

GEOMETRIC MECHANICS

TULCZYJEW TRIPLES, ALGEBROIDS & DIRAC STRUCTURES

KATARZYNA GRABOWSKA



- STATICS OF PHYSICAL SYSTEMS
- VARIATIONAL PRINCIPLE AS AN INSPIRATION
- TULCZYJEW ISOMORPHISM AND TULCZYJEW TRIPLE
- EXAMPLES: RELATIVISTIC PARTICLES, CLASSICAL PHOTONS

IN GEOMETRIC MECHANICS THERE APPEAR VERY DIFFERENT STRUCTURES

- SYMPLECTIC
- POISSON
- CONTACT
- DIRAC
- COSYMPLECTIC
- ...

USED FOR:

- DERIVING EQUATIONS
- STUDYING PROPERTIES OF EQUATIONS
- SOLVING EQUATIONS
- NUMERICALLY SOLVING EQS

DOUBLE VECTOR BUNDLES
AND LINEARITY
CRASH COURSE

GEOMETRIC MECHANICS

AN OVERVIEW OF STRUCTURES AND METHODS
THAT CAN BE USED TO AVOID INFINITE
DIMENSIONAL GEOMETRY

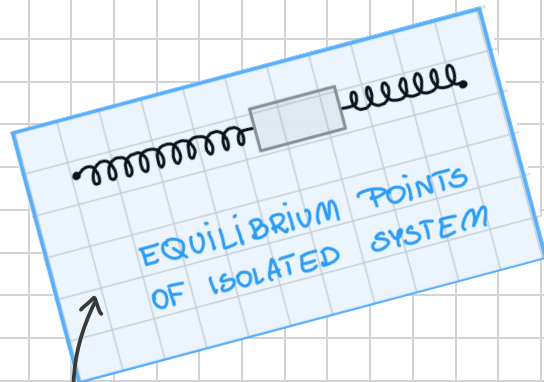
TULCZYJEW TRIPLE GENERALIZED EVEN MORE

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- DIRAC ALGEBROID
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MECHANICAL SYSTEM REVISITED
CONSTRAINED SYSTEM WITH
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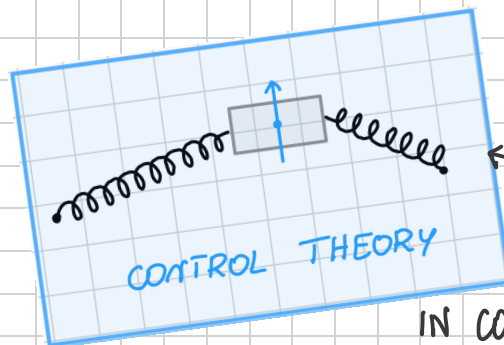
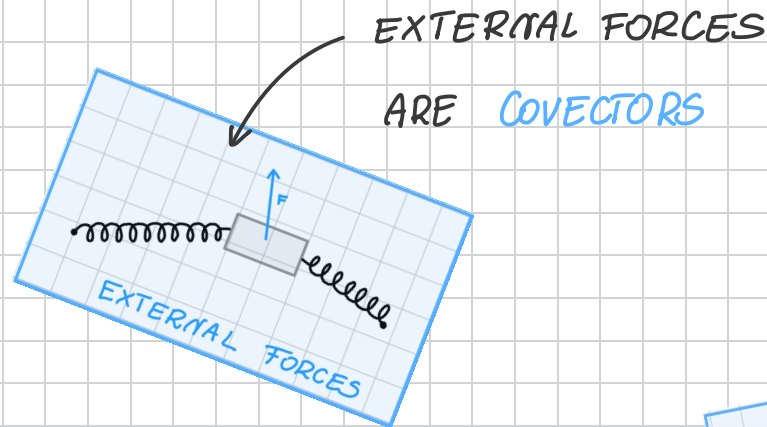
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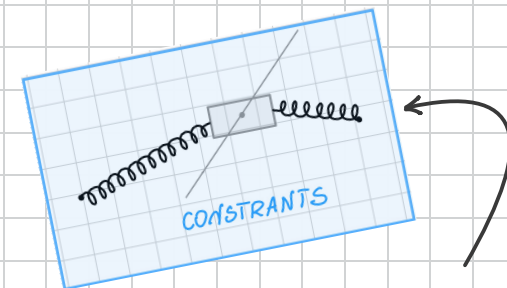
STATICS OF PHYSICAL SYSTEMS IS ESSENTIALLY
DESCRIBED IN THE LANGUAGE OF DIFFERENTIAL
GEOMETRY



CONFIGURATIONS ARE
POINTS ON A MANIFOLD

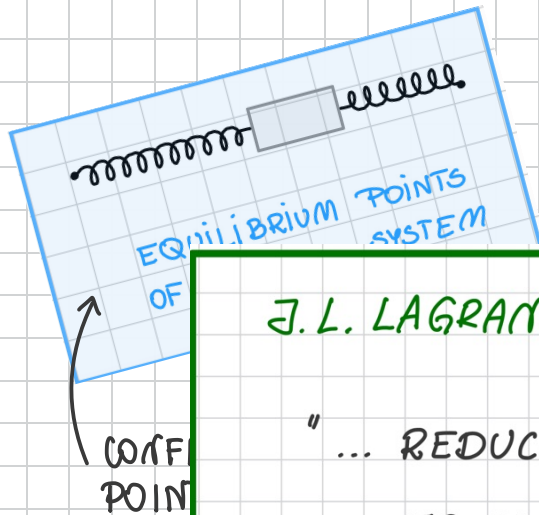


IN CONTROL THEORY
WE DEAL WITH DISTRIBUTIONS
AND CODISTRIBUTIONS



VIRTUAL DISPLACEMENTS ARE
TANGENT VECTORS, CONSTRAINTS
ARE DESCRIBED IN TERMS OF
DISTRIBUTIONS AND SUBMANIFOLDS

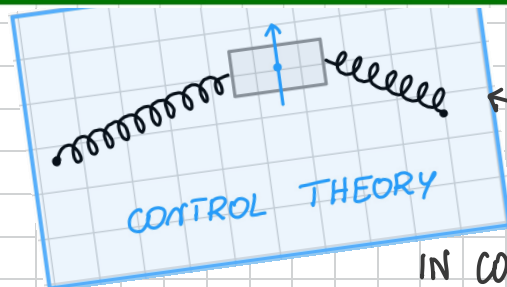
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EXTERNAL FORCES

J. L. LAGRANGE

"... REDUCE ALL THE LAWS OF MOTION OF BODIES TO THOSE OF THEIR
EQUILIBRIUM AND THUS BRING DYNAMICS BACK INTO STATICS."



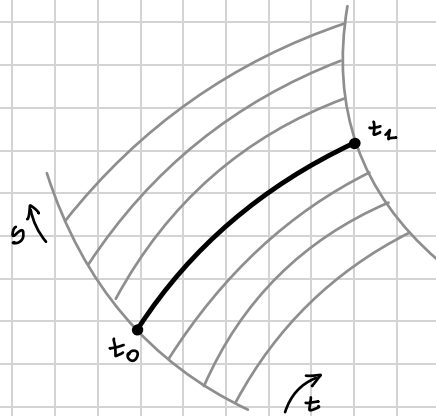
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"... REDUCE ALL THE LAWS OF MOTION OF BODIES TO THOSE OF THEIR EQUILIBRIUM AND THUS BRING DYNAMICS BACK INTO STATICS."



$$\{q: [t_0, t_1] \rightarrow Q\} = M$$

$$S(q) = \int_{t_0}^{t_1} L(q(t), \dot{q}(t)) dt \quad \text{a function on } M$$

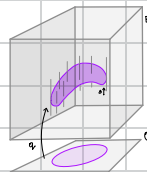
$$s \mapsto \chi(\cdot, s) \quad \text{a curve in } M$$

DESCRIBE WHAT IS GOING ON IN TERMS
OF FINITE DIMENSIONAL DIFFERENTIAL
GEOMETRY $\rightarrow$ GEOMETRIC MECHANICS

MAKE SENSE OF IT IN TERMS
OF INFINITE DIMENSIONAL
GEOMETRY OF M WITH
PROPER STRUCTURE

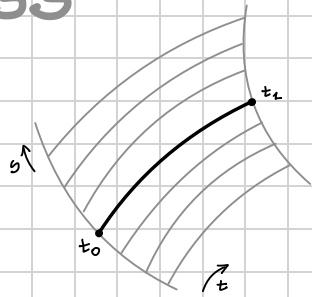


WŁODZIMIERZ M. TULCZYJEW



THERE EXISTS

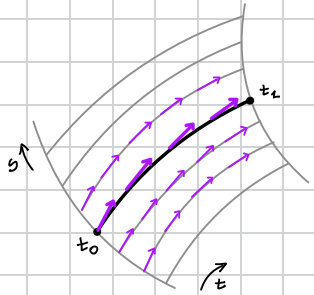
A FIELD THEORETIC VERSION



$$\{q: [t_0, t_1] \rightarrow Q\} = M$$

$$S(q) = \int_{t_0}^{t_1} L(\dot{q}(t), q(t)) dt \quad \text{a function on } M$$

$$s \mapsto \chi(\cdot, s) \quad \text{a curve in } M$$



$$S(q) = \int_{t_0}^{t_1} L(\dot{q}(t)) dt \quad \dot{q}: [t_0, t_1] \rightarrow TQ$$

$$\chi: \mathbb{R}^2 \rightarrow Q \quad \chi(t, s)$$

$$q(t) = \chi(t, 0) \quad \dot{q} = t^{(1,0)} \chi(t, 0)$$

$$\frac{d}{ds} \Big|_{s=0} S(\chi(\cdot, s)) = \int_{t_0}^{t_1} \frac{d}{ds} \Big|_{s=0} L(t^{(1,0)} \chi(t, s)) dt = \int_{t_0}^{t_1} \langle \underbrace{dL(\dot{q}(t))}_{T^*TQ}, \underbrace{t^{(1,0)} \chi(t, 0)}_{TTQ} \rangle dt =$$

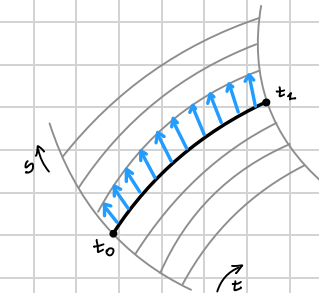
$$\begin{array}{c} TTQ \\ \downarrow \kappa_Q \\ TTQ \end{array} \quad (q^i, \dot{q}^i, \delta q^k, \delta \dot{q}^l)$$

$$TTQ \quad (q^i, \delta q^i, \dot{q}^k, \delta \dot{q}^l)$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} \underbrace{\frac{\partial \chi^i}{\partial s}}_{\delta q^i} + \frac{\partial L}{\partial \dot{q}^i} \underbrace{\frac{\partial}{\partial s} \frac{\partial \chi^i}{\partial t}}_{\delta \dot{q}^i} \right) dt =$$

