Dirac fields on Kerr spacetime and the Hawking radiation II

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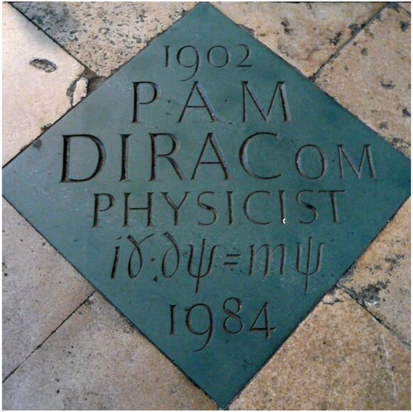
Part II : Scattering theory for Dirac fields on the Kerr metric (with Jean-Philippe Nicolas)



(commemorative marker in Westminster Abbey)

II.1 Dirac fields on globally hyperbolic manifolds

Spinor bundles A spinor bundle is a vector bundle $\mathcal{S} \xrightarrow{\pi} M$ with the following objects ($\mathcal{S}^*$ is the anti-dual bundle):

1. a linear map $\gamma : C^\infty(M; TM) \to C^\infty(M; \mathrm{End}(\mathcal{S}))$ such that,
$$\gamma(X)\gamma(Y) + \gamma(Y)\gamma(X) = 2X \cdot gY\mathbf{1},\ X, Y \in C^\infty(M; TM), \tag{1}$$
and for each $x \in M$, γ_x induces a faithful irreducible representation of the Clifford algebra $\mathrm{Cl}(T_xM, g_x)$ in $\mathcal{S}_x$;
2. a section $\beta \in C^\infty(M; \mathrm{End}(\mathcal{S}, \mathcal{S}^*))$ such that β_x is Hermitian non-degenerate for each $x \in M$ and
$$\begin{aligned} &i) \quad \gamma(X)^*\beta = -\beta\gamma(X), \quad \forall X \in C^\infty(M; TM), \\ &ii) \quad \imath\beta\gamma(e) > 0, \text{ for } e \text{ a time-like, future directed vector field on } M; \end{aligned} \tag{2}$$
3. a section $\kappa \in C^\infty(M; \mathrm{End}(\mathcal{S}, \bar{\mathcal{S}}))$ such that
$$\kappa\gamma(X) = \gamma(X)\kappa, \quad \kappa^2 = \mathbf{1}; \tag{3}$$

Connections

A connection $\nabla^{\mathcal{S}}$ on $\mathcal{S}$, called a spin connection, such that:

$$\begin{aligned} i) \quad & \nabla^{\mathcal{S}}_X(\gamma(Y)\psi) = \gamma(\nabla_X Y)\psi + \gamma(Y)\nabla^{\mathcal{S}}_X\psi, \\ ii) \quad & X(\bar{\psi}\cdot\beta\psi) = \overline{\nabla^{\mathcal{S}}_X\psi}\cdot\beta\psi + \bar{\psi}\cdot\beta\nabla^{\mathcal{S}}_X\psi, \\ iii) \quad & \kappa\nabla^{\mathcal{S}}_X\psi = \nabla^{\mathcal{S}}_X\kappa\psi, \end{aligned} \tag{4}$$

for all $X, Y \in C^\infty(M; TM)$ and $\psi \in C^\infty(M; \mathcal{S})$, where ∇ is the Levi-Civita connection on (M, g). The rank of $\mathcal{S}$ is necessarily equal to 4 and if (2) holds for some time-like future directed vector field e, then it holds for all such vector fields.

Remark

A linear map γ as in (1) is called a Clifford representation. *A section β as in (2) is called a positive energy Hermitian form (for the Clifford representation γ), while a section κ as in* (3) *is called a charge conjugation (for the Clifford representation γ). The properties listed in* (4) *are usually summarized by saying that γ, β, κ are covariantly constant w.r.t. the connection $\nabla^{\mathcal{S}}$.*

Weyl spinors

Volume element $\eta = \gamma(e_0)\cdots\gamma(e_3)$ (independent on the choice of local oriented orthonormal frame e_0, e_1, e_2, e_3).

