The resonable ineffectiveness of non-toric geometry

[McGovern-JK arXiv:2504.06147]

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The Unreasonable Effectiveness of Toric Geometry
ESI, 07/07/2026

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Outline

Geometry and GLSMs

Abelian model

Non-abelian model I

Non-abelian model II

Conclusion

"Geometry" for the purpose of this talk

- We study compact Calabi-Yau threefolds.
- Geometry:
  - Complete intersections in (non-)toric spaces
  - Non-complete intersections, eg. determinantal varieties
  - Non-commutative resolutions of singular varieties
  - Loci in the moduli space that have maximally unipotent monodromy
- Not today:
  - Singular geometries
  - Landau-Ginzburg models
  - Hybrids, pseudo-hybrids, exoflops, and other exotics
  - Calabi-Yaus with $h^{1,1} \neq 1$
- GLSMs
  - We construct geometries as phases of GLSMs. [Witten 93]

GLSM data

- Symmetries
  - 2D $\mathcal{N} = (2,2)$-SUSY
  - Gauge symmetry: gauge group G (not necessarily abelian)
- Field content
  - Chiral multiplets: $\Phi^i = (\phi^i, \ldots)$
    - transform in some representation ρ_V of G, $\phi^i \in V$
    - gauge charges Q_i^a: weights of ρ_V
    - CY: $\rho_V : G \to SL(V)$
  - Vector multiplets: $\Sigma_a = (\sigma_a, \ldots)$, $\sigma_a \in \mathfrak{g}_{\mathbb{C}}$
- Parameters
  - gauge couplings
  - FI-theta parameters (ζ^a, θ^a)
    - not renormalised in CY case
    - $t^a = \zeta^a - i\theta^a$: coordinates on $\mathcal{M}_K$, the stringy Kähler moduli space
- Superpotential $W(\Phi)$

Classical vacua and phases

- Classical vacua are determined by the zeroes of the scalar potential

$$U = \frac{1}{8e^2}|[\sigma,\overline{\sigma}]|^2 + \frac{1}{2}\left(|\langle Q,\sigma\rangle\phi|^2 + |\langle Q,\overline{\sigma}\rangle\phi|^2\right) + \frac{e^2}{2}D^2 + |F|^2$$

  - D-term:

$$D = \mu(\phi) - \zeta, \qquad \mu : V \to \mathfrak{g}^*$$

  - F-term:

$$F = dW(\phi)$$

- Phase of a GLSM: Low-energy effective theory associated to a solution of the vacuum equations, potentially with quantum corrections.

Classical geometric phases

- We find a classical geometric (perturbative) Calabi-Yau phase when:
  - All σ-fields are zero. Gauge sector decouples.
  - The classical vacuum is given by the solution of the D-term and F-term equations:

$$X_\zeta = \mu^{-1}(\zeta)/G \cap dW(0)^{-1} \simeq (V - F_\zeta)/G_{\mathbb{C}} \cap dW(0)^{-1}$$

  - The gauge symmetry is broken completely.
  - The vacuum manifold X_ζ is a smooth Calabi-Yau.
  - The phase/low-energy effective theory is an $N = (2,2)$ non-linear sigma model with target X_ζ.
- Most GLSM phases are not of this type.
  - Different vacua: Coulomb branches, mixed Coulomb/Higgs branches, strongly coupled branches, etc.
  - Different phases: Landau-Ginzburg, hybrid, SQCDs, or worse.
- Rest of the talk: How to get Calabi-Yau geometries that are not complete intersections in toric spaces?