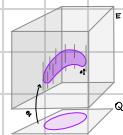
$$= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q^i} \frac{\partial \chi^i}{\partial s} + \frac{\partial L}{\partial \dot{q}^i} \underbrace{\frac{\partial}{\partial t} \frac{\partial \chi^i}{\partial s}}_{(\delta \dot{q}^i)'} \right) dt = \dots \text{TO EULER LAGRANGE EQUATIONS}$$

$$= \int_{t_0}^{t_1} \langle \underbrace{\alpha_Q(dL)}_{T^*TQ}, \underbrace{t^{(0,1)} \chi(t, 0)}_{TTQ} \rangle dt$$



$$\left(\begin{array}{ccc} TTQ & \xleftarrow{\kappa_Q} & TTQ \\ \downarrow \tau_{TQ} & & \downarrow \tau_{TQ} \\ TQ & \xleftarrow{=} & TQ \end{array} \right)^*$$

$$\begin{array}{ccc} T^*TQ & \xrightarrow{\alpha_Q} & T^*T^*Q \\ \downarrow & & \downarrow \\ TQ & \xleftarrow{=} & TQ \end{array}$$



THERE EXISTS
A FIELD THEORETIC VERSION

$$\int_{t_0}^{t_1} \langle \underbrace{dL(\dot{q}(t))}_{T^*TQ}, \underbrace{t \mapsto \chi(t,0)}_{TTQ} \rangle dt = \int_{t_0}^{t_1} \langle \underbrace{\alpha_Q(dL)}_{TT^*Q}, \underbrace{t \mapsto \chi(t,0)}_{TTQ} \rangle dt$$

EULER LAGRANGE EQUATIONS

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \right) = f_i, \quad \mathcal{E}_L \subset T^2Q$$

(1) DEFINITION OF MOMENTA

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad TQ \ni \dot{q} \mapsto \lambda_L(\dot{q}) \in T^*Q$$

PHASE EQUATIONS

$$\dot{q}^i = \dots, \quad \dot{p}_i = \dots, \quad \mathcal{D}_L \subset TT^*Q$$

ALL OF THEM CAN
BE DERIVED
GEOMETRICALLY
FROM $dL(TQ) \subset T^*TQ$

(2) $\mathcal{D}_L = \alpha_Q^{-1}(dL(TQ))$ MAY NOT BE AN IMAGE OF A VECTOR FIELD, BUT STILL IS A FIRST ORDER (IMPLICIT) DIFFERENTIAL EQUATION, $\mathcal{E}_L \subset T^2Q$ MAY ALSO BE IMPLICIT
 $\longrightarrow$ SOME POTENTIAL FOR GENERALIZATIONS (E.G. USE OTHER LAGRANGIAN SUBMANIFOLDS THEN $dL(TQ)$).

(3) IT MIGHT BE USEFUL TO KNOW MORE ABOUT DOUBLE VECTOR BUNDLES

CRASH COURSE
IN D.V.B

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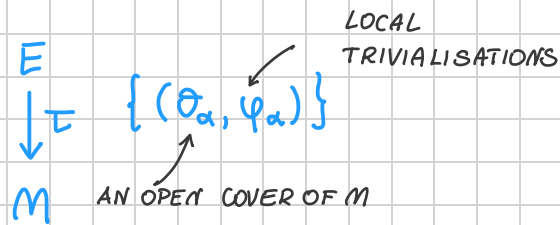
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CRASH COURSE ON DOUBLE VECTOR BUNDLES

ONE VECTOR BUNDLE



$(E, M, \tau, \{(\sigma_\alpha, \varphi_\alpha)\})$
 A VECTOR BUNDLE

$$\varphi_\alpha: \tau^{-1}(\sigma_\alpha) \longrightarrow \sigma_\alpha \times \mathbb{R}^n$$

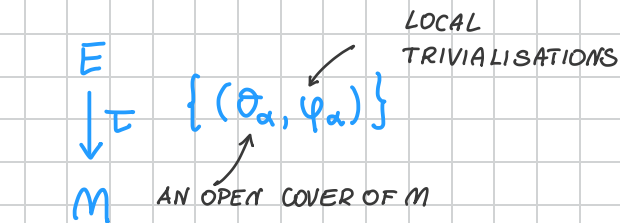
$$\begin{array}{ccc} \tau^{-1}(\sigma_\alpha \cap \sigma_\beta) & \xrightarrow{\varphi_\alpha} & \sigma_\alpha \cap \sigma_\beta \times \mathbb{R}^n \\ & \searrow & \downarrow \varphi_\beta \circ \varphi_\alpha^{-1} \\ & & \sigma_\alpha \times \sigma_\beta \times \mathbb{R}^n \end{array}$$

$$\forall x \quad \varphi_\beta \circ \varphi_\alpha^{-1}(x, \cdot) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

LINEAR!

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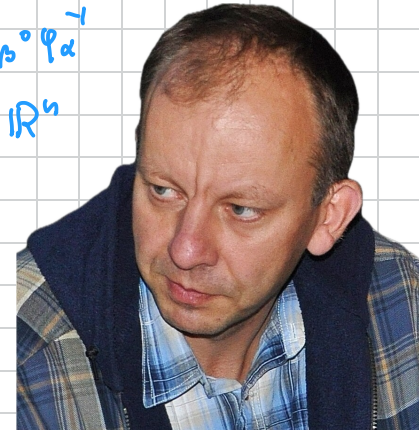
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LINEAR!



JANUSZ GRABOWSKI



MIKOŁAJ ROTKIEWICZ

MULTIPLICATION BY REAL NUMBERS

$$\begin{array}{ccc} E & & E \\ \downarrow \tau & \xrightarrow{h} & \downarrow \\ M & \xrightarrow{=} & M \end{array} \quad \begin{array}{l} h(v, t) = tv \\ h_t(v) = h(v, t) \\ h_t \circ h_s = h_{ts} \end{array}$$

MONOID ACTION



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Graded bundles and homogeneity structures ☆

Janusz Grabowski ^a ✉, Mikołaj Rotkiewicz ^b ✉

THEOREM SMOOTH MANIFOLD WITH A SMOOTH ACTION h OF MULTIPLICATIVE REALS SUCH THAT $\frac{d}{dt}\big|_{t=0} h_t(e) = 0$ ONLY FOR $e \in h_0(E)$ IS A VECTOR BUNDLE SUCH THAT h_t IS THE MULTIPLICATION BY t

(E, h) $\swarrow$ A VECTOR BUNDLE

ELEMENTS OF THE STRUCTURE

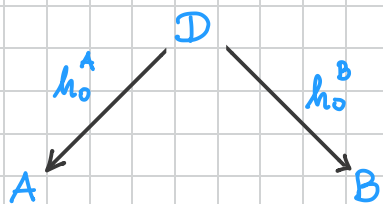
- $\tau = h_0$
- $M = h_0(E) \subset E$ ZERO SECTION IDENTIFIED WITH THE BASE
- EULER VECTOR FIELD $\nabla \in \chi(E)$ $\nabla(e) = \frac{d}{dt}\big|_{t=1} h_t(e)$
- $VE = \{v \in TE : \tau_*(v) = 0\}$ $VE = E \times_M E$

COORDINATES LINEAR IN FIBRES
i.e. 1-HOMOGENEOUS

$$y^a(h_t(e)) = t y^a(e)$$

$$(x^i, y^a) \quad \nabla(x^i, y^a) = y^a \frac{\partial}{\partial y^a}$$

DEFINITION: A DOUBLE VECTOR BUNDLE IS A MANIFOLD D TOGETHER WITH TWO COMPATIBLE VECTOR BUNDLE STRUCTURES h^A, h^B , WHICH MEANS THAT THEY COMMUTE $\forall s, t \in \mathbb{R} \quad h_s^A \circ h_t^B = h_t^B \circ h_s^A$



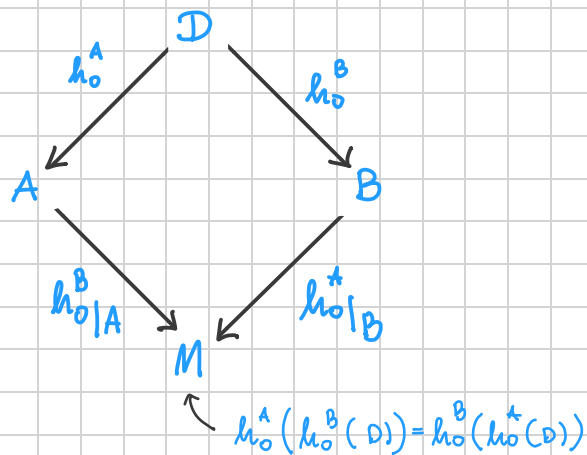
$$A = h_0^A(D)$$

$$a \in A \quad h_t^B(a) = h_t^B(h_0^A(v)) = h_0^A(h_t^B(v)) \in A$$

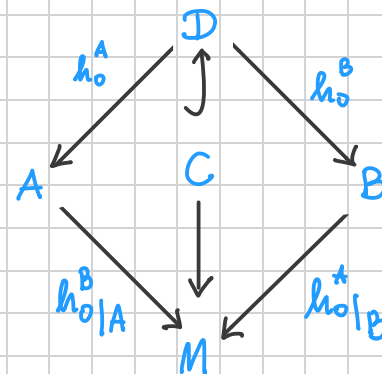
A IS INVARIANT WITH RESPECT TO $\left. \begin{matrix} h_t^B \\ h_t^A \end{matrix} \right\} \Rightarrow$
— II —

$$(A, h_{|A}^B) \quad (B, h_{|B}^A)$$

ARE VECTOR BUNDLES



$C = \{v \in D : h_0^A(v) = h_0^B(v) \in M\}$
 THE CORE INVARIANT WITH RESPECT TO h^A AND h^B
 MOREOVER $h^A|_C = h^B|_C$
 $(C, h^A|_C = h^B|_C)$ IS A VECTOR BUNDLE



CANONICAL EXAMPLES OF DOUBLE VECTOR BUNDLES

TE

(E, h) VECTOR BUNDLE
 $E \downarrow s = h_0$
 M

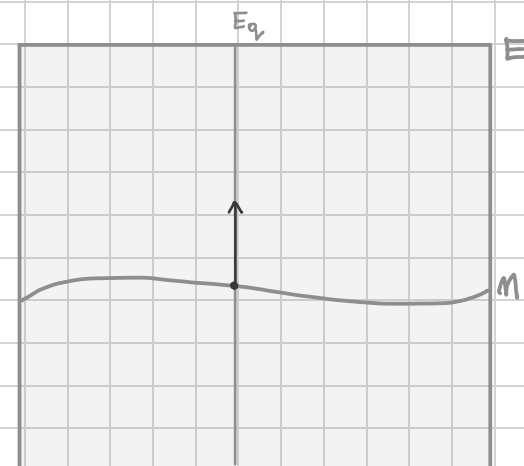
(TE, h^T) THE 'USUAL' TANGENT BUNDLE

$(TE, d_T h)$ THE TANGENT STRUCTURE

$$(d_T h)_t = Th_t \quad (d_T h)_0 = Th_0 = T_s \quad T_s(TE) = TM$$

CORE: VERTICAL VECTORS AT ZERO-SECTION OF τ_E

COORDINATES: OF E (x^i, y^a) $x^i \circ h_t = x^i$ $y^a \circ h_t = t y^a$
 ON TE $(x^i, y^a, \dot{x}^i, \dot{y}^a)$