$$\eta^2 = -\mathbf{1}, \quad \eta\gamma(X) = -\gamma(X)\eta. \tag{5}$$

$\mathcal{S}_x = \mathcal{W}_{\mathrm{e},x} \oplus \mathcal{W}_{\mathrm{o},x}$, $\mathcal{W}_{\mathrm{e},x} = \mathrm{Ker}(\imath\eta(x) - \mathbf{1})$, $\mathcal{W}_{\mathrm{o},x} = \mathrm{Ker}(\imath\eta(x) + \mathbf{1})$.
The bundle $\mathcal{S}$ splits as $\mathcal{W}_\mathrm{e} \oplus \mathcal{W}_\mathrm{o}$, where $\mathcal{W}_{\mathrm{e}/\mathrm{o}}$ is the bundle of *even/odd Weyl spinors*.
$\mathcal{W}^{\#}$: dual bundle of $\mathcal{W}$, $\mathcal{W}^* = \overline{\mathcal{W}^{\#}}$. Setting $\mathbb{S} := \mathcal{W}_\mathrm{e}^*$, one identifies $\mathcal{S}$ with $\mathbb{S}^* \oplus \mathbb{S}^{\#}$ by the map

$$C^\infty(M;\mathcal{S}) \ni \psi \mapsto \psi_\mathrm{e} \oplus \kappa\psi_\mathrm{o} =: \chi \oplus \phi \in C^\infty(M;\mathbb{S}^*) \oplus C^\infty(M;\mathbb{S}^{\#}).$$

$\mathbb{S}$ is equipped with the symplectic form

$$\epsilon := \frac{1}{\sqrt{2}}(\beta\kappa)^{-1} \in C^\infty(M;\mathrm{End}(\mathbb{S},\mathbb{S}^{\#})).$$

Null tetrads and associated frames

There is an isomorphism

$$\tau : C^\infty(M; \mathbb{C}TM) \ni v \mapsto \beta\gamma(v) \in C^\infty(M; \mathbb{S} \otimes \bar{\mathbb{S}})$$

If we extend g to $\mathbb{C}TM$ as a *bilinear* (not sesquilinear) form, one can show that

$$\tau^\# \circ (\epsilon \otimes \bar{\epsilon}) \circ \tau = g.$$

A normalized null tetrad (Newman-Penrose tetrad) is a global frame $(l, n, m, \bar{m})$ of $\mathbb{C}TM$ such that:

$$l, n \text{ are real}, \quad l \cdot gl = n \cdot gn = 0, \quad l \cdot gn = -1$$
$$m \cdot gm = l \cdot gm = n \cdot gm = 0, \quad m \cdot g\bar{m} = 1.$$

Spin frames

$$\imath\tau(l) = o \otimes \bar{o}, \quad \imath\tau(n) = \imath \otimes \bar{\imath},$$

$$\imath\tau(m) = o \otimes \bar{\imath}, \quad \imath\tau(\bar{m}) = \imath \otimes \bar{o},$$

$$o \cdot \epsilon\imath = 1.$$

$$\Gamma(X) = \beta\gamma(X) \in C^\infty(M, \mathrm{End}(\mathbb{S}^*, \mathbb{S})), \quad X \in C^\infty(M; TM).$$

Dirac operators

If $\mathcal{S} \xrightarrow{\pi} M$ is a spinor bundle, the Dirac operator $\not{D}$ acting on $C^\infty(M;\mathcal{S})$ is the differential operator defined as:

$$\not{D} = g^{\mu\nu}\gamma(e_\mu)\nabla^{\mathcal{S}}_{e_\nu}.$$

where $(e_0,\dots,e_3)$ is a local frame of TM.

1. The advantage of $\not{D}$ (massless) over $\not{D}+\lambda$ with $\lambda \neq 0$ (massive) is conformal invariance: $g \to c^2 g$ corresponds to $\not{D} \to c^{-2}\not{D}c$.
2. Weyl equation ($\mathcal{W}_{\mathrm{e}}$ even Weyl spinors).

   $$\mathbb{S} := \mathcal{W}_{\mathrm{e}}^*,\ \Gamma(X) = \beta\gamma(X), \mathbb{D} = g^{\mu\nu}\Gamma(e_\mu)\nabla^{\mathcal{S}}_{e_\nu},\ \hat{\mathbb{D}} = c^{-3}\mathbb{D}c.$$

   Weyl equation $\mathbb{D}\phi = 0\,(\mathbb{D} : C^\infty(M;\mathbb{S}^*) \to C^\infty(M;\mathbb{S}))$.
3. In the following we only consider massless Dirac fields (Weyl equation).

