Symmetry Relations between Feynman Integrals

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Motivation

It is well-known that Feynman integrals satisfy linear relations, the integration-by-parts identities. However, in many cases there are further relations, which simply relate two Feynman integrals via a judicious choice of integration variables. The existence of these so-called *symmetry relations* is well-known and they can often be identified from the graph. There are however some open questions, which we aim to address [1].

- Can we give a complete description of the symmetry group and devise an algorithm to find it?
- What is the origin of the kinematics-dependent symmetry transformations discussed in [2]?
- How are symmetry transformations reflected in the various integral representations, and do they agree? In particular it would be natural to formulate a notion of symmetry transformations in the framework of twisted cohomology.
- Can we, a priori, compute the number of master integrals, taking the symmetry group into account?

Integral representations

$$\text{Loop-momentum:}\quad I_{\boldsymbol{\nu}}(\boldsymbol{s},\varepsilon) = \int \left(\prod_{j=1}^{L} \frac{\mathrm{d}^D k_j}{i\pi^{\frac{D}{2}}}\right) \frac{1}{(q_1^2-m_1^2)^{\nu_1}\dots(q_P^2-m_P^2)^{\nu_P}}\,,$$

$$\text{Lee-Pomeransky:}\quad I_{\boldsymbol{\nu}}(\boldsymbol{s},\varepsilon) \sim \left(\prod_{i=1}^{P}\int_0^\infty \mathrm{d}z_i\, z_i^{\nu_i-1}\right)\mathcal{G}(\boldsymbol{z},\boldsymbol{s})^{-\frac{D}{2}}\,,$$

$$\text{(Democratic) Baikov:}\quad I_{\boldsymbol{\nu}}(\boldsymbol{s},\varepsilon) \sim \int_{\mathcal{C}} \mathrm{d}^n z\,[\mathcal{B}(\boldsymbol{z})]^{\frac{D-L-E-1}{2}} \prod_{s=1}^{P} z_s^{-\nu_s}\,.$$

Symmetry Transformations from Graph Theory

Consider a Feynman integral family in loop-momentum space and group the integrals into sectors, according to which propagators are active. Choose two sectors Θ_1 and Θ_2 and parametrize the edge momenta as

$$\boldsymbol{q}_i(\boldsymbol{k},\boldsymbol{p}) = \boldsymbol{C}_i^T\boldsymbol{k} + \boldsymbol{E}_i^T\boldsymbol{p}\,,\quad i=1,2\,.$$

Then, we define a symmetry transformation from Θ_1 to Θ_2 as a transformation

$$\boldsymbol{k}_1 = \boldsymbol{L}^T\boldsymbol{k}_2 + \boldsymbol{M}^T\boldsymbol{p}_2\,,\qquad \boldsymbol{p}_1 = \boldsymbol{N}^T\boldsymbol{p}_2\,,$$

such that

- The Jacobian is trivial: $\det\boldsymbol{L} = \pm 1$.
- There is a bijection α on the edge momenta: $(\boldsymbol{q}_1)_i(\boldsymbol{k_1},\boldsymbol{p}_1) = \pm(\boldsymbol{q}_2)_{\alpha(i)}(\boldsymbol{k}_2,\boldsymbol{p}_2)$.
- The external scalar products are invariant: $(\boldsymbol{p}_1)_i\cdot(\boldsymbol{p}_1)_j = (\boldsymbol{p}_2)_i\cdot(\boldsymbol{p}_2)_j$.

We can explicitly describe the set $\mathrm{Sym}(\Theta_1,\Theta_2)$ of all symmetry transformations as [1]

$$\mathrm{Sym}(\Theta_1,\Theta_2) = \mathbb{S}(\mathcal{G}_1,\mathcal{G}_2)\times(\text{trivial})\,,$$

where we are suppressing factors that act trivially, e.g., by flipping the signs of all momenta. The non-trivial part is given by symmetry transformations between the Lee-Pomeransky polynomials

$$\mathbb{S}(\mathcal{G}_1,\mathcal{G}_2) = \{\alpha\in S_P\,|\,\mathcal{G}_1(\alpha(\boldsymbol{z})) = \mathcal{G}_2(\boldsymbol{z})\}\,.$$