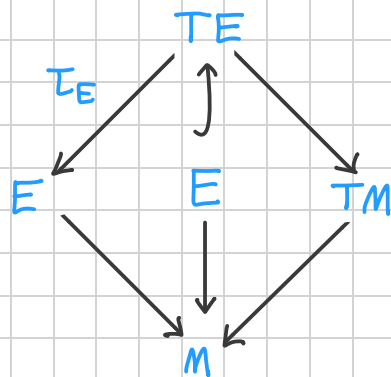


$C \simeq E$

COORDINATES : ON E (x^i, y^a) $x^i \circ h_t = x^i$ $y^a \circ h_t = t y^a$
 ON TE $(x^i, y^a, \dot{x}^i, \dot{y}^a)$

$$(x^i, y^a, \dot{x}^i, \dot{y}^b) \circ h_t^T = (x^i, y^a, t \dot{x}^i, t \dot{y}^b)$$

$$(x^i, y^a, \dot{x}^i, \dot{y}^b) \circ (d_T h)_t = (x^i, t y^a, \dot{x}^i, t \dot{y}^b)$$



$$\nabla^T = \dot{x}^i \frac{\partial}{\partial \dot{x}^i} + \dot{y}^a \frac{\partial}{\partial \dot{y}^a}$$

$$d_T \nabla = y^a \frac{\partial}{\partial y^a} + \dot{y}^b \frac{\partial}{\partial \dot{y}^b}$$

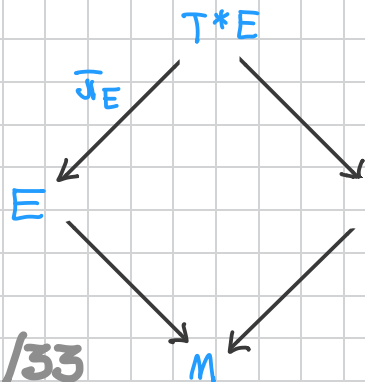
TANGENT LIFT OF THE EULER
VECTOR FIELD ON E

$$\nabla = y^a \frac{\partial}{\partial y^a}$$

COMPATIBILITY OF h^T AND $d_T h$
MEANS THAT

$$[\nabla^T, d_T \nabla] = 0$$

T^*E



E (E, h) VECTOR BUNDLE
 $\downarrow \tau = h_0$
 M

(T^*E, h^{T*}) THE 'USUAL' COTANGENT BUNDLE

COORDINATES : ON E (x^i, y^a)

ON T^*E $(x^i, y^a, \bar{\pi}_i, \xi_a)$

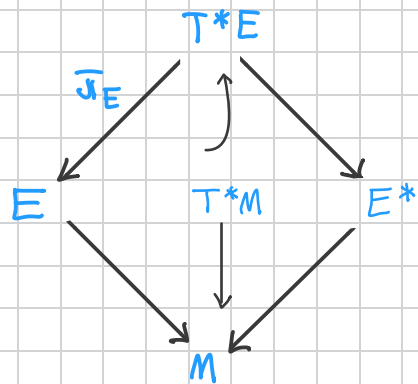
$$(x^i, y^a, \bar{\pi}_i, \xi_b) \circ h^{T*}_t = (x^i, y^a, t \bar{\pi}_i, t \xi_a)$$

$$\text{On } E: (x^i, y^a) \circ h_t = (x^i, t y^a) \quad x^i, t y^a, \frac{t}{\bar{y}_j}, \cancel{\frac{t}{\bar{\xi}_b}}$$

$$t \neq 0 \quad (x^i, y^a, \bar{y}_j, \bar{\xi}_b) \circ T^* h_t = (x^i, \frac{1}{t} y^a, \bar{y}_j, t \bar{\xi}_b)$$

$$\mathcal{L}_T^* h_t = h_t^{T^*} \circ T^* h_{\frac{1}{t}}$$

$$(x^i, y^a, \bar{y}_j, \bar{\xi}_b) \circ h_t^{T^*} \circ T^* h_{\frac{1}{t}} = (x^i, y^a, t \bar{y}_j, t \bar{\xi}_b) \circ T^* h_{\frac{1}{t}} = \underbrace{(x^i, t y^a, t \bar{y}_j, \bar{\xi}_b)}_{\text{WORKS FOR } t=0}$$



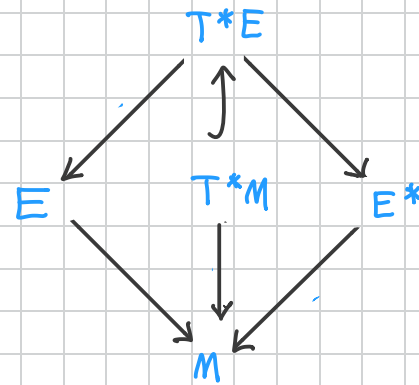
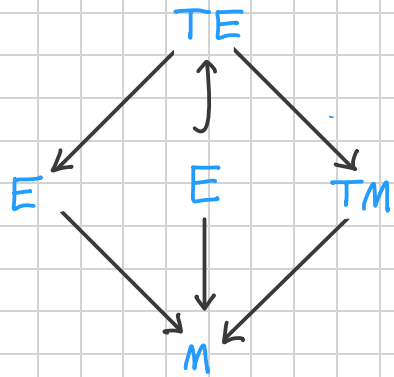
$$(T^*E, h^{T^*}) : (x^i, y^a, \bar{y}_j, \bar{\xi}_b) \circ h_t^{T^*} = (x^i, y^a, t \bar{y}_j, t \bar{\xi}_a)$$

$$(T^*E, \mathcal{L}_T^* h) : (x^i, y^a, \bar{y}_j, \bar{\xi}_b) \circ \mathcal{L}_T^* h_t = (x^i, t y^a, t \bar{y}_j, \bar{\xi}_b)$$

$$\nabla^{T^*} = \bar{y}_j \frac{\partial}{\partial \bar{y}_j} + \bar{\xi}_a \frac{\partial}{\partial \bar{\xi}_a}$$

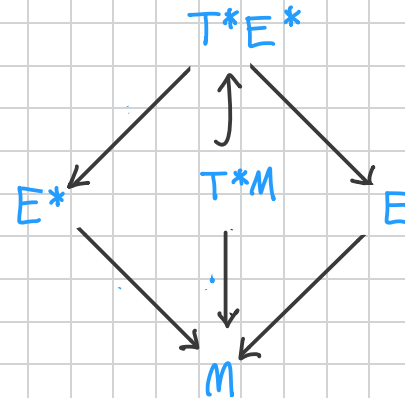
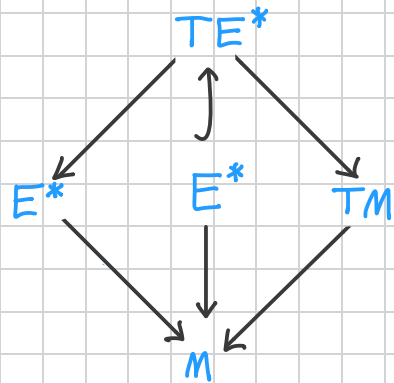
$$d_T^* \nabla = y^a \frac{\partial}{\partial y^a} + \bar{y}_j \frac{\partial}{\partial \bar{y}_j}$$

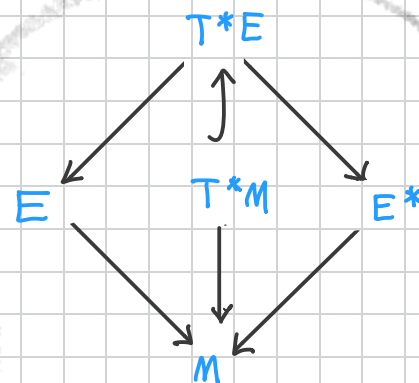
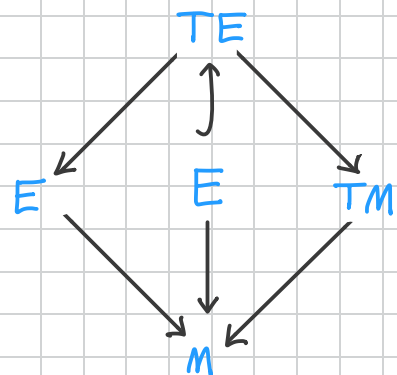
$$[d_T^* \nabla, \nabla^{T^*}] = 0$$



APPLY T AND T^* TO E^*

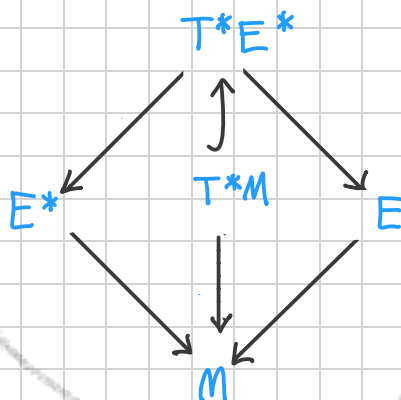
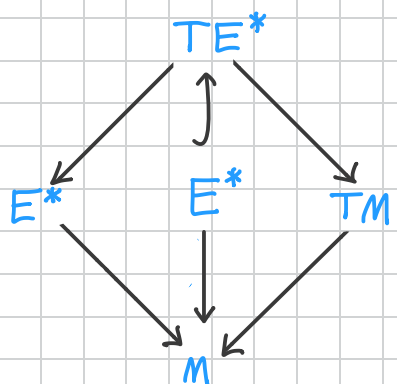
$E^* \downarrow$
 M





APPLY T AND T^* TO

$$E^* \xrightarrow{\varphi^*} M$$



THE TWO DOUBLE VECTOR BUNDLES
DIFFER BY REFLECTION ONLY!

$$T^*E \cong T^*E^*$$

CANONICAL ISOMORPHISM
OF D.V.B

(ANTI) SYMPLECTOMORPHISM

$$\mathcal{R}_E : T^*E^* \longrightarrow T^*E$$

$$(x^i, \zeta_a, p_j, q^b) \quad (x^i, y^a, \bar{p}_j, \bar{\zeta}_b)$$

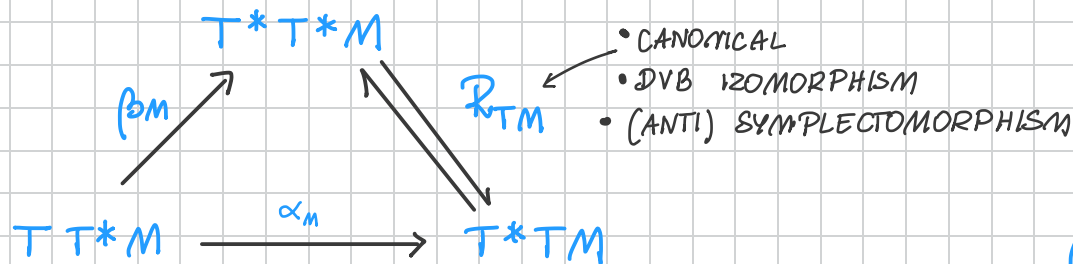
$$(x^i, y^a, \bar{p}_j, \bar{\zeta}_b) \circ \mathcal{R}_E = (x^i, q^b, -p_j, \zeta_a)$$