Conserved current

$$(\phi|v)_M = \overline{(v|\phi)_M} := \int_M \bar{\phi}(x) \cdot v(x) \, d\mathrm{vol}_g, \tag{6}$$

(M, g) globally hyperbolic spacetime (of dimension 4) with a spin structure and denote by $\mathbb{D}$ the associated Weyl operator.
Space-compact solutions Let $\mathrm{Sol}_{\mathrm{sc}}(M)$ be the space of smooth space-compact solutions of $\mathbb{D}\phi = 0$, $\phi \in C^\infty(M; \mathbb{S}^*)$.
The current $J(\phi_1, \phi_2) \in C^\infty(M; T^*M)$ defined by

$$J(\phi_1, \phi_2) \cdot X := \bar{\phi}_1 \cdot \Gamma(X)\phi_2, \quad X \in C^\infty(M; TM), \ \phi_i \in \mathrm{Sol}_{\mathrm{sc}}(M),$$

it satisfies

$$\nabla^a J_a(\phi_1, \phi_2) = 0, \quad \phi_i \in \mathrm{Sol}_{\mathrm{sc}}(M).$$

Stokes :

$$\int_{\partial U} i^*(g^{-1} J(\phi_1, \phi_2) \lrcorner \Omega_g) = 0, \ \phi_i \in \mathrm{Sol}_{\mathrm{sc}}(M).$$

L^2 solutions

Hilbertian scalar product Let now $S \subset M$ be any smooth Cauchy surface.

$$(\phi_1|\phi_2)_{\mathbb{D}} := \imath \int_S i^*(g^{-1}J(\phi_1,\phi_2) \lrcorner \Omega_g) = \imath \int_S \bar{\phi}_1 \cdot \Gamma(g^{-1}\nu)\phi_2 \;\; i_l^* d\mathrm{vol}_g, \tag{7}$$

where $S = \mathrm{Ker}\,\nu$, l transverse to S future pointing with $\nu \cdot l = 1$, $i_l^* d\mathrm{vol}_g = |i^*(l \lrcorner \Omega_g)|$. The r.h.s. in (7) is independent on the choice of the Cauchy surface S. If S is space-like, we obtain ($l = n$, $\nu = -gn$):

$$(\phi_1|\phi_2)_{\mathbb{D}} = \imath \int_S \bar{\phi}_1 \cdot \Gamma(n)\phi_2 \, d\mathrm{vol}_h. \tag{8}$$

By (2) $\imath\Gamma(n)$ is positive definite, which shows that $(\cdot|\cdot)_{\mathbb{D}}$ is a Hilbertian scalar product on $\mathrm{Sol}_{\mathrm{sc}}(M)$.

Definition

The Hilbert space $\mathrm{Sol}_{\mathrm{L}^2}(M)$, called the *space of L^2 solutions*, is the completion of $\mathrm{Sol}_{\mathrm{sc}}(M)$ for the scalar product $(\cdot|\cdot)_{\mathbb{D}}$.

Use of null tetrads

Let $(l, n, m, \bar{m})$ be a normalized null tetrad and $(o, \imath)$ the associated frame of $\mathbb{S}$. For $\phi \in C^\infty(M; \mathbb{S}^*)$ one sets then:

$$\phi_0 = \phi \cdot o, \quad \phi_1 = \phi \cdot \imath, \quad U\phi = \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} \in C^\infty(M; \mathbb{C}^2),$$

so that $\phi = \phi_0 o^* + \phi_1 \imath^*$ if $(o^*, \imath^*)$ is the dual frame of $\mathbb{S}^*$. If the tetrad is chosen such that $l + n$ is normal to a space-like Cauchy surface S , then we obtain

$$(\phi|\phi)_{\mathbb{D}} = \frac{1}{\sqrt{2}} \int_S \left(|\phi_0|^2 + |\phi_1|^2 \right) d\mathrm{vol}_h. \tag{9}$$

We say that a tetrad is adapted to a foliation if $l + n$ is normal to all slices of the foliation.