The model

- Consider a GLSM with $G = U(1)$ and field content

[Căldăraru-Distler-Hellerman-Pantev-Sharpe 07]

ϕ_i	$p^{1,\dots,4}$	$x_{1,\dots,8}$	FI
$U(1)$	-2	1	ζ

- Superpotential

$$W = \sum_{k=1}^{4} p^k G_{2,k}(x) = \sum_{k=1}^{4} p^k \sum_{i,j=1}^{8} A_k^{ij} x_i x_j$$

- Vacuum equations

$$\sum_{i=1}^{4} -2|p_i|^2 + \sum_{j=1}^{8} |x_i|^2 = \zeta, \qquad dW = 0$$

Large volume: classical geometry

- $\zeta > 0$-phase
  - Deleted set $F_{\zeta>0} = \{x_1 = \ldots = x_8 = 0\}$.
  - Vacuum equations solved by $p^1 = \ldots = p^4 = 0$:

$$X_{\zeta>0} = (\mathbb{C}^8 - F_{\zeta>0})/(x_i \sim \lambda x_i) \cap \{G_{2,k}(x) = 0\} \qquad \lambda \in \mathbb{C}^*$$
$$= \mathbb{P}^7[2,2,2,2]$$

  - The vacuum is a smooth Calabi-Yau threefold with $(h^{1,1}, h^{2,1}) = (1, 65)$.
- Abelian GLSMs: Complete intersections in toric ambient spaces can be realised as "large volume" phases of GLSMs with abelian gauge group.
  - x-fields and their charges: toric variety
  - p-fields and their charges: codimension and degrees of complete intersection
  - Classification: classify matter content and $U(1)$-representations

"Small volume": non-CY vacuum

- The Picard-Fuchs operator (AESZ3) has a second MUM point. What's going on? [Almkvist-van Enckevort-van Straten-Zudilin 05]
- $\zeta < 0$-vacuum
  - Deleted set $F_{\zeta<0} = \{p^1 = \ldots = p^4 = 0\}$.
  - Vacuum equations solved by $x_1 = \ldots = x_8 = 0$:

$$X_{-\zeta>0} = (\mathbb{C}^4 - F_{\zeta<0})/(p^i \sim \lambda^2 p^i) = \mathbb{P}^3_{2222}$$

  - The vacuum manifold is not Calabi-Yau.
  - There is and unbroken $\mathbb{Z}_2$ symmetry ($\mathbb{Z}_2$-gerbe).
  - To get a full understanding of the phase, we have to work harder.

"Small volume": non-perturbative geometry

- $\zeta < 0$-phase
  - Turning on classical fluctuations of the x_i generates a superpotential

$$W^{LG} = \sum_{i,j=1}^{8} A^{ij}(\langle p \rangle) x_i x_j$$

  - Hybrid: $\mathbb{Z}_2$ Landau-Ginzburg orbifold fibered over $\mathbb{P}^3$
  - But the theory is massive (quadratic potential!) and hence IR trivial, unless the rank of the mass matrix drops:

$$Y = \{p^k \in \mathbb{P}^3 | \mathrm{det} A(p) = 0\}$$

  - Geometry: branched double cover of $\mathbb{P}^3$
  - Non-perturbative geometry: The geometry does not arise as a classical vacuum but is hidden in a rank condition on a mass matrix.

"Small volume" non-commutative resolution

- The branched double cover is given by a determinantal octic with nodal singularities.
- The GLSM sees a non-commutative resolution:
  - Let $\hat{X}$ be a non-commutative (non-Kähler) resolution of the geometry with order N torsion: $\mathrm{Tors} H_2(\hat{X}, \mathbb{Z}) \simeq \mathbb{Z}_N$. (here: $N = 2$)
  - The singularities are resolved by turning on a discrete B-field.

    [Vafa-Witten 94]
  - This modifies the topological string partition function, leading to the definition of torsion-refined Gopakumar-Vafa invariants.

    [Schimannek 21][Katz-Klemm-Schimannek-Sharpe 22]
  - Many new examples of Calabi-Yaus.