JEAN-PAUL DUFOUR

LET US NOW COME BACK TO THE RESULTS OF OUR VARIATIONAL CALCULATIONS:

$$E = TM, \quad E^* = T^*M, \quad TT^*M, \quad T^*TM, \quad T^*T^*M$$



- α_M IS A DVB ISOMORPHISM,
DUAL TO κ WHICH IS CANONICAL
- THIS ALSO IS CANONICAL
- SYMPLECTOMORPHISM
- WE GOT IT FROM VARIATIONAL CONSIDERATIONS

- CANONICAL
- DVB ISOMORPHISM
- (ANTI) SYMPLECTOMORPHISM

$$TT^*M \quad (q^i, p_j, \dot{q}^k, \dot{p}_j)$$

$$T^*TM \quad (q^i, \dot{q}^j, \varphi_i, \psi_j)$$

$$T^*T^*M \quad (q^i, p_j, a_k, b^l)$$

$$(q^i, \dot{q}^j, \varphi_k, \psi_l) \circ \alpha_M = (q^i, \bar{q}^j, \dot{p}_k, p_l)$$

$$(q^i, p_j, a_k, b^l) \circ R_{TM} = (q^i, \psi_j, -\varphi_k, \dot{q}^l)$$

$$\beta_M = R_{TM} \circ \alpha_M$$

$$(q^i, p_j, a_k, b^l) \circ \beta_M = (q^i, p_j, -\dot{p}_k, \dot{q}^l)$$

WE DEFINITELY RECOGNIZE THIS!

HERE WE HAVE RECOVERED
THE CANONICAL
SYMPLECTIC STRUCTURE
OF T^*M

$$\begin{aligned} \beta_M \left(\dot{q}^l \frac{\partial}{\partial q^l} + \dot{p}_k \frac{\partial}{\partial p_k} \right) &= -\dot{p}_k dq^k + \dot{q}^l dp_l = \\ &= - \left(\dot{q}^l \frac{\partial}{\partial q^l} + \dot{p}_k \frac{\partial}{\partial p_k} \right) \lrcorner dp_i \wedge dq^i \end{aligned}$$

$$\beta_M(v) = \omega_M(\cdot, v)$$

- ✓ STATICS OF PHYSICAL SYSTEMS
- ✓ VARIATIONAL PRINCIPLE AS AN INSPIRATION
- ✓ TULCZYJEW ISOMORPHISM AND TULCZYJEW TRIPLE

- EXAMPLES: RELATIVISTIC PARTICLES, CLASSICAL PHOTONS

IN GEOMETRIC MECHANICS THERE APPEAR VERY DIFFERENT STRUCTURES

- SYMPLECTIC
- POISSON
- CONTACT
- DIRAC
- COSYMPLECTIC
- ...

USED FOR:

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- NUMERICALLY SOLVING EQS

- ✓ DOUBLE VECTOR BUNDLES AND LINEARITY CRASH COURSE

GEOMETRIC MECHANICS

AN OVERVIEW OF STRUCTURES AND METHODS THAT CAN BE USED TO AVOID INFINITE DIMENSIONAL GEOMETRY

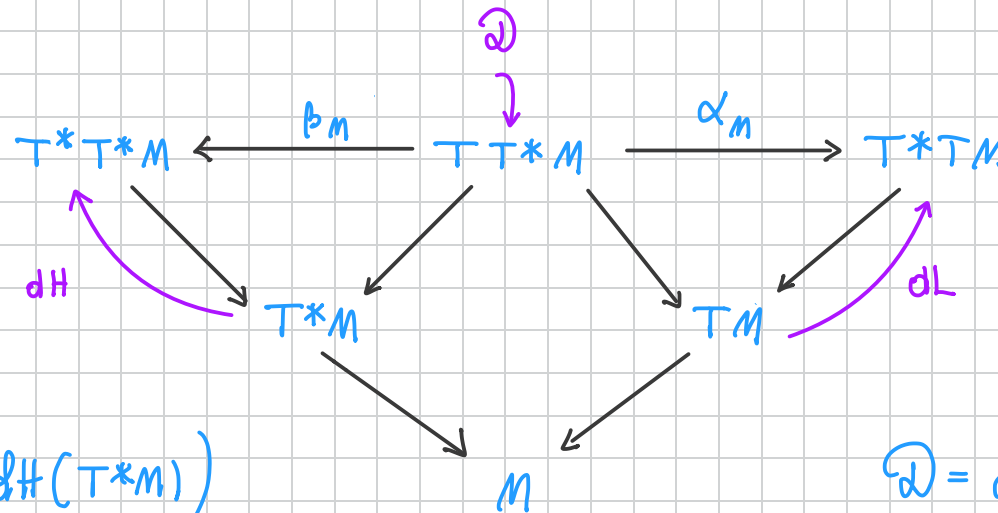
TULCZYJEW TRIPLE GENERALIZED

- LIE AND GENERAL ALGEBROIDS
- TULCZYJEW TRIPLE FOR IN AN ALGEBROID SETTING
- EXAMPLES: MECHANICS ON A LIE ALGEBRA, MECHANICAL SYSTEM WITH CONSTRAINTS

TULCZYJEW TRIPLE GENERALIZED EVEN MORE

- DIRAC STRUCTURE
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THE CLASSICAL TULCZYJEW TRIPLE



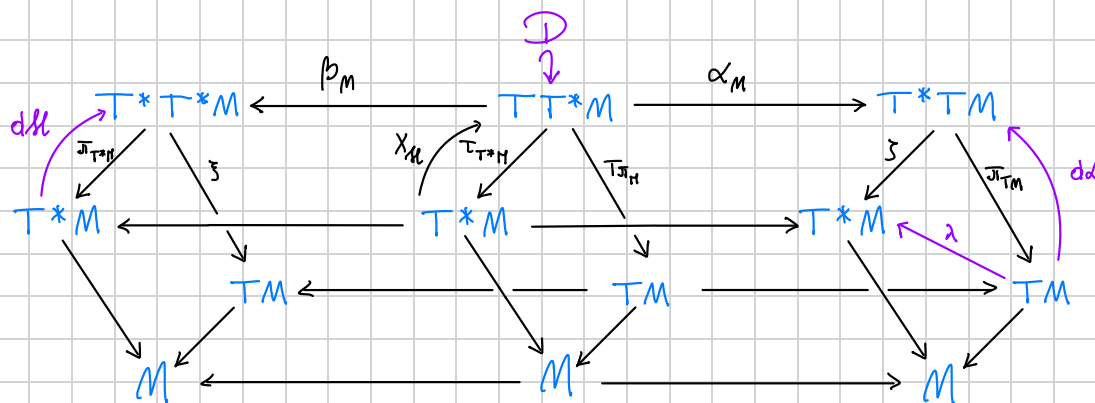
$$\mathcal{D} = \beta_M^{-1} (dH(T^*M))$$

THE DYNAMICS OBTAINED FROM
HAMILTONIAN IS ALWAYS EXPLICIT

$$\mathcal{D} = \alpha_M^{-1} (dL(TM))$$

THE DYNAMICS OBTAINED
FROM LAGRANGIAN CAN BE IMPLICIT

IT IS EXPLICIT FOR REGULAR LAGRANGIANS



N.M. TULCZYJEW

EXAMPLES THAT ARE SIMPLE, NOT REGULAR, AND MAKE SENSE

(M, η) THE AFFINE MINKOWSKI SPACE

$$\tilde{\eta}: V \rightarrow V^*, \quad \tilde{\eta}(v) = \eta(v, \cdot) \quad V^+ = \{v \in V : \eta(v, v) > 0\}$$

FREE RELATIVISTIC PARTICLE

$$\begin{array}{ccccc} T^*T^*M & & TT^*M & & T^*TM \\ M \times V^* \times V^* \times V & \xleftarrow{\beta_M} & M \times V^* \times V \times V^* & \xrightarrow{\alpha_M} & M \times V \times V^* \times V^* \\ (x, p, -\dot{p}, v) & \xleftarrow{\quad} & (x, p, v, \dot{p}) & \xrightarrow{\quad} & (x, v, \dot{p}, p) \end{array}$$

THE DYNAMICS IS NOT
NOT GIVEN BY ANY
HAMILTONIAN
FUNCTION BUT...

$$\mathcal{H}(x, p, \alpha) = \alpha(\|p\|^2 - m^2 c^2)$$

$$\mathcal{D} = \left\{ (x, p = \frac{mc}{\|v\|} \tilde{\eta}(v), v, \dot{p} = 0) \right\} =$$

$$= \left\{ (x, p, \frac{\alpha}{mc} \eta(p), 0) : \right. \\ \left. \eta(p, p) = m^2 c^2 \right\}$$

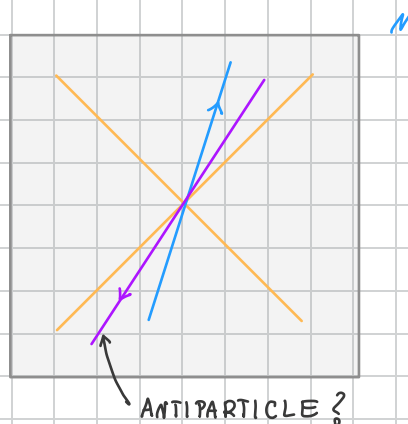
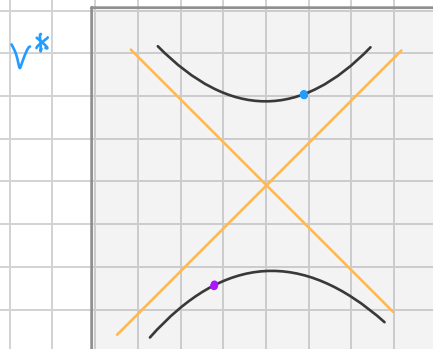
$$L: M \times V^+ \rightarrow \mathbb{R}$$

$$L(x, v) = mc \sqrt{\eta(v, v)} \quad \leftarrow \|v\|$$

- DEFINED ON $V^+ \subset V$
- POSITIVELY HOMOGENEOUS
- THE ACTION FUNCTIONAL DOES NOT DEPEND ON PARAMETERIZATION OF THE TRAJECTORY

$$dL(M \times V^+) = \left\{ (x, v, 0, \frac{mc}{\|v\|} \tilde{\eta}(v)) \right\}$$

$$\lambda(v) = \frac{mc}{\|v\|} \tilde{\eta}(v) \quad \text{NOT INVERTIBLE}$$



$$\mathcal{Q} = \left\{ (x, p, \frac{1}{mc} \eta(p, \cdot), 0) : \eta(p, p) = m^2 c^2 \right\} \subset M \times V^* \times V \times V^* \cong TT^*M$$