II.2 The Cauchy problem for the Weyl equation on Kerr spacetime

Evolutionary form of the Weyl equation in $\mathrm{M_I}$

Kinnersley's tetrad The Weyl equation $\mathbb{D}\phi = 0$ can be reduced to an equation of the form $\partial_t \Psi - \imath H \Psi = 0$ with H a t-independent differential operator.

$$l = \frac{1}{\sqrt{2\Delta\rho^2}} \left((r^2 + a^2)\partial_t + \Delta\partial_r + a\partial_\varphi \right),$$

$$n = \frac{1}{\sqrt{2\Delta\rho^2}} \left((r^2 + a^2)\partial_t - \Delta\partial_r + a\partial_\varphi \right),$$

$$m = \frac{1}{\sqrt{2}p} \left(\imath a \sin\theta \partial_t + \partial_\theta + \frac{\imath}{\sin\theta}\partial_\varphi \right),$$

for $p = r + \imath a \cos\theta$. Note that l, n are up to scaling the principal null directions. We set

$$\phi_0 = \phi \cdot o, \quad \phi_1 = \phi \cdot \imath.$$

Tetrad adapted to the foliation

A null tetrad $(\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}})$ adapted to the foliation of $\mathrm{M_I}$ by the hypersurfaces $\Sigma_s = \{t = s\}$ is used for the scattering theory arguments, with the property that

$$\mathbf{l} + \mathbf{n} = 2\mathbf{T},\ \mathbf{l},\ \mathbf{n} \in Span\{\mathbf{T},\ \partial_r\},$$

where $\mathbf{T}$ is the future directed unit normal vector field to this foliation. Concretely,

$$\mathbf{l} = \frac{\sigma}{\sqrt{2\Delta\rho^2}}\left(\partial_t + \frac{2aMr}{\sigma^2}\partial_\varphi\right) + \sqrt{\frac{\Delta}{2\rho^2}}\partial_r,$$

$$\mathbf{n} = \frac{\sigma}{\sqrt{2\Delta\rho^2}}\left(\partial_t + \frac{2aMr}{\sigma^2}\partial_\varphi\right) - \sqrt{\frac{\Delta}{2\rho^2}}\partial_r,$$

$$\mathbf{m} = \frac{1}{\sqrt{2\rho^2}}\left(\partial_\theta + \frac{\imath\rho^2}{\sigma\sin\theta}\partial_\varphi\right).$$

If $(\mathbf{o}, \mathbf{i})$ is the spinor basis associated to $(\mathbf{l}, \mathbf{n}, \mathbf{m}, \bar{\mathbf{m}})$, one sets then:

$$\Psi := \left(\frac{\Delta\rho^2\sigma^2}{(r^2+a^2)^2}\right)^{\frac{1}{4}} \begin{pmatrix} \phi\cdot\mathbf{o} \\ \phi\cdot\mathbf{i} \end{pmatrix} = \begin{pmatrix} \Psi_0 \\ \Psi_1 \end{pmatrix} =: \mathcal{V}\phi,$$

and uses the coordinates $(t, r_*, \theta, \varphi)$.

$$\begin{pmatrix} \Psi_0 \\ \Psi_1 \end{pmatrix} = \left(\frac{\Delta\rho^2\sigma^2}{(r^2+a^2)^2}\right)^{\frac{1}{4}} \mathbf{U} \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}.$$

Finally, one sets

$$D = \mathcal{V}\mathbb{D}\mathcal{V}^{-1}, \quad (\Psi|\Psi)_D = \frac{1}{\sqrt{2}} \int_\Sigma \left(|\Psi_0|^2 + |\Psi_1|^2\right) dx d^2\omega.$$

The equation $D\Psi = 0$ can be rewritten as

$$\partial_t \Psi - \imath H \Psi = 0,$$

which we will call the reduced Weyl equation.

The Cauchy problem

We denote by $Sol_{L^2}(\mathrm{M_I})$ the closure of the space $Sol_{\mathrm{sc}}(\mathrm{M_I})$ of space-compact solutions of $D\Psi = 0$ for the scalar product $(\cdot|\cdot)_D$.

$$\mathcal{V} : \mathrm{Sol_{L^2}}(\mathrm{M_I}) \xrightarrow{\sim} Sol_{L^2}(\mathrm{M_I}) \text{ is unitary.}$$

Cauchy evolution Let $\mathcal{H}$ be the Hilbert space associated to $(\Psi|\Psi)_D$.

Lemma

$(H, D(H))$ is selfadjoint on $\mathcal{H}$.