    [Katz-Schimannek 23][Lee-Lian-Romo 23][Schimannek 25][McGovern-JK 25]
- The two geometries are not birational, but related by homological projective duality. [Kuznetsov 05]

Rødland model

- Consider a GLSM with $G = U(2)$ and field content

[Rødland 98][Hori-Tong 06]

ϕ_i	$p^{1,\dots,7}$	$x_{1,\dots,7}$	FI
$U(2)$	$\det^{-1}$	$\square$	ζ

  - Note: Only one FI-parameter due to $\mathbb{Z}_2$ Weyl symmetry.
- Superpotential

$$W = \sum_{ijk=1}^{1} \sum_{ab=1}^{2} A_k^{ij} p^k x_i^a \epsilon_{ab} x_j^b = \sum_{ij=1}^{7} A^{ij}(p)[x_i x_j]$$

- Vacuum equations

$$-|p|^2 \mathrm{id}_{2\times 2} + x x^\dagger = \zeta \mathrm{id}_{2\times 2}, \qquad dW = 0$$

$\zeta > 0$: complete intersection in $G(2,7)$

- $\zeta > 0$-phase
  - Deleted set $F_{\zeta>0} = \{\mathrm{rk} x < 2\}$. ($x_i^a$: 2×7-matrix)
  - Vacuum equations solved by $p^1 = \ldots = p^7 = 0$.
  - The gauge symmetry is broken completely.
  - The "mesons" $[x_i x_j]$ are Plücker coordinates of the Grassmannian $G(2,7)$.
  - The vacuum manifold is:

$$X_{\zeta>0} = \{[x_i x_j] \in G(2,7) | A_k^{ij}[x_i x_j] = 0\}$$

  - This is a smooth Calabi-Yau threefold with $(h^{1,1}, h^{2,1}) = (1, 50)$.
  - The Calabi-Yau arises as the classical vacuum but the ambient space is not toric.
- Classification: Representation theory of non-abelian gauge groups is richer than representation theory of $U(1)^k$. Non-abelian GLSMs can have exotic matter.

$\zeta < 0$: strongly coupled phase

- The Picard-Fuchs operator (AESZ27/243) has a second MUM point.
- $\zeta < 0$-vacuum
  - Deleted set $F_{\zeta<0} = \{p^1 = \ldots = p^7 = 0\}$.
  - Vacuum equations solved by $x = 0$:

$$X_{\zeta<0} = \{p \in \mathbb{P}^6\}$$

  - The classical vacuum is not Calabi-Yau.
  - There is an unbroken $SU(2)$ gauge symmetry.
  - Mathematically, we are dealing with an Artin stack rather than a Deligne-Mumford stack.
- $\zeta < 0$-phase
  - The low-energy effective theory is an $SU(2)$ SQCD.
  - This is a strongly coupled phase. It has σ-fields and ϕ-fields.
  - Where is the geometry?

$\zeta < 0$: non-perturbative geometry reloaded

- Deep in the IR, we can make a Born-Oppenheimer approximation whereupon the gauge degrees of freedom decouple. [Hori-Tong 06]
- Non-perturbative geometry
  - There is a superpotential

$$W = \sum_{ij=1}^{7} A^{ij}(\langle p \rangle)[x_i x_j]$$

  - As before, the x_i are massive unless

$$Y = \{p \in \mathbb{P}^6 | \mathrm{rk} A(p) = 4\}.$$

  Y is a smooth non-complete intersection Calabi-Yau threefold.
- $X_{\zeta>0}$ and Y are not birational but categorically equivalent.

[Rødland 98][Kuznetsov 05][Donovan-Segal 12][EHKR asymptoting completion]

Non-abelian duality

- There is a dual GLSM with gauge group $G^\vee = (U(1) \times USp(4))/(\pm 1, \pm \mathbf{1})$ and matter content. [Hori 11]

ϕ_i	$p^{1,\dots,7}$	$\widetilde{x}^{1,\dots,7}$	$a_{ij} = -a_{ji}$	FI
$U(1)$	-2	-1	2	$\widetilde{\zeta}$
$USp(4)$	1	$\square$	1	$-$

- The superpotential is

$$\widetilde{W} = \sum_{i,j=1}^{7} [\widetilde{x}^i \widetilde{x}^j] a_{ij} + A^{ij}(p) a_{ij}$$