LAGRANGIAN SUBMANIFOLD $\leftarrow \beta_M(\mathcal{Q}) \subset T^*T^*M$

LAGRANGIAN SUBMANIFOLDS BY GENERATING FAMILIES (OF FUNCTIONS)

$$\begin{array}{ccc} F & \xrightarrow{h} & \mathbb{R} \\ \downarrow \vartheta & & \\ N & & \end{array} \quad T^*F \supset d\mathcal{H}(F) \cap (VF)^\circ$$

REDUCTION

$$S_h \subset T^*N$$

ISOTROPIC SUBMANIFOLD

+ CONDITIONS FOR h TO ENSURE THAT S_h IS A SUBMANIFOLD

UNIVERSAL HAMILTONIAN GENERATING FAMILY

$$\begin{array}{ccc} F := T^*M \times_M TM & \xrightarrow{\mathcal{H}} & \mathbb{R} \\ \downarrow & & \\ T^*M & & \end{array}$$

$$\mathcal{H}(p, v) = \langle p, v \rangle - L(v)$$

COMPOSING SYMPLECTIC RELATION
THIS GENERATES R_{TM}

$$d\mathcal{H}(p, v) \in (VF)^\circ \text{ IF } \frac{\partial \mathcal{H}}{\partial v} = 0 \text{ I.E. } \frac{\partial L}{\partial v} = p \quad \leftarrow \text{LEGENDRE MAP}$$

GRAPH OF R_{TM} IS GENERATED BY

$$\begin{array}{ccc} T^*T^*M & \xleftrightarrow{R_{TM}} & T^*TM \\ \nwarrow \beta_M & & \nearrow \alpha_M \\ & TT^*M & \end{array} \quad T^*M \times TM \supset T^*M \times_M TM \ni (p, v) \mapsto \langle p, v \rangle$$

MASSLESS RELATIVISTIC PARTICLE

$$\begin{array}{ccccc}
 T^*T^*M & & T^*T^*M & & T^*TM \\
 M \times V^* \times V^* \times V & \xleftarrow{\beta_M} & M \times V^* \times V \times V^* & \xrightarrow{\alpha_M} & M \times V \times V^* \times V^* \\
 (x, p, -\dot{p}, v) & \xleftarrow{\quad} & (x, p, v, \dot{p}) & \xrightarrow{\quad} & (x, v, \dot{p}, p)
 \end{array}$$

$$\mathcal{Q} = \{ (x, p, y \tilde{\eta}^T(p), 0) : \|p\| = 0 \}$$

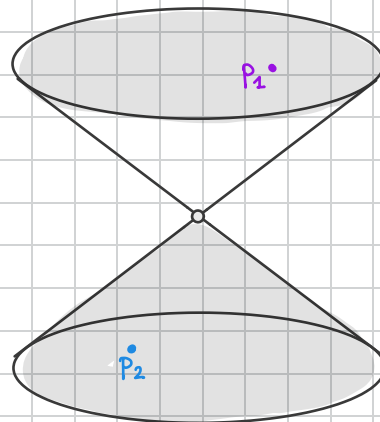
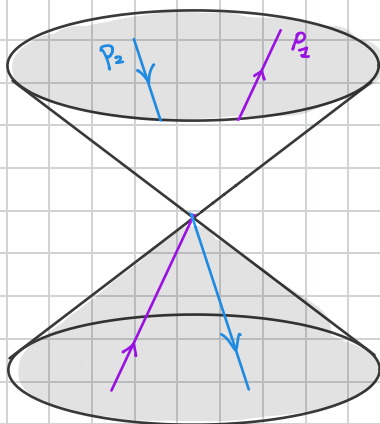
$$h: M \times (V^*)^x \times \mathbb{R}_+ \longrightarrow \mathbb{R}$$

$$h(x, p, \tau) = \frac{1}{2} y \eta(p, p)$$

$$l: M \times V^* \times \mathbb{R}_+ \longrightarrow \mathbb{R}$$

$$l(x, v, y) = \frac{1}{2y} \eta(v, v)$$

RELATED BY LEGENDRE TRANSFORMATION



THESE STRUCTURES SERVE FOR ALL SYSTEMS THAT ARE NOT TOO COMPLICATED

- AUTONOMOUS
- NO CONSTRAINTS
- NOT REDUCED H.R.T. SYMMETRIES

AN APPROPRIATE VERSION EXISTS

SYSTEMS WITH SYMMETRIES & CONSTRAINTS

REDUCTION?

PHASE DYNAMICS?



$$\begin{array}{ccccc}
 T T^* M & & T T^* M & & T^* T M \\
 M \times V^* \times V^* \times V & \xleftarrow{\beta_M} & M \times V^* \times V \times V^* & \xrightarrow{\alpha_M} & M \times V \times V^* \times V^* \\
 (x, p, -\dot{p}, v) & \xleftarrow{\quad} & (x, p, v, \dot{p}) & \xrightarrow{\quad} & (x, v, \dot{p}, p)
 \end{array}$$

W.M. Tulczyjew, P. Urbański

A Slow and Careful Legendre Transformation for Singular Lagrangians

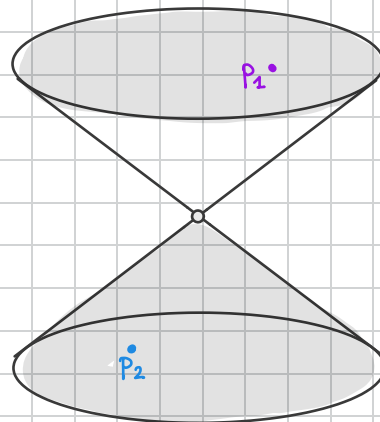
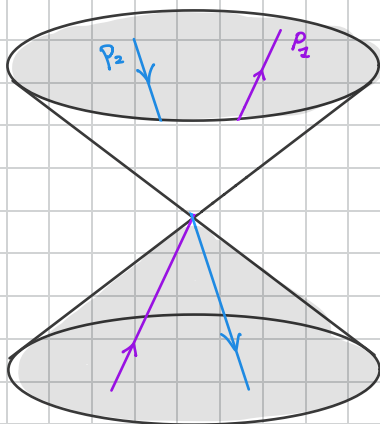
vol. 30, p. 2909 (70 pages)

[abstract](#) • [links/reference](#)

$$\mathcal{L}(x, p, v) = \frac{1}{2} g(p, p) - g(v, v)$$

$$\mathcal{H}(x, v, p) = \frac{1}{2} g(v, v) + g(p, p)$$

RELATED BY LEGENDRE TRANSFORMATION



THESE STRUCTURES SERVE FOR ALL SYSTEMS

Vol. **30** (1999), No. 10, pp. 2853 – 3027 NOT TOO COMPLICATED

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SYSTEMS WITH SYMMETRIES & CONSTRAINTS

REDUCTION?

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HAMILTONIAN MECHANICS

- ASSOCIATED WITH A PHASE SPACE $\mathcal{P}$ WHICH IS SYMPLECTIC $(\mathcal{P}, \omega)$ OR POISSON $(\mathcal{P}, \Delta)$ MANIFOLD

- ω OR Δ ARE USED TO PRODUCE A VECTOR FIELD FROM A HAMILTONIAN FUNCTION $\mathcal{H}: \mathcal{P} \longrightarrow \mathbb{R}$

SYMPLECTIC $X_H \lrcorner \omega = d\mathcal{H}$

POISSON $d\mathcal{H} \lrcorner \Delta = X_H$

IN DARBOUX COORDINATES $\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i} \quad \dot{p}^j = -\frac{\partial \mathcal{H}}{\partial q^j}$

USING POISSON BRACKET $\dot{f} = \{f, \mathcal{H}\}$

LAGRANGIAN MECHANICS

- „LIVES ON“ THE TANGENT BUNDLE TM

- LAGRANGIAN FUNCTION $\mathcal{L}: TM \longrightarrow \mathbb{R}$
+ VARIATIONAL CALCULUS GIVES EULER-LAGRANGE EQUATIONS AND DEFINITION OF MOMENTA

OR

- LAGRANGIAN FUNCTION + TULCZYJEW MAP $\alpha_M: TT^*M \longrightarrow T^*TM$
GIVES PHASE EQUATIONS ON T^*M

QUESTION: WHAT GEOMETRIC STRUCTURE IS „RESPONSIBLE“ FOR LAGRANGIAN MECHANICS? $\longrightarrow$ ALGEBROID

ALGEBROIDS

THERE EXISTS SEVERAL EQUIVALENT DEFINITIONS OF A LIE ALGEBROID ON A VECTOR BUNDLE $\tau: E \longrightarrow M$

$$\textcircled{1} [\cdot, \cdot]: \text{Sec}(\tau) \times \text{Sec}(\tau) \longrightarrow \text{Sec}(\tau), \quad \begin{array}{ccc} E & \xrightarrow{\rho} & TM \\ \downarrow & & \downarrow \\ M & \xrightarrow{=} & M \end{array} \quad \text{VB MORPHISM}$$

ANTISYMMETRIC $[X, Y] = -[Y, X],$

JACOBI IDENTITY $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]],$

$$[X, fY] = f[X, Y] + \rho(X)(f)Y, \quad f \in C^\infty(M).$$

EXAMPLES: $\left(\begin{array}{c} TM \\ \downarrow \\ M \end{array}, [\cdot, \cdot], \text{id}_{TM} \right)$
 $\left(\begin{array}{c} \mathfrak{g} \\ \downarrow \\ \{*\} \end{array}, [\cdot, \cdot], 0 \right)$

FOR A PRINCIPAL BUNDLE $\begin{array}{c} P \\ \downarrow \\ M \end{array} \supset G$ WE DEFINE $AP = TP/G$

WITH SECTIONS BEING INVARIANT VECTOR FIELDS AND ρ COMING FROM $T\tau$

ALGEBROIDS

THERE EXISTS SEVERAL EQUIVALENT DEFINITIONS OF A LIE ALGEBROID ON A VECTOR BUNDLE $\tau: E \longrightarrow M$

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$$[X, fY] = f[X, Y] + \rho(X)(f)Y, \quad f \in C^\infty(M).$$

$$\textcircled{2} \text{ A LINEAR POISSON STRUCTURE } \Lambda \text{ ON } E^* \left(\{ \cdot, \cdot \} \right)$$

COMPATIBLE WITH A VB STRUCTURE
HOMOGENEOUS OF DEGREE $-1: \mathcal{L}_{\nabla_{E^*}} \Lambda = -\Lambda$