We set

$$\rho_\Sigma : Sol_{L^2}(\mathrm{M_I}) \ni \psi \mapsto \psi(0) \in \mathcal{H}$$

and denote by $\Psi = U_\Sigma f$ the unique solution of the Cauchy problem

$$\begin{cases} D\Psi = 0, \\ \rho_\Sigma \Psi = f \in \mathcal{H}. \end{cases}$$

II.3 Scattering theory for the Weyl equation on Kerr spacetime

Decomposition of $Sol_{L^2}(\mathrm{M_I})$

Theorem (H-Nicolas '03)

There exists a selfadjoint operator $P^- \in B(\mathcal{H})$, called the past asymptotic velocity such that:

$$\chi(P^-) = \mathrm{s}-\lim_{t\to-\infty} \mathrm{e}^{-\imath tH}\chi\left(\frac{r_*}{t}\right)\mathrm{e}^{\imath tH}, \quad \forall \chi \in C_{\mathrm{c}}^{\infty}(\mathbb{R}).$$

We have $\sigma(P^-) = \{-1, 1\}$, $[P^-, H] = [P^-, \partial_\varphi] = 0$.

We set

$$\pi_{\mathscr{H}^-} := \mathbf{1}_{\{1\}}(P^-), \quad \pi_{\mathscr{I}^-} := \mathbf{1}_{\{-1\}}(P^-),$$
$$Sol_{L^2}(\mathrm{M_I}) = Sol_{L^2,\mathscr{H}^-}(\mathrm{M_I}) \oplus Sol_{L^2,\mathscr{I}^-}(\mathrm{M_I}),$$

where $Sol_{L^2,\mathscr{H}^-/\mathscr{I}^-}(\mathrm{M_I}) := U_\Sigma \circ \pi_{\mathscr{H}^-/\mathscr{I}^-}\mathcal{H}$.
Let $\Pi_{\mathscr{H}^-/\mathscr{I}^-} = U_\Sigma \circ \pi_{\mathscr{H}^-/\mathscr{I}^-} \circ \rho_\Sigma$ (orthogonal projections on $Sol_{L^2,\mathscr{H}^-/\mathscr{I}^-}(\mathrm{M_I})$). Note : $\imath^{-1} v_{\mathscr{H}}$, resp. $\imath^{-1} v_{\mathscr{I}}$ preserves $Sol_{L^2,\mathscr{H}^-}(\mathrm{M_I})$ resp. $Sol_{L^2,\mathscr{I}^-}(\mathrm{M_I})$.

Traces at infinities

Traces on $\mathscr{H}^-$ For $\Psi \in Sol_{\rm sc}(\mathrm{M_I})$, the trace

$$T_{\mathscr{H}^-}\Psi := \Psi_{1|\mathscr{H}^-} \in C^\infty(\mathscr{H}^-;\mathbb{C})$$

is well defined.

Proposition (H-Nicolas '03)

$T_{\mathscr{H}^-}$ uniquely extends as a bounded operator $T_{\mathscr{H}^-} : Sol_{L^2}(\mathrm{M_I}) \to L^2(\mathscr{H}^-, \mathrm{dvol}_{\mathscr{H}^-})$, where $\mathscr{H}^-$ is identified with $\mathbb{R}_{^*t} \times \mathbb{S}^2_{\theta, ^*\varphi}$, and $\mathrm{dvol}_{\mathscr{H}^-} = \sin\theta d^*t d\theta d^*\varphi$. One has:

$$\mathrm{Ker}\, T_{\mathscr{H}^-} = Sol_{L^2,\mathscr{I}^-}(\mathrm{M_I}), \quad \mathrm{Ran}\, T_{\mathscr{H}^-} = L^2(\mathscr{H}^-, \mathrm{dvol}_{\mathscr{H}^-}),$$

$$(\Psi|\Psi)_D = \frac{1}{\sqrt{2}}\int_{\mathscr{H}^-} |T_{\mathscr{H}^-}\Psi|^2 \,\mathrm{dvol}_{\mathscr{H}^-}, \quad \Psi \in Sol_{L^2,\mathscr{H}^-}(\mathrm{M_I}).$$

Traces on $\mathscr{I}^-$

For $\Psi \in Sol_{sc}(\mathrm{M_I})$, the trace

$$T_{\mathscr{I}^-}\Psi := \Psi_{0|\mathscr{I}^-} \in C^\infty(\mathscr{I}^-;\mathbb{C})$$

is well defined.