This set includes graph isomorphisms but also more general *matroid* isomorphisms. It can be determined by using Pak's algorithm [3], for example. From an element $\alpha\in\mathbb{S}(\mathcal{G}_1,\mathcal{G}_2)$, one can then construct the corresponding symmetry transformation as follows: First, lift α to a signed permutation $\sigma = (\boldsymbol{\kappa},\alpha)$ with associated (signed) permutation matrix $(P_\sigma)_{ij} = \kappa_j\delta_{i\alpha(j)}$. Then the symmetry transformation is given by [1]

$$\begin{aligned}\boldsymbol{L}_\sigma &= \boldsymbol{C}_2\boldsymbol{P}_\sigma\boldsymbol{C}_1^T(\boldsymbol{C}_1\boldsymbol{C}_1^T)^{-1}\,,\\ \boldsymbol{N}_\sigma &= \boldsymbol{E}_2\boldsymbol{\Pi}^\perp_{C_2}\boldsymbol{P}_\sigma\boldsymbol{\Pi}^\perp_{C_1}\boldsymbol{E}_1^T(\boldsymbol{E}_1\boldsymbol{\Pi}^\perp_{C_1}\boldsymbol{E}_1^T)^{-1}\,,\\ \boldsymbol{M}_\sigma &= \boldsymbol{E}_2\boldsymbol{P}_\sigma\boldsymbol{C}_1^T(\boldsymbol{C}_1\boldsymbol{C}_1^T)^{-1} - \boldsymbol{N}_\sigma\boldsymbol{E}_1\boldsymbol{C}_1^T(\boldsymbol{C}_1\boldsymbol{C}_1^T)^{-1}\,,\end{aligned}\qquad \boldsymbol{\Pi}^\perp_{C_i} = \mathbb{1} - \boldsymbol{C}_i^T(\boldsymbol{C}_i\boldsymbol{C}_i^T)^{-1}\boldsymbol{C}_i\,.$$

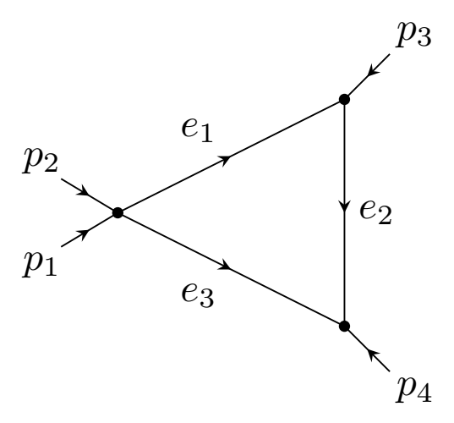
For sectors with momentum groupings (multiple external momenta entering at same the vertex), we first associate auxiliary sectors without momentum groupings, by adding up the respective momenta $\widetilde{\boldsymbol{p}}_i = \boldsymbol{S}_i\boldsymbol{p}_i$. Given the transformation between these sectors with matrices $\widetilde{\boldsymbol{L}}_\sigma,\widetilde{\boldsymbol{M}}_\sigma,\widetilde{\boldsymbol{N}}_\sigma$, we can construct the transformation between the original sectors via [1]

$$\begin{pmatrix}\boldsymbol{L}_\sigma & 0\\ \boldsymbol{M}_\sigma & \boldsymbol{N}_\sigma\end{pmatrix} = \begin{pmatrix}\mathbb{1} & 0\\ 0 & \boldsymbol{S}_2\\ 0 & \boldsymbol{S}_2'(\boldsymbol{s})\end{pmatrix}\begin{pmatrix}\widetilde{\boldsymbol{L}}_\sigma & 0 & 0\\ \widetilde{\boldsymbol{M}}_\sigma & \widetilde{\boldsymbol{N}}_\sigma & 0\\ 0 & 0 & \overline{\boldsymbol{O}}\end{pmatrix}\begin{pmatrix}\mathbb{1} & 0\\ 0 & \boldsymbol{S}_1\\ 0 & \boldsymbol{S}_1'(\boldsymbol{s})\end{pmatrix}^{-1}\,,\qquad \boldsymbol{S}_i\boldsymbol{G}(\boldsymbol{p}_i)\boldsymbol{S}_i'^T(\boldsymbol{s}) = 0\,,$$

where $\overline{\boldsymbol{O}}$ is unimportant and only reflects a "gauge choice". Note that the matrices $\boldsymbol{S}_i'(\boldsymbol{s})$ introduce kinematics dependence into the transformation via $\boldsymbol{N}_\sigma$. This explains the observations of [2] in a systematic way.

Example: A Four-Point Triangle Integral

Consider a triangle subsector of the on-shell massless box with edge momenta



$$\boldsymbol{q} = \begin{pmatrix}k+p_1+p_2\\ k+p_1+p_2+p_3\\ -k\end{pmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix}k + \begin{pmatrix}1&1&0\\1&1&1\\0&0&0\end{pmatrix}\begin{pmatrix}p_1\\p_2\\p_3\end{pmatrix}.$$