CLOSED ON LINEAR FUNCTIONS

$$\iota_X: E^* \longrightarrow \mathbb{R}, \quad \iota_X(\varphi) = \langle \varphi, X \rangle$$

$$\{ \iota_X, \iota_Y \} = \iota_{[X, Y]}$$

$$\{ \iota_X, \pi^* f \} = \pi^* (\rho(X)f)$$

$$\pi: E^* \longrightarrow M$$

- ① BRACKET OF $\text{Sec}(\tau)$ & ANCHOR
- ② LINEAR POISSON STRUCTURE ON E^*
- ③ A HOMOLOGICAL DERIVATION $d_E : \mathcal{A}(E^*) \longrightarrow \mathcal{A}(E^*)$ OF DEGREE 1
IN THE GRASSMANN ALGEBRA $\mathcal{A}(E^*)$ OF E^*

$$d_E : \mathcal{A}^i(E^*) \longrightarrow \mathcal{A}^{i+1}(E^*),$$

$$d_E^2 = 0,$$

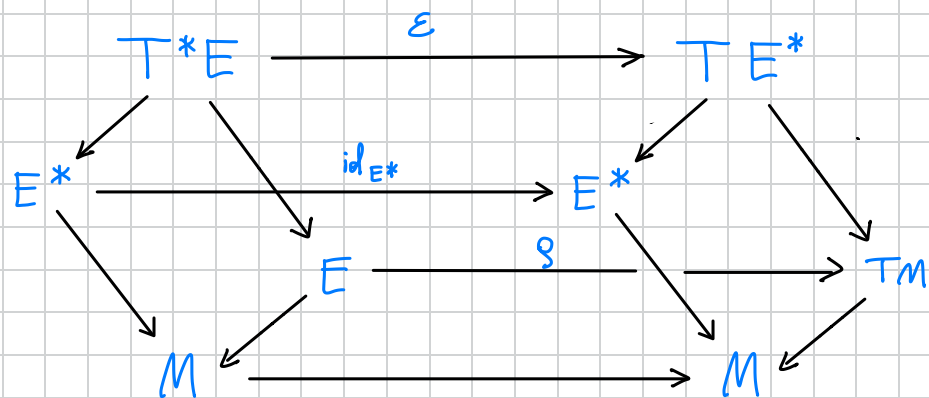
$$d_E(\alpha \wedge \beta) = (d_E \alpha) \wedge \beta + (-1)^a \alpha \wedge d_E(\beta), \quad \alpha \in \mathcal{A}^a(E^*).$$

RELATION WITH THE BRACKET

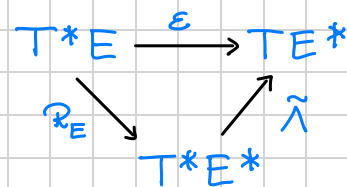
$$f \in C^\infty(M) = \mathcal{A}^0(E^*), \quad d_E f \in \mathcal{A}^1(E^*) \quad \langle d_E f, X \rangle = g(X)f,$$

$$\alpha \in \mathcal{A}^1(E^*) \quad d_E \alpha(X, Y) = g(X) \langle \alpha, Y \rangle - g(Y) \langle \alpha, X \rangle - \langle \alpha, [X, Y] \rangle.$$

- ① BRACKET OF $\text{Sec}(\tau)$ & ANCHOR
- ② LINEAR POISSON STRUCTURE ON E^*
- ③ A HOMOLOGICAL DERIVATION d_E OF DEGREE 1
- ④ A DVB MORPHISM $\varepsilon: T^*E \longrightarrow TE^*$ OVER THE IDENTITY ON E^*



RELATION WITH LINEAR POISSON STRUCTURE



COORDINATES

$$\begin{aligned}
 M &: (q^i) & TE^* & (q^i, \xi_a, \dot{q}^i, \dot{\xi}_b) \\
 E &: (q^i, y^a) & T^*E & (q^i, y^a, p_i, \eta_b) \\
 E^* &: (q^i, \xi_a)
 \end{aligned}$$

$$\begin{aligned}
 q^i \circ \varepsilon &= q^i \\
 \xi_a \circ \varepsilon &= \eta_b \\
 \dot{q}^i \circ \varepsilon &= g_a^i(q) y^a \\
 \dot{\xi}_b \circ \varepsilon &= G_{db}^a(q) y^d \eta_a + g_a^i(q) p_i
 \end{aligned}$$

$$\varepsilon(q^i, y^a, p_i, \eta_b) = (q^i, \eta_b, g_a^i(q) y^a, G_{db}^a(q) y^d \eta_a + g_a^i(q) p_i)$$

- ① BRACKET OF $\text{Sec}(\tau)$ & ANCHOR
- ② LINEAR POISSON STRUCTURE ON E^*
- ③ A HOMOLOGICAL DERIVATION d_E OF DEGREE 1
- ④ A DVB MORPHISM $\varepsilon: T^*E \longrightarrow TE^*$
- ⑤ IN SUPERGEOMETRIC LANGUAGE :

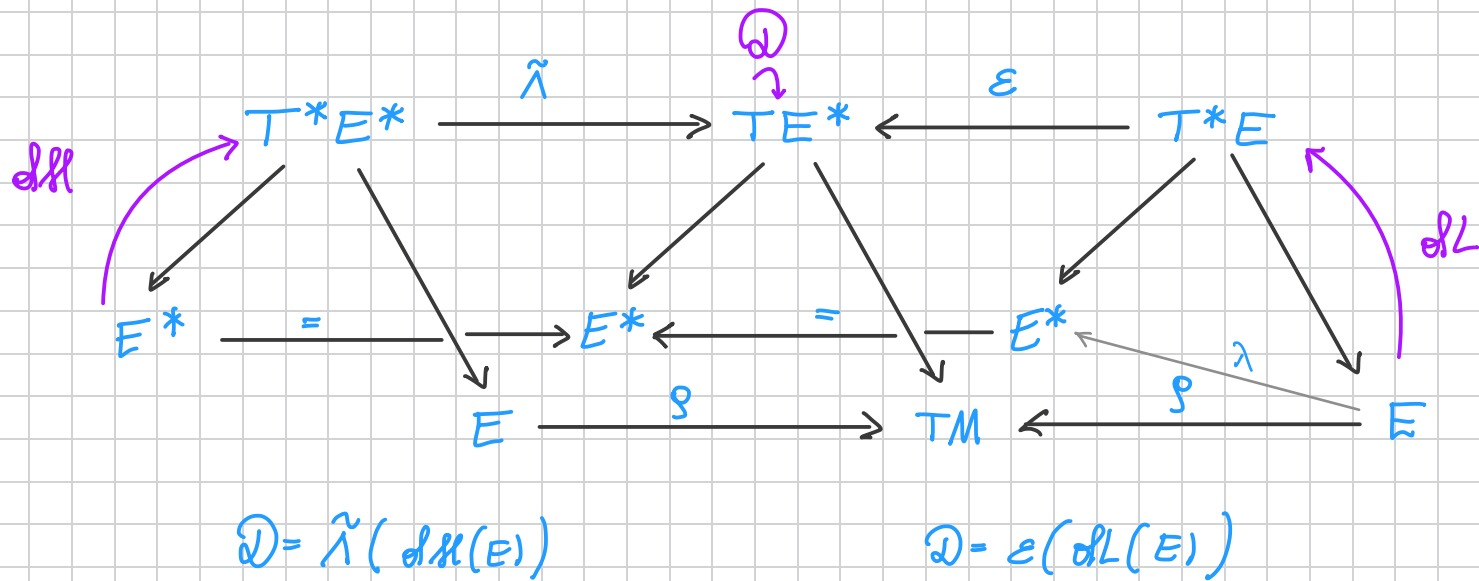
A HOMOLOGICAL VECTOR FIELD ON THE SUPERMANIFOLD
TIE ASSOCIATED TO THE V.B. $E \longrightarrow M$

$$M = TE$$

$$C^\infty(M) = A(E^*)$$

VECTOR FIELD d_E

USING DEFINITIONS (2) AND (4) WE CAN WRITE DOWN THE TULCZYJEW TRIPLE FOR AN ALGEBROID

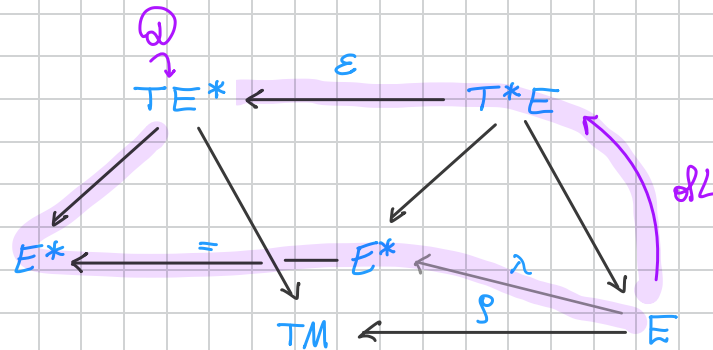


EULER LAGRANGE EQUATION?

$$\begin{aligned}
 \mathcal{Q} &\subset TT^*M \\
 T\mathcal{Q} &\subset TTT^*M \\
 &\quad \downarrow \cup \\
 T\mathcal{Q} &\cap T^2T^*M \\
 &\quad \downarrow \\
 \mathcal{E}_L &\subset T^2M
 \end{aligned}$$

$$T^2M \longleftrightarrow$$

$$\begin{aligned}
 v \in TE : \quad \tau_E(v) &= T\tau(v) \\
 &= (q^i, y^a, \dot{q}^i, \dot{y}^a) \\
 \text{ADMISSIBILITY CONDITION} &\rightarrow y^a = \rho^a_i \dot{q}^i
 \end{aligned}$$



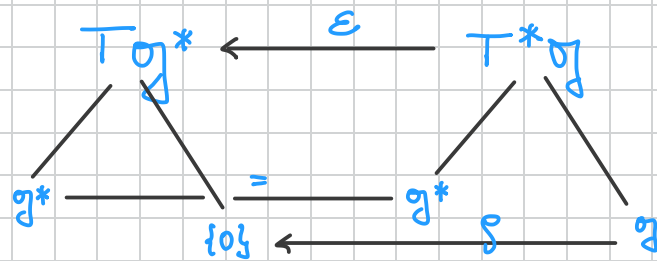
$$\begin{aligned}
 \gamma: I &\rightarrow E \\
 \varepsilon(dL(\gamma(t))) &= \frac{d}{dt} \lambda(\gamma(t))
 \end{aligned}$$

EXAMPLE

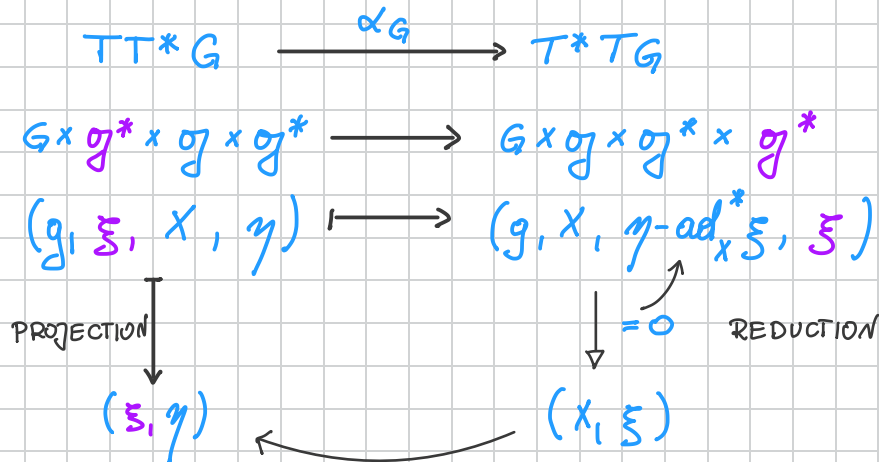
G - LIE GROUP

$L: TG \rightarrow \mathbb{R}$ INVARIANT, i.e. $L(gv) = L(v)$

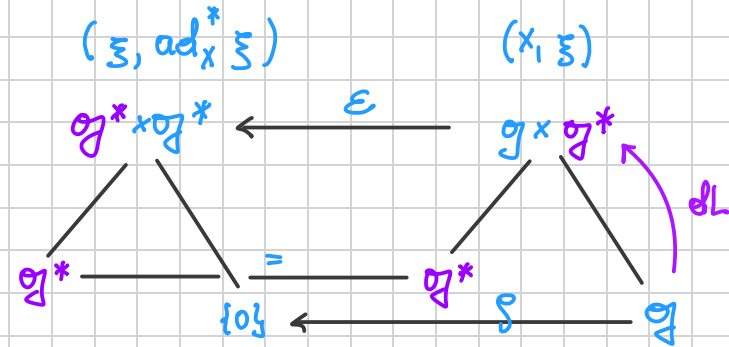
$\parallel$
 $G \times \mathfrak{g} \quad l: \mathfrak{g} \rightarrow \mathbb{R}$
 REDUCTION WITH RESPECT
 TO THE GROUP ACTION
 LEADS TO MECHANICS ON
 THE LIE ALGEBROID $\mathfrak{g} \rightarrow \{0\}$



HOW DO WE GET ϵ ?