Proposition (H-Nicolas '03)

$T_{\mathscr{I}^-}$ uniquely extends as a bounded operator

$$T_{\mathscr{I}^-} : Sol_{L^2}(\mathrm{M_I}) \to L^2(\mathscr{I}^-, \mathrm{dvol}_{\mathscr{I}^-}),$$

where $\mathscr{I}-$ is identified with $\mathbb{R}_{t^*} \times \mathbb{S}^2_{\theta,\varphi^*}$ and $\mathrm{dvol}_{\mathscr{I}^-} = \sin\theta \mathrm{d}t^* d\theta d\varphi^*$. One has

$$\mathrm{Ker}\, T_{\mathscr{I}^-} = Sol_{L^2,\mathscr{H}^-}(\mathrm{M_I}), \quad \mathrm{Ran}\, T_{\mathscr{I}^-} = L^2(\mathscr{I}^-, \mathrm{dvol}_{\mathscr{I}_-})$$
$$(\Psi|\Psi)_D = \frac{1}{\sqrt{2}} \int_{\mathscr{I}^-} |T_{\mathscr{I}^-}\Psi|^2 \,\mathrm{dvol}_{\mathscr{I}^-}, \quad \Psi \in Sol_{L^2,\mathscr{I}^-}(\mathrm{M_I}).$$

Killing vector fields

Remark

Note that at $\mathscr{H}^-$, only the Ψ_1 component is relevant for the trace, whereas at $\mathscr{I}^-$ only the Ψ_0 component is relevant.

Killing vector field on $\mathscr{H}^-$ The Killing vector field $v_{\mathscr{H}}$ is null and tangent to $\mathscr{H}$. On $\mathscr{H}^-$ it equals $\partial_{*t} + \Omega_{\mathscr{H}}\partial_{*\varphi}$, resp. $-\kappa_+ U\partial_U$ in star-Kerr, resp. KBL coordinates.
If we also denote by $\imath^{-1}v_{\mathscr{H}}$ its selfadjoint realization on $L^2(\mathscr{H}^-, \mathrm{dvol}_{\mathscr{H}^-})$ we have:

$$T_{\mathscr{H}^-} \circ (\imath^{-1}v_{\mathscr{H}}) = (\imath^{-1}v_{\mathscr{H}}) \circ T_{\mathscr{H}^-} \text{ on } Sol_{L^2,\mathscr{H}^-}(\mathrm{M_I}).$$

Killing vector field on $\mathscr{I}^-$ The Killing vector field $v_{\mathscr{I}}$ is null and tangent to $\mathscr{I}$. On $\mathscr{I}^-$ it equals ∂_{t^*} in Kerr-star coordinates. If we also denote by $\imath^{-1}v_{\mathscr{I}}$ its selfadjoint realization on $L^2(\mathscr{I}^-, \mathrm{dvol}_{\mathscr{I}^-})$, we have:

$$T_{\mathscr{I}^-} \circ (\imath^{-1}v_{\mathscr{I}}) = (\imath^{-1}v_{\mathscr{I}}) \circ T_{\mathscr{I}^-} \text{ on } Sol_{L^2,\mathscr{I}^-}(\mathrm{M_I}).$$

Asymptotic completeness

Theorem (H-Nicolas '03)

$T = T_{\mathscr{H}^-} \oplus T_{\mathscr{I}^-}$ *from*
$Sol_{L^2}(\mathrm{M_I}) = Sol_{L^2,\mathscr{H}^-}(\mathrm{M_I}) \oplus Sol_{L^2,\mathscr{I}^-}(\mathrm{M_I})$ *to*
$L^2(\mathscr{H}^-, \mathrm{dvol}_{\mathscr{H}^-}) \oplus L^2(\mathscr{I}^-, \mathrm{dvol}_{\mathscr{I}_-})$ *is unitary with*

$$T\,\imath^{-1}v_{\mathscr{H}}\Pi_{\mathscr{H}^-} = (\imath^{-1}v_{\mathscr{H}} \oplus 0)T, \quad T\,\imath^{-1}v_{\mathscr{I}}\Pi_{\mathscr{I}^-} = (0 \oplus \imath^{-1}v_{\mathscr{I}})T.$$

