theorem. Every $\underline{x} = (x_1, \dots, x_n) \in \mathcal{C}$ gives unique extended TQFT:

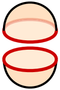
$$\mathrm{Bord}^{\mathrm{or}}_{2,1,0} \longrightarrow \mathcal{C}$$

$$+ \longmapsto \underline{x}$$

$$= \widetilde{\mathrm{ev}}_+ \longmapsto \underline{a} \cdot (\underline{x} - \underline{x}')$$

$$= \widetilde{\mathrm{ev}}_+ \otimes {}^\dagger\widetilde{\mathrm{ev}}_+ = S^1 \longmapsto (\underline{a} - \underline{a}') \cdot (\underline{x} - \underline{x}')$$

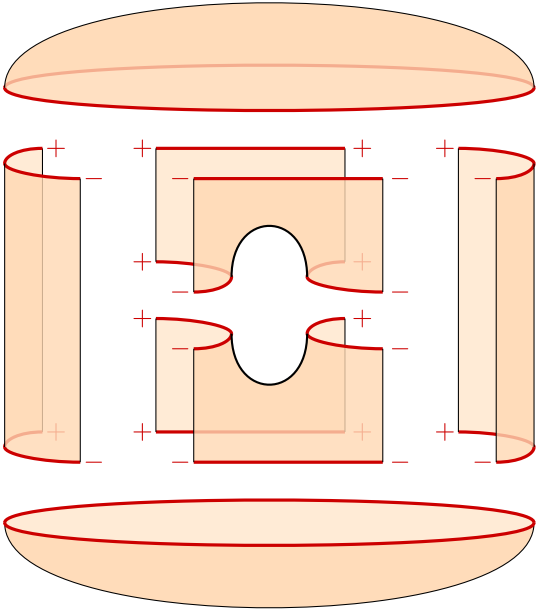
$$= \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_+} \longmapsto \Big[\underline{a} - \underline{a}', \underline{x} - \underline{x}'\Big]$$



$$= \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_+} \circ \mathrm{coev}_{\widetilde{\mathrm{ev}}_+} = S^2 \longmapsto \mathbb{C}[\underline{a}, \underline{x}]$$

Truncated affine Rozansky–Witten models

$T^2 \quad =$



Truncated affine Rozansky–Witten models

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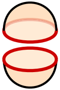

$$= \widetilde{\mathrm{ev}}_{\widetilde{\mathrm{ev}}_+} \circ \mathrm{coev}_{\widetilde{\mathrm{ev}}_+} = S^2 \longmapsto \mathbb{C}[\underline{a}, \underline{x}]$$

$$\Sigma_g \longmapsto \mathbb{C}[\underline{a}, \underline{x}] \otimes \big(\mathbb{C} \oplus \mathbb{C}[1]\big)^{\otimes 2ng}$$

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Truncated affine Rozansky–Witten models

Theorem. Every $\underline{x} \equiv (x_1, \dots, x_n) \in \mathcal{C}$ gives unique extended TQFT:

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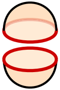

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($\lambda = I_{1_{\underline{x}}}$ and $\lambda = I_{1_{\underline{x}}}[1]$ give equivalent TQFTs.)

obtain Rozansky–Witten **state spaces** from extended TQFT

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Truncated affine Rozansky–Witten models

Theorem′. Every $\underline{x} = (x_1, \dots, x_n) \in \mathcal{C}$ gives unique extended TQFT:

$$
\begin{aligned}
\mathrm{Bord}^{\mathrm{or}}_{2,1,0} &\longrightarrow \mathcal{C} \\
+ &\longmapsto (x_1, \dots, x_n) \\
S^1 &\longmapsto (\underline{a} - \underline{a}') \cdot (\underline{x} - \underline{x}') \\
&\qquad \widehat{=} \ \mathrm{hmf}\Big(\mathbb{C}[\underline{a}, \underline{d}, \underline{x}, \underline{y}];\ (\underline{a} - \underline{d}) \cdot (\underline{x} - \underline{y})\Big)^{\omega} \\
&\qquad \cong \ \mathrm{mod}^{\mathbb{Z}_2}(\mathbb{C}[\underline{a}, \underline{x}]) \ =: \ \mathcal{A}_n \\
S^2 &\longmapsto \mathbb{C}[\underline{a}, \underline{x}] \\
T^2 &\longmapsto \mathbb{C}[\underline{a}, \underline{x}] \otimes \big(\mathbb{C} \oplus \mathbb{C}[1]\big)^{\otimes 2g} \ \cong \ \mathrm{HH}_\bullet(\mathcal{A}_n) \\
\Sigma_g &\longmapsto \mathbb{C}[\underline{a}, \underline{x}] \otimes \big(\mathbb{C} \oplus \mathbb{C}[1]\big)^{\otimes 2ng}
\end{aligned}
$$

Can also compute **state spaces with defects**.

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Further directions

Option 1. $\mathcal{C}$ symmetric monoidal $(\infty, 2)$-category

$\Longrightarrow$ obtain **mapping class group** representations (?)

Option 2.

- Encorporate **flavour and R-charge** into new 2-category $\mathcal{C}^{\mathrm{gr}}$:
  - objects $\underline{x}$: all variables x_i have bidegree $(+1, -1)$
  - 1-cells $(\underline{a}; W)$ with a_k arbitrary bidegree in $\mathbb{Q} \times \mathbb{Q}$,
    W homogeneous of bidegree $(2, 0)$
  - 2-cells are $(\mathbb{Z}_2 \times \mathbb{Q} \times \mathbb{Q})$-graded matrix factorisation classes
- Every $\underline{x} \in \mathcal{C}^{\mathrm{gr}}$ fully dualisable, $S_{\underline{x}}$ trivialisable.
- Get extended TQFT $\mathcal{Z}_n^{\mathrm{gr}} \colon \mathrm{Bord}_{2,1,0}^{\mathrm{or}} \longrightarrow \mathcal{C}^{\mathrm{gr}}$ with (✓)

$$\mathcal{Z}_n^{\mathrm{gr}}(\Sigma_g) = \Big(\big(\mathbb{C} \oplus \mathbb{C}[1]_{\{0,\,1\}}\big)^{\otimes n} \otimes \big(\mathbb{C} \oplus \mathbb{C}[1]_{\{0,\,-1\}}\big)^{\otimes n}{}_{\{1,\,0\}} \Big)^{\otimes g} \otimes \mathbb{C}[\underline{a}, \underline{x}]_{\{-1,\,0\}}$$

Option 3.

Construction for target $T^*\mathbb{CP}^{n-1}$ via **$U(1)$-equivariantisation**. . . (✓ wip)

Option 4.

Consider all Rozansky–Witten models with compact target (?)

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Summary

Theorem.
Affine **Landau–Ginzburg models** give spin extended TQFTs

$$\mathrm{Bord}_{2,1,0}^{\mathrm{spin}} \longrightarrow \mathcal{LG}$$

$$+ \longmapsto W$$

 $\longmapsto \mathrm{Jac}_W$

 $\longmapsto \mathrm{Res}\left[\frac{(-)\,\mathrm{d}x}{\partial_{x_1}W \dots \partial_{x_n}W}\right]$

Theorem.
Affine **Rozansky–Witten models** give oriented extended TQFTs

$$\mathrm{Bord}_{2,1,0}^{\mathrm{or}} \longrightarrow \mathcal{C} = \mathrm{T}(\mathcal{RW}^{\mathrm{aff}})$$

$$+ \longmapsto \underline{x} = (x_1, \dots, x_n)$$

$$S^1 \longmapsto (\underline{a} - \underline{a}') \cdot (\underline{x} - \underline{x}')$$

$$\Sigma_g \longmapsto \mathbb{C}[\underline{a}, \underline{x}] \otimes \big(\mathbb{C} \oplus \mathbb{C}[1]\big)^{\otimes 2ng}$$

Carqueville/Montiel Motoya 2018, Carqueville/Szegedy 2021, Brunner/Carqueville/Roggenkamp 2022