- REDUCTION WRT A SUBGROUP $H \subset G$
- REDUCTION WRT GROUP ACTION ON M
- PRINCIPAL BUNDLE
 $\hookrightarrow$ MECHANICS ON ATIYAH ALGEBROID



$$\mathcal{D} = \{ (\xi, \dot{\xi}) : \xi = dL(x), \dot{\xi} = \text{ad}_x^* dL(x) \}$$

$$\gamma: I \rightarrow \mathfrak{g}$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial x}(\gamma(t)) \right) = \text{ad}_{\gamma(t)}^* dL(\gamma(t))$$

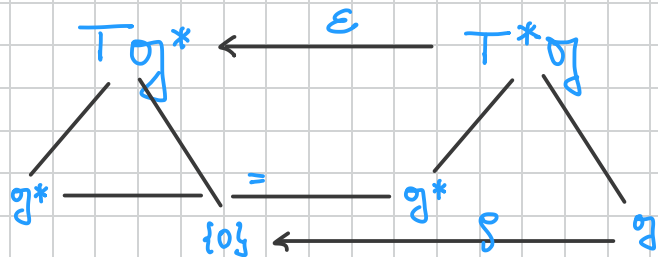
EULER POINCARÉ EQUATION

EXAMPLE

G - LIE GROUP

$$L: TG \longrightarrow \mathbb{R} \text{ INVARIANT, i.e. } L(gv) = L(v)$$
$$G \times \mathfrak{g} \quad l: \mathfrak{g} \rightarrow \mathbb{R}$$

REDUCTION WITH RESPECT
TO THE GROUP ACTION
LEADS TO MECHANICS ON
THE LIE ALGEBROID $\mathfrak{g} \rightarrow \mathfrak{h}$



HOW DO WE GET

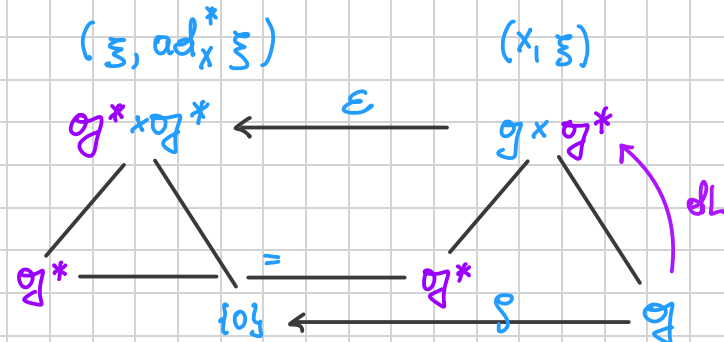
$$T T^* G \quad \underline{\alpha_G}$$
$$G \times g^* \times g \times g^* \longrightarrow$$
$$(x, \xi, x, y) \mapsto$$

PROJECTION

 (ξ, η) (x_1, ξ)

THE RESULTING RELATION IS A MAP IN THE OTHER DIRECTION

- REDUCTION W.R.T A SUBGROUP $H \subset G$
- REDUCTION W.R.T GROUP ACTION ON M
- PRINCIPAL BUNDLE
 $\searrow$ MECHANICS ON ATIYAH ALGEBROID



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The Tulczyjew triple in mechanics on a Lie group

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EXAMPLE MECHANICS ON A SKEW ALGEBROID (NO JACOBI IDENTITY)

SUCH A STRUCTURE CAN BE USED TO DESCRIBE A SYSTEM WITH NONHOLONOMIC

CONSTRAINTS $C \subset TM$ ($C \subset E$) AND MECHANICAL LAGRANGIAN

NOT A SUBALGEBROID
JUST A SUBBUNDLE

$$g: E \times_M E \longrightarrow \mathbb{R}, \quad L(v) = \frac{m}{2} g(v, v) - V(\pi(v))$$

BILINEAR, SYMMETRIC
POSITIVE DEFINITE

$$E = C \oplus_M C^\perp$$

$$(q^i, y^a)$$

$$i: C \hookrightarrow E$$

$$p: E \longrightarrow C$$

$$i^*: E^* \longrightarrow C^*$$

$$TP: TE \longrightarrow TC$$

$$T^*p: T^*_C E \longleftarrow T^*C$$

$$(q^i, y^a, y^b)$$

$$C^A_{BD} y^D \xi_A + g^i_B a_i$$

$$C^A_{BD} y^D \xi_A + g^i_B a_i$$

FORGOTTEN
BY THE
PROJECTION

$$(q^i, \xi_A, 0, g^i_B y^B + g^i_K \cancel{a_K})$$

$$C^A_{BD} y^D \xi_A + g^i_B a_i$$

$$(q^i, y^A, 0, a_j, \xi_B, 0)$$

$$TE^* \xleftarrow{\epsilon} T^*_C E$$

$$Ti^*$$

$$T^*p$$

$$\epsilon_C(q^i, y^A, a_j, \xi_B) = (q^i, \xi_A, g^j_B y^B, C^A_{BD} y^D \xi_A + g^k_B a_k)$$

NO JACOBI IDENTITY!

$$TC^* \xleftarrow{\epsilon_C} T^*C$$

$$(q^i, \xi_A, g^j_B y^B, C^A_{BD} y^D \xi_A + g^k_B a_k)$$

$$(q^i, y^A, a_j, \xi_B)$$

- WE HAVE C^* AS THE PHASE SPACE
- WE CAN USE L_C ONLY

DOES IT PRODUCE GOOD RESULTS?

E-L VIA SKEW ALGEBROID

$$\dot{q}^i = \rho_A^i y^A$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial y^B} \right) = C_{BD}^A y^D \frac{\partial L}{\partial y^A} + \rho_B^i y^B$$

FROM VARIATIONAL PRINCIPLE

$$y^\alpha = 0$$

$$\dot{q}^i = \rho_A^i y^A$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial y^B} \right) = C_{BD}^A y^D \frac{\partial L}{\partial y^A} + C_{BD}^\alpha y^D \frac{\partial L}{\partial y^\alpha} + \rho_B^i y^B$$

FOR MECHANICAL LAGRANGIAN

$$L(q^i, y^A, y^\alpha) = \frac{m}{2} (g_{AB} y^A y^B + g_{\alpha\beta} y^\alpha y^\beta) - V(q^i)$$

$$\frac{\partial L}{\partial y^\alpha} = m g_{\alpha\beta} y^\beta$$

= 0

$$\frac{d}{dt} \left(\frac{\partial L}{\partial y^B} \right) = C_{BD}^A y^D \frac{\partial L}{\partial y^A} + C_{BD}^\alpha y^D \frac{\partial L}{\partial y^\alpha} + \rho_B^i y^B$$

- MECHANICAL SYSTEMS WITH CONSTRAINTS LIVE ON SKEW ALGEBROIDS

- WHAT TO DO FOR OTHER SYSTEMS?

MAGNETIC FORCE
GYROSCOPIC FORCE

DIRAC STRUCTURE

- WE HAVE C^* AS THE PHASE SPACE
- WE CAN USE $L|_C$ ONLY

} DOES IT PRODUCE GOOD RESULTS?

E-L VIA SKEW ALGEBROID

$$\dot{q}^i = \rho_A^i y^A$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial y^B} \right) = C_{BD}^A y^D \frac{\partial L}{\partial y^A} + \rho_B^i \dot{y}^i$$

FROM VARIATIONAL PRINCIPLE

$$y^\alpha = 0$$

$$\dot{q}^i = \rho_A^i y^A$$

$$\frac{d}{dt} \left(\frac{\partial L}{\partial y^B} \right) = C_{BD}^A y^D \frac{\partial L}{\partial y^A} + C_{BD}^\alpha y^D \frac{\partial L}{\partial y^\alpha} + \rho_B^i \dot{y}^i$$

FOR MECHANICAL LAGRANGIAN

$$L(q^i, y^A, y^\alpha) = \frac{m}{2} (y^A y^B + y^\alpha y^\beta) - V(q^i)$$

- MECHANICAL SYSTEMS WITH CONSTRAINTS LIVE ON SKEW ALGEBROIDS

- WHAT TO DO FOR OTHER

MAGNETIC
GYROSCOPE

DIRAC STRUCTURE


Journal of Mathematical Physics

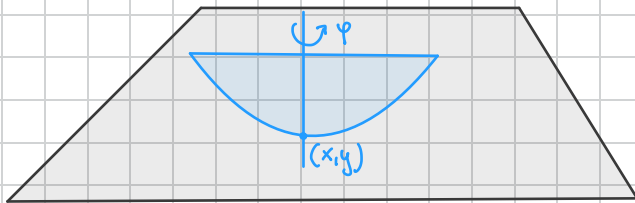
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Volume 50, Issue 1
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RESEARCH ARTICLE | JANUARY 13 2009
Nonholonomic constraints: A new viewpoint
J. Grabowski; M. de León; J. C. Marrero; D. Martín de Diego
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J. Math. Phys. 50, 013520 (2009)
<https://doi.org/10.1063/1.3049752> Article history