Scattering theory was formulated for vectors $\Psi \in Sol_{L^2}(\mathrm{M_I})$. We will now re-express these results as the existence of traces on the horizon and infinity for spinors $\phi \in \mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})$. We will use the decomposition

$$\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I}) = \mathrm{Sol}_{\mathrm{L}^2,\mathscr{H}^-}(\mathrm{M_I}) \oplus \mathrm{Sol}_{\mathrm{L}^2,\mathscr{I}^-}(\mathrm{M_I}),$$

where

$$\mathrm{Sol}_{\mathrm{L}^2,\mathscr{H}^-/\mathscr{I}^-}(\mathrm{M_I}) := \mathcal{V}Sol_{L^2,\mathscr{H}^-/\mathscr{I}^-}(\mathrm{M_I}).$$

Results for spinors

Traces on $\mathscr{H}^-$ For $\phi \in \mathrm{Sol}_{\mathrm{sc}}(\mathrm{M_I})$ we set

$$\mathrm{T}_{\mathscr{H}^-}\phi = \phi_{|\mathscr{H}^-} \in C^\infty(\mathscr{H}^-;\mathbb{C}^2).$$

We denote by $\mathrm{L}^2(\mathscr{H}^-)$ the completion of $C_{\mathrm{c}}^\infty(\mathscr{H}^-;\mathbb{C}^2)$ for

$$(\phi|\phi)_{\mathscr{H}^-} = -\mathrm{ı}\int_{\mathscr{H}^-} \bar{\phi}\cdot\Gamma(\nabla V)\phi|g|^{\frac{1}{2}}\mathrm{d}U d\theta d\varphi^\#.$$

Traces on $\mathscr{I}^-$ Conformal rescaling

$$\hat{g} = c^2 g, \quad c = r^{-1},$$

$$\mathrm{T}_{\mathscr{I}^-}\phi := \hat{\phi}_{|\mathscr{I}^-}, \quad \hat{\phi} = c^{-1}\phi \in \mathrm{Sol}_{\mathrm{sc}}(\hat{\mathbb{D}}).$$

We denote by $\mathrm{L}^2(\mathscr{I}^-)$ the completion of $C_{\mathrm{c}}^\infty(\mathscr{I}^-;\mathbb{C}^2)$ for the scalar product:

$$(\hat{\phi}|\hat{\phi})_{\mathscr{I}^-} = -\mathrm{ı}\int_{\mathscr{I}^-} \bar{\hat{\phi}}\cdot\hat{\Gamma}(\hat{\nabla}c)\hat{\phi}|\hat{g}|^{\frac{1}{2}}\mathrm{d}t^* d\theta d\varphi^*.$$

Asymptotic completeness for spinors

$$\mathcal{S}_{\mathscr{H}^-}\phi = \mathrm{T}_{\mathscr{H}^-}\phi\cdot \mathfrak{i}, \quad \mathcal{S}_{\mathscr{I}^-}\phi = \mathrm{T}_{\mathscr{I}^-}\phi\cdot \hat{o}.$$

$\imath^{-1}\mathcal{L}_{\mathscr{H}} = \imath^{-1}\mathcal{L}_{v_{\mathscr{H}}}$ and $\imath^{-1}\mathcal{L}_{\mathscr{I}} = \imath^{-1}\mathcal{L}_{v_{\mathscr{I}}}$

Proposition

1. *The map* $\mathrm{T}_{\mathrm{M_I}} = \mathrm{T}_{\mathscr{H}^-} \oplus \mathrm{T}_{\mathscr{I}^-}$ *from* $Sol_{\mathrm{L}^2}(\mathrm{M_I})$ *to* $\mathrm{L}^2(\mathscr{H}^-) \oplus \mathrm{L}^2(\mathscr{I}^-)$ *is unitary.*
2. *The map* $\mathcal{S}_{\mathrm{M_I}} = \mathcal{S}_{\mathscr{H}^-} \oplus \mathcal{S}_{\mathscr{I}^-}$ *from* $\mathrm{Sol}_{\mathrm{L}^2}(\mathrm{M_I})$ *to* $L^2(\mathscr{H}^-;\mathbb{C}) \oplus L^2(\mathscr{I}^-;\mathbb{C})$ *is unitary.*
3. *One has*

$$\mathcal{S}_{\mathscr{H}^-} \circ \imath^{-1}\mathcal{L}_{\mathscr{H}} = -\imath^{-1}\kappa_+(U\partial_U + \tfrac{1}{2}) \circ \mathcal{S}_{\mathscr{H}^-},$$

$$\mathcal{S}_{\mathcal{I}_-} \circ \imath^{-1}\mathcal{L}_{\mathscr{I}} = \imath^{-1}\partial_{t^*} \circ \mathcal{S}_{\mathscr{I}_-},$$

in the sense of unitary equivalence of selfadjoint operators.