Consider the auxiliary sector with external momenta $\widetilde{p}_1 = p_1+p_2$, $\widetilde{p}_2 = p_3$. From

$$\mathcal{G}(\boldsymbol{z}) = z_1+z_2+z_3-sz_1z_3\,,\qquad s = (p_1+p_2)^2 = \widetilde{p}_1^2\,,$$

we see that the sector admits a symmetry transformation, that exchanges e_1 and e_3. Lifting the permutation (13) to the signed permutation $\sigma = ((1,-1,1),(13))$, and plugging into the above formulas, we find

$$\widetilde{\boldsymbol{L}}_\sigma = (-1)\,,\quad \widetilde{\boldsymbol{M}}_\sigma = \begin{pmatrix}-1\\0\end{pmatrix},\quad \widetilde{\boldsymbol{N}}_\sigma = \begin{pmatrix}1&-1\\0&-1\end{pmatrix},\qquad \text{i.e.,}\qquad k = -k'-\widetilde{p}_1'\,,\qquad \begin{aligned}\widetilde{p}_1 &= \widetilde{p}_1'\,,\\ \widetilde{p}_2 &= -\widetilde{p}_1'-\widetilde{p}_2'\,.\end{aligned}$$

This can now be lifted to the original sector to find (with $x = t/s$, $t = (p_1+p_3)^2$)

$$\begin{pmatrix}p_1\\p_2\\p_3\end{pmatrix} = \begin{pmatrix}-(1+o)x & -(x+o(x+1)) & -(1+o)(1+2x)\\ -(1+x+ox) & (1+o)(1+x) & (1+o)(1+2x)\\ -1 & -1 & -1\end{pmatrix}\begin{pmatrix}p_1'\\p_2'\\p_3'\end{pmatrix},\qquad o\in\{\pm1\}\,.$$

In particular, for the gauge choice $o = 1$, we obtain a kinematics-dependent transformation.

Twisted Symmetries

Twisted cohomology deals with integrals of the form

$$I = \langle\varphi|\gamma] = \int_\gamma \varphi\,\Psi\,,$$

with the *twist* Ψ a multivalued function, and the integration contour and integrand living in *twisted (co)homology groups*. Focusing on the Baikov representation, for example, we need to consider

$$\gamma\in H_n(X,\check{\mathcal{L}}_\omega)\,,\quad \varphi\in H_n(X,\nabla_\omega)\,,\qquad\text{with}\qquad X = \mathbb{C}^n\setminus(\Sigma_\Psi\cup D_-)\,,$$

and

$$\Sigma_\Psi = \{\boldsymbol{z}\in\mathbb{C}^n\,|\,\Psi(\boldsymbol{z}) = 0\}\,,\qquad D_- = \bigcup_{i=1}^{P} D_-^{(i)}\,,\qquad D_-^{(i)} = \{\boldsymbol{z}\in\mathbb{C}^n\,|\,z_i = 0\}\,,$$

where we recall that $z_1,\dots,z_P$ correspond to the propagators, while $z_{P+1},\dots,z_n$ are irreducible scalar products (ISPs). Furthermore, $\check{\mathcal{L}}_\omega = \mathcal{L}_{-\omega}$ is a local system, which keeps track of the branch of the twist, and $\nabla_\omega = \mathrm{d}+\omega\wedge$ is a covariant derivative determined by $\omega = \mathrm{dlog}\,\Psi$.

We can define a notion of *sectors* of the twisted cohomology group, with a sector Θ consisting of forms with singularities on a prescribed subset of components of D_-, mirroring the corresponding definition for Feynman integrals. Given two sectors Θ_1,Θ_2 we define a *twisted symmetry* to be an affine transformation

$$f: X_{-,\Theta_1}\to X_{-,\Theta_2}\,,\quad \boldsymbol{z}\mapsto\boldsymbol{A}\boldsymbol{z}+\boldsymbol{b}\,,$$

with

$$X_{-,\Theta_i} = \mathbb{C}^n\setminus(\Sigma_\Psi\cup D_{-,\Theta_i})\,,\quad D_{-,\Theta_i} = \cup_{i\in\Theta_i}D_-^{(i)}\,,$$

such that

- The Jacobian is trivial: $\det\boldsymbol{A} = \pm1$.
- The twist is invariant: $\Psi(f(\boldsymbol{z})) = \Psi(\boldsymbol{z})$.
- The components of D_- are permuted, i.e., there is a bijection α such that $f(D_-^{(i)}) = D_-^{(\alpha(i))}$ for $i\in\Theta_1$.