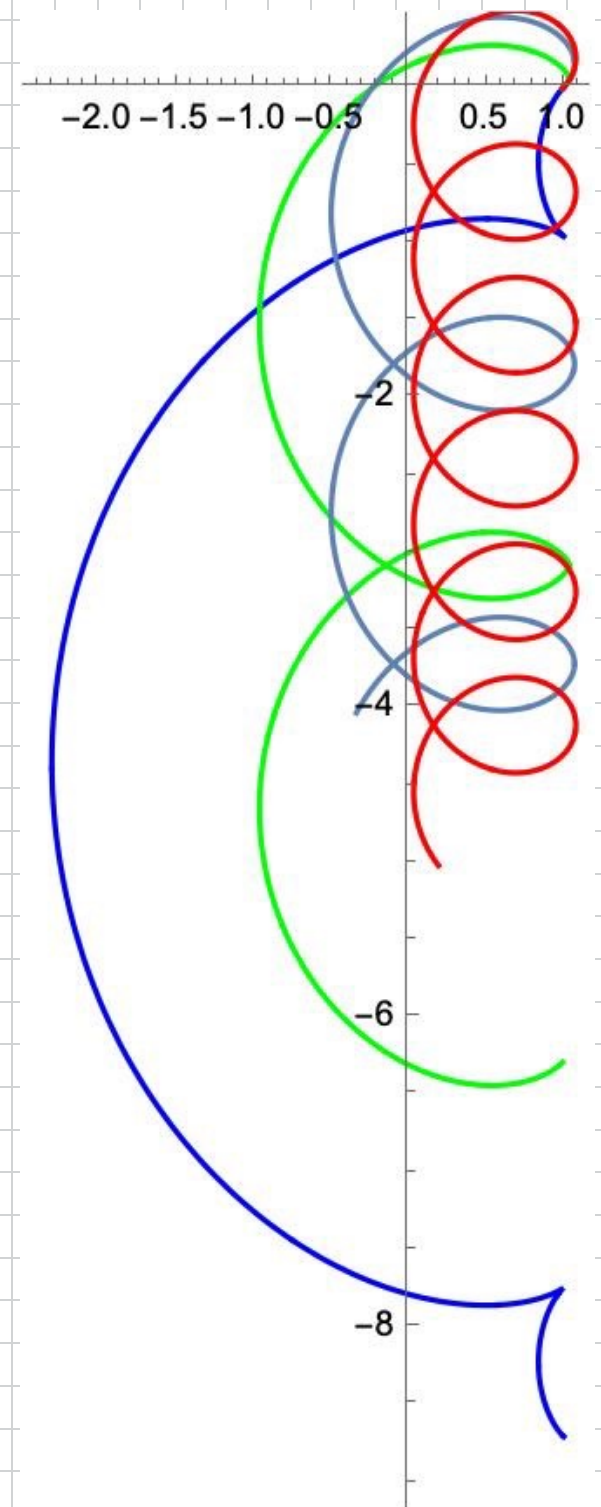
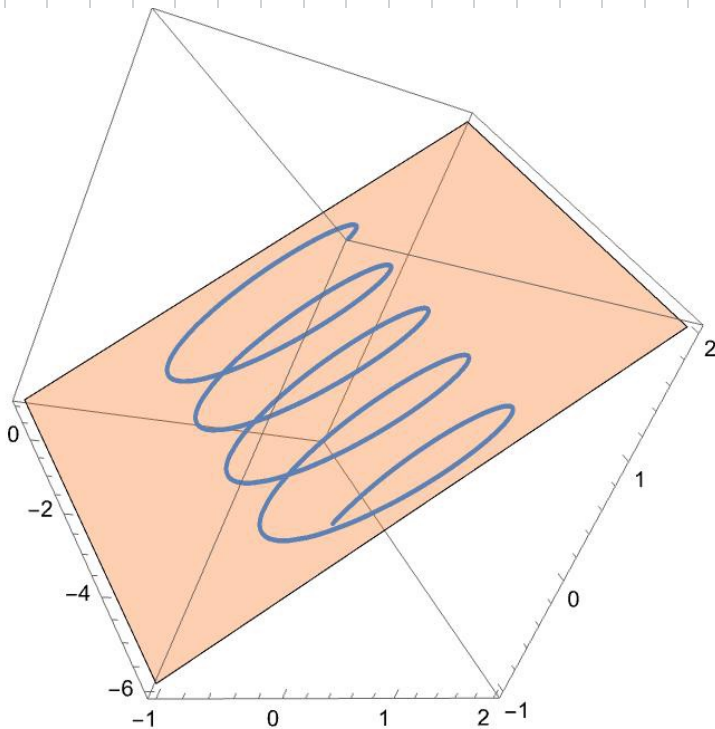
EXAMPLE : ASKATE



$$M = \mathbb{R}^2 \times S^1$$

$$C = \{ y \cos \varphi - x \sin \varphi = 0 \}$$

$$L(x, y, \varphi, \dot{x}, \dot{y}, \dot{\varphi}) = \frac{m}{2} (\dot{x}^2 + \dot{y}^2) + \frac{I}{2} \dot{\varphi}^2 - \lambda x$$



- ✓ STATICS OF PHYSICAL SYSTEMS
- ✓ VARIATIONAL PRINCIPLE AS AN INSPIRATION
- ✓ TULCZYJEV ISOMORPHISM AND TULCZYJEV TRIPLE
- ✓ EXAMPLES: RELATIVISTIC PARTICLES, CLASSICAL PHOTONS

IN GEOMETRIC MECHANICS THERE APPEAR VERY DIFFERENT STRUCTURES

- SYMPLECTIC
- POISSON
- CONTACT
- DIRAC
- COSYMPLECTIC
- ...

USED FOR:

- DERIVING EQUATIONS
- STUDYING PROPERTIES OF EQUATIONS
- SOLVING EQUATIONS
- NUMERICALLY SOLVING EQS

- ✓ DOUBLE VECTOR BUNDLES AND LINEARITY CRASH COURSE

GEOMETRIC MECHANICS

AN OVERVIEW OF STRUCTURES AND METHODS THAT CAN BE USED TO AVOID INFINITE DIMENSIONAL GEOMETRY

TULCZYJEV TRIPLE GENERALIZED EVEN MORE

- DIRAC STRUCTURE
- DIRAC ALGEBROID
- EXAMPLES:
MECHANICAL SYSTEM REVISITED
CONSTRAINED SYSTEM WITH
MAGNETIC LAGRANGIAN

TULCZYJEV TRIPLE GENERALIZED

- ✓ LIE AND GENERAL ALGEBROIDS
- ✓ TULCZYJEV TRIPLE FOR IN AN ALGEBROID SETTING
- ✓ EXAMPLES: MECHANICS ON A LIE ALGEBRA, MECHANICAL SYSTEM WITH CONSTRAINTS

N - A MANIFOLD

IN $TN \oplus_N T^*N$ WE CONSIDER A SYMMETRIC PAIRING

$$(X_1 + \alpha_1, X_2 + \alpha_2) = \langle \alpha_1, X_2 \rangle + \langle \alpha_2, X_1 \rangle$$

ON SECTIONS OF $TN \oplus_N T^*N \longrightarrow N$ THERE IS THE DORFMAN BRACKET

$$[X_1 + \alpha_1, X_2 + \alpha_2] = [X_1, X_2] + \mathcal{L}_{X_1} \alpha_2 - i_{X_2} d\alpha_1$$

AN ALMOST DIRAC STRUCTURE IS A MAXIMALLY ISOTROPIC SUBBUNDLE OF $TN \oplus_N T^*N$. IF IT IS CLOSED W.R.T THE BRACKET IT IS CALLED A DIRAC STRUCTURE

• $\Lambda \in \text{Sec}(\wedge^2 TN)$, $\tilde{\Lambda} : T^*N \ni \alpha \mapsto \alpha \lrcorner \Lambda \in TN$, $\mathcal{D}_\Lambda = \text{graph}(\tilde{\Lambda})$ IS ALMOST DIRAC
AND DIRAC WHEN Λ IS POISSON

• $\omega \in \Omega^2(N)$, $\tilde{\omega} : TN \ni v \mapsto v \lrcorner \omega \in T^*N$, $\mathcal{D}_\omega = \text{graph}(\tilde{\omega})$ IS ALMOST DIRAC
AND DIRAC WHEN $d\omega = 0$

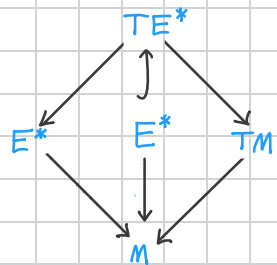
• $\Delta \subset TN$ A DISTRIBUTION $\Delta \oplus \Delta^\circ$ IS ALMOST DIRAC AND DIRAC WHEN Δ IS INTEGRABLE

IN GEOMETRIC MECHANICS WE CONSIDER **LINEAR** ALMOST DIRAC STRUCTURES

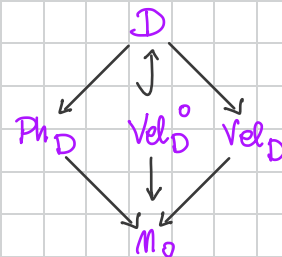
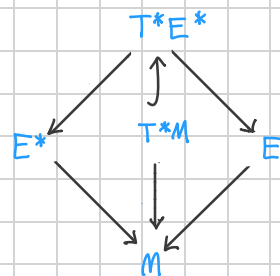
$$N = E^* \quad (N = T^*M)$$

DIRAC ALGEBROID IS A **LINEAR** ALMOST DIRAC STRUCTURE ON E^* . IT IS CALLED A **DIRAC-LIE ALGEBROID** WHEN IT IS A DIRAC STRUCTURE

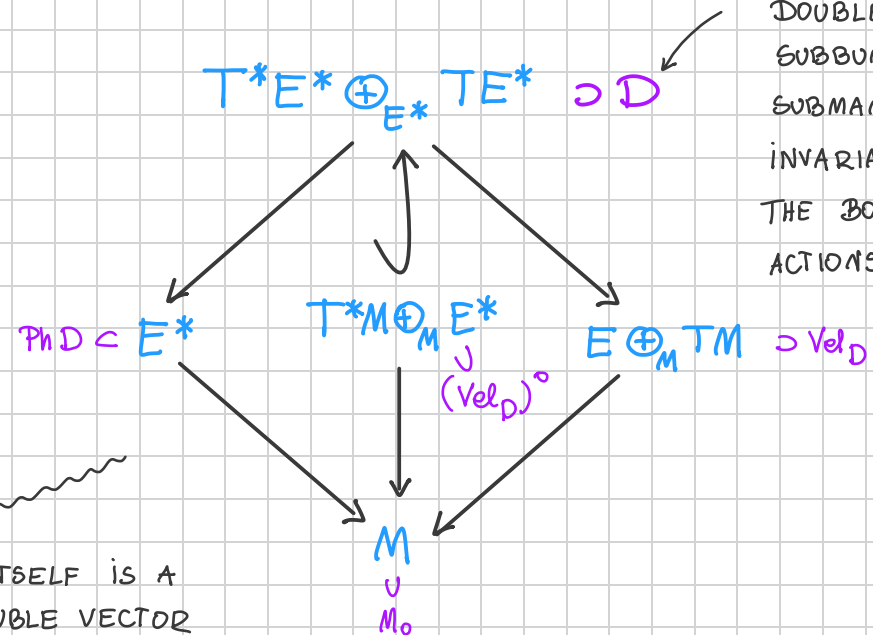
IT IS A DOUBLE VECTOR SUBBUNDLE
OF A DOUBLE VECTOR BUNDLE $T^*E^* \oplus_{E^*} TE^*$



$\oplus$