Phases of the dual theory

- $\widetilde{\zeta} > 0$-phase
  - Strongly coupled $USp(4)$ phase.
  - $X_{\zeta>0}$ is realised as a non-perturbative geometry.
- $\widetilde{\zeta} < 0$-phase
  - Gauge symmetry is broken completely.
  - Y is realised as the classical vacuum in the phase.
- Good news:
  - Strongly coupled phases map to weakly coupled duals.
  - Non-perturbative geometries can be realised as dual perturbative geometries.
- Bad news:
  - There is a new type of redundancy in the description of the geometries.
  - For example, the Rødland model can be described by four different GLSMs that are pairwise Hori-dual. [Hori-JK 13]

A non-abelian model with a discrete group

- One can also build non-abelian models with discrete groups.

[Hosono-Takagi 11][Hori 11]

- Consider a GLSM with gauge group $G = U(1)^3 \rtimes \mathbb{Z}_3$.

[McGovern-JK 25]

- Chiral matter

ϕ_i	p^1, p^2, p^3	x_1, x_2, x_3	y_1, y_2, y_3	z_1, z_2, z_3	FI
$U(1)_1$	-1	1	0	0	ζ
$U(1)_2$	-1	0	1	0	ζ
$U(1)_3$	-1	0	0	1	ζ

$$\mathbb{Z}_3 : \qquad x_i \to y_i \to z_i \to x_i$$

- Superpotential

$$W = \sum_{ijkl=1}^{3} S_l^{ijk} p^l x_i y_j z_k = \sum_{ijk=1}^{3} S^{ijk}(p) x_i y_j z_k$$

$\zeta > 0$-phase: CICY quotient

- Solving the D-term and F-term equations in the $\zeta > 0$-phase we get:

$$X_{\zeta>0} = \{(x_i, y_i, z_i) \in (\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2)/\mathbb{Z}_3 | S_l^{ijk} x_i y_j z_k = 0\}$$

  - This is a non-simply-connected Calabi-Yau threefold with

$$h^{11} = 1, \qquad H^3 = 30, \qquad c_2 \cdot H = 36, \qquad \chi = -30$$

[Candelas-Dale-Lutken-Schimmrigk 87][Candelas-Davies 08][Constantin-Gray-Lukas 16][...]

  - CICY quotients can arise from non-abelian GLSMs.
- The associated Calabi-Yau operator AESZ 17 has two MUM points, so we expect that the GLSM has a second geometric phase.

$\zeta < 0$-phase: vacuum

- Vacuum
  - At $\sigma_i = 0$, the D-terms and F-terms are solved by setting $x_i = y_i = z_i = 0$.
  - The p-fields take values in a $p^i \in \mathbb{P}^2$.
  - The gauge symmetry is broken to

$$G_{\zeta<0} = (U(1) \times U(1)) \rtimes \mathbb{Z}_3$$

    where "determinantal" $U(1) \subset G$ is broken.
  - We have a continuous unbroken non-abelian symmetry.
  - The phase is strongly coupled.

$\zeta < 0$-phase: superpotential

- The low energy theory has a superpotential with a non-abelian "orbifold" action:

$$W_{\zeta<0} = \sum_{ijk} S^{ijk}(\langle p \rangle) x_i y_j z_k$$

- This is an interacting theory.
- There is no mass term for the chirals.
  - The mechanism that leads to the geometry in other known examples does not work.
- This is a hybrid theory. It is not clear how a geometry is created. (hyperdeterminants?)
- But there is more...

$\zeta < 0$-phase: here be dragons

- The non-abelian theory in the $\zeta < 0$-phase has a Coulomb branch.
- Coulomb and Higgs branch are not scale-separated.
- Such a theory is called non-regular. [Hori-Tong 06][Hori 11][Hori-JK 13]
- Thanks to quantum effects due to the broken symmetry, the Coulomb branch is lifted except at $\zeta \to -\infty$. [Hori-Tong 06]
- Nevertheless, the Born-Oppenheimer approximation is invalid. We cannot isolate the matter sector.
- There is no obvious way to extract a geometry in the IR.
- Still, there is a MUM point. Can we nevertheless assign a geometry to this phase?