There are pairings between twisted (co)homology groups (and their duals), which in a chosen set of bases, are represented by certain matrices. Explicitly, we have the (co)homology intersection matrices $\boldsymbol{C}$ and $\boldsymbol{H}$ as well as the period matrix $\boldsymbol{P}$ and its dual $\check{\boldsymbol{P}}$. If we focus on a particular sector, the symmetries in this sector form a group G and induce a representation $\boldsymbol{R}$ on the (dual) (co)homology groups. Then the pairing matrices (in that sector) are invariant in the sense that for all $g\in G$ [1]

$$\boldsymbol{R}(g)\boldsymbol{P}\boldsymbol{R}(g)^{-1} = \boldsymbol{P}\,,\qquad \boldsymbol{R}(g)\check{\boldsymbol{P}}\boldsymbol{R}(g)^{-1} = \check{\boldsymbol{P}}\,,\qquad \boldsymbol{R}(g)\boldsymbol{C}\boldsymbol{R}(g)^{-1} = \boldsymbol{C}\,,\qquad \boldsymbol{R}(g)\boldsymbol{H}\boldsymbol{R}(g)^{-1} = \boldsymbol{H}\,.$$

In particular, we can further rotate the basis such that $\boldsymbol{R}$ decomposes into irreducible representations of G. Then, the pairing matrices take a block-diagonal form, with the blocks corresponding to the inequivalent irreducible representations and the entire twisted cohomology framework "factorizes" [1]. In particular it makes sense to focus on the trivial representation, i.e., the G-invariant twisted cohomology group $H^n_G(X,\nabla_\omega)$, etc.

All of the above statements can be extended to more general twisted cohomology setups, covering also other representations of Feynman integrals, such as the Lee-Pomeransky or Feynman representation [1]. The main novelty there, is the presence of *boundaries* on which the integration cycles are allowed to end, which are appropriately described in the framework of *relative* twisted cohomology.

Twisted Symmetries of Feynman Integrals

Feynman integrals are pairings of twisted cohomology elements, with a distinguished twisted homology element, the *Feynman contour*.
Examples:

$$\gamma_{\text{loop}} = \mathbb{R}^{LD}\,,\qquad \gamma_{\text{Baikov}} = \{\boldsymbol{z}\in\mathbb{R}^n\,|\,\mathcal{B}(\boldsymbol{z})\geq0\}\,,\qquad \gamma_{\text{LP}} = \mathbb{R}^P_{\geq0}\,.$$

Hence, a Feynman integral family should not be identified with a twisted cohomology group but rather as the set of twisted periods $\langle\varphi|\gamma]$ for φ in the twisted cohomology group and γ the Feynman contour. Symmetry transformations between integrals are twisted symmetries that leave the Feynman contour invariant

$$\mathrm{Sym}(\Theta_1,\Theta_2) = \{f\in\mathrm{TSym}(\Theta_1,\Theta_2)\,|\,f_*\gamma = \gamma\}\,.$$

One can show that the group $\mathrm{Sym}(\Theta_1,\Theta_2)$ agrees in the Baikov, the Lee-Pomeransky and the Feynman representation (up to trivial factors) [1]. Also, it matches the definition in loop-momentum space, given in the left column.

Consider the group G of symmetry transformations within a particular sector. The Feynman contour is always in the trivial representation of G and hence the master integrals need to have integrands in the G-invariant twisted cohomology group $H^n_G(X,\nabla_\omega)$, in order for the integrals to be non-trivial, c.f., [4].

The Euler Characteristic

Recall that (working e.g., in a fixed sector),

$$\dim H^n(X,\nabla_\omega) = |\chi(X)|\,.$$

Taking a symmetry group G into account we are interested in the dimension of $H^n_G(X,\nabla_\omega)$, which can be computed by an *orbit space Euler characteristic* [1]

$$\dim H^n_G(X,\nabla_\omega) = |\chi(X/G)| = \frac{1}{|G|}\sum_{[\sigma]\in G} n_{[\sigma]}|\chi(X_\sigma)|\,,$$

where the sum is over all conjugacy classes $[\sigma]$ of G, we denote their sizes by $n_{[\sigma]}$ and X_σ is the fixed-point locus of σ in X. Note that we can compute the terms in the sum by counting critical points of the twist restricted to X_σ.

Example: The Equal-Mass Sunrise

$$\mathcal{G}(\boldsymbol{z}) = (z_1z_2+z_1z_3+z_2z_3)(1+m^2(z_1+z_2+z_3)) - p^2z_1z_2z_3\,.$$

Critical point counting yields $|\chi(X)| = |\chi(X_{\mathbb{1}})| = 4$ and hence $\dim H^3(X,\nabla_\omega) = 4$. Restricting to the fixed-point loci yields

$$\begin{aligned}\mathcal{G}_{(12)}(\boldsymbol{z}) &= z_1(2z_2+z_1(1-p^2z_2)+m^2(2z_1^2+5z_1z_2+2z_2^2)\,,\\ \mathcal{G}_{(123)}(\boldsymbol{z}) &= z_1^2(3-p^2z_1+9m^2z_1)\,.\end{aligned}$$

Critical point counting gives $|\chi(X_{(12)})| = 2$, $|\chi(X_{(123)})| = 1$. Thus

$$\dim H^3_G(X,\nabla_\omega) = \frac{1}{6}(1\times4+3\times2+2\times1) = 2\,.$$

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