Elements of the proof

Wave operators

$$P_N = \gamma D_{r_*} - \frac{a}{r^2+a^2} D_\varphi,\ \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathcal{H}_{out} := \{(\psi_0, 0) \in \mathcal{H}\},\ \mathcal{H}_{in} = \{(0, \psi_1) \in \mathcal{H}\}.$$

The group e^{itP_N} represents transport along the principal null directions.

Theorem (H-Nicolas '03)

The following strong limits exist:

$$W^-_{\mathcal{H},pn} = s - \lim_{t\to-\infty} e^{-itH} e^{itP_N} \mathbf{1}_{\mathcal{H}_{in}},$$

$$\Omega^-_{\mathcal{H}^-,pn} = s - \lim_{t\to-\infty} e^{-itP_N} e^{itH} \pi_{\mathscr{H}^-},$$

$$W^-_{\mathscr{I}^-,pn} = s - \lim_{t\to-\infty} e^{-itH} e^{itP_N} \mathbf{1}_{\mathcal{H}_{out}},$$

$$\Omega^-_{\mathscr{I}^-,pn} = s - \lim_{t\to-\infty} e^{-itP_N} e^{itH} \pi_{\mathscr{I}^-}.$$

Link with the trace operators and choice of the tetrad

Link with the trace operators Let $\Sigma = \{t = 0\}$ and $\mathcal{F}_{\mathcal{H}^-}$ the diffeomorphism which identifies points on $\mathcal{H}^-$ and Σ by following outgoing principal null geodesics. We define $\mathcal{F}_{\mathscr{I}^-}$ in the same way. Then

$$T_{\mathcal{H}^-} = \mathcal{F}^*_{\mathcal{H}^-} \Omega^-_{\mathcal{H}^-,pn}, \; T_{\mathscr{I}^-} = \mathcal{F}^*_{\mathscr{I}^-} \Omega^-_{\mathscr{I}^-,pn}.$$

Choice of the tetrad If we choose a tetrad adapted to the foliation, then

$$H = hH_0h + V_\varphi D_\varphi + V,$$

$h^2 - 1$, V_φ, V short range and H_0 spherically symmetric.

The Mourre estimate

Let $\mathcal{H}$ be a Hilbert space and $(H, D(H)), (A, D(A))$ selfadjoint operators s. t.
(M1) e^{isA} preserves $D(H)$
(M2) $|[iH, A](u, v)| \lesssim \|(H+i)u\|\|v\|$
(M3) $|[[H, A], A](u, v)| \lesssim \|(H+i)u\|\|(H+i)v\|$.
(M4) $\mathbf{1}_\Delta(H)[iH, A]\mathbf{1}_\Delta(H) \geq \delta \mathbf{1}_\Delta(H) + \mathbf{1}_\Delta(H)K\mathbf{1}_\Delta(H)$;
$\delta > 0$, Δ open interval, K compact operator. Then

1. In Δ the point spectrum of H is finite.
2. For each closed interval $[a, b] \subset \Delta \cap \sigma_c(H)$ we have for $\mu > 1/2$:

$$\sup_{\operatorname{Re} z \in [a,b], \operatorname{Im} z > 0} \|\langle A\rangle^{-\mu}(H-z)^{-1}\langle A\rangle^{-\mu}\| < \infty$$

$$\langle A\rangle := \sqrt{1+A^2}.$$

Toy model near the horizon

Toy model :

$H = \gamma D_r + e^{\kappa r} \not{D}_{S^2}$, $\kappa > 0$, $\mathbb{R}_- \times S^2$, $\gamma = Diag(1, -1)$.

- Try $A = \gamma r$. We obtain : $[iH, A] = \mathbf{1} + r e^{\kappa r} [\not{D}_{S^2}, \gamma]$.

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- Spin weighted spherical harmonics :
  $$\hat{H}^{nl} = \gamma D_r + e^{\kappa r} \tau, \quad \tau = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$
- Now A works and the estimates are uniform in n, l because everything is independent of n, l ! $\hat{A} := U A U^*$.

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