Mirror symmetry/topological strings to the rescue

- Despite what the GLSM says, the Picard-Fuchs equation indicates that that the topological string should behave as if there were a geometry.
- Assume this is the case, and extract information from the mirror, monodromies, genus-g free energies, etc.

[BCOV 93][Huang-Klemm-Quackenbush 06]

- First attempt: assume the phase is a non-linear sigma model on a smooth geometry.
  - Monodromies, genus-0 free energy etc. would imply fractional triple intersection numbers: $H^3 = \frac{\chi}{13}$.
  - Something must be wrong.
- Second attempt: assume the phase is a non-commutative resolution of a singular geometry, leading to a theory with a B-field. [Vafa-Witten 94][Schimannek 21][Katz-Klemm-Schimannek-Sharpe 22]
  - This seems to work.

Geometry/Topology bootstrap I

- Consider a Calabi-Yau which has m nodal singularities.
- Let $\hat{X}$ be a non-commutative (non-Kähler) resolution of the geometry with order N torsion: $\mathrm{Tors} H_2(\hat{X}, \mathbb{Z}) \simeq \mathbb{Z}_N$.
- This modifies the topological string partition function.

$$F^{(0)}|_{const} = \frac{\zeta(3)}{(2\pi i)^3}\left(\frac{\chi}{2} + mC_N\right)$$

  - Explicit formula for C_N from the Donaldson-Thomas partition function. [Katz-Klemm-Schimannek-Sharpe 22]
  - We expect $H^3 = \frac{2\left(\frac{\chi}{2}+mC_N\right)}{13}$.
  - Must determine m, N, H^3.

Geometry/Topology bootstrap II

- We get $c_2 = 32$ from the genus 1 free energy.

$$F^1(t) = -\frac{c_2}{24}t + O(e^{2\pi i t})$$

- Get $H^3 = 2$ from Yukawa couplings.
- Search databases for a suitable geometry and find N, m such that we get integer higher genus invariants (up to genus 4):

$$\mathbb{P}^5_{111223}[4,6], \qquad m = 63, \qquad N = 3$$

- Conjecture: This is the geometry we are looking for (assuming simply-connectedness).

Refined invariants

- Assuming the conjecture is true, we can compute $\mathbb{Z}_3$-torsion-refined GV-invariants.

[Schimannek 21][Katz-Klemm-Schimannek-Sharpe 22][see also Schimannek 25]

$$
\begin{aligned}
F_k &= \sum_{g=0}^{\infty} F_k^g(t)\lambda^{2g-2} \\
&= \sum_{g=0}^{\infty}\sum_{l=0}^{N-1}\sum_{\beta\in H_2(X,\mathbb{Z})}\sum_{m=1}^{\infty}\frac{n_{\beta,l}^{(g)}}{m}\left(2\sin\frac{m\lambda}{2}\right)^{2g-2} e^{2\pi ikl/N}e^{2\pi im\beta t}
\end{aligned}
$$

  - $[k]\in\mathbb{Z}_N\ldots$ fractional B-field
- Under these assumptions, we find integer invariants up to genus 11 out of an overdetermined system.

Conclusion

- Geometries can arise from non-perturbative effects in GLSMs.
- The GLSM sees non-commutative resolutions of singular geometries.
- Non-abelian GLSMs lead to geometries in non-toric spaces.
- Strongly coupled phases of non-abelian GLSMs can encode geometries.
  - These are often non-complete intersections.
  - Is there always a weakly-coupled, perturbative dual?
- There is at least one non-regular GLSM with a MUM-point, where it is not clear if the geometry can be described directly.
  - Information can be extracted indirectly through the moduli space.
  - Could there be a dual model which is regular?
  - What does this mean for classification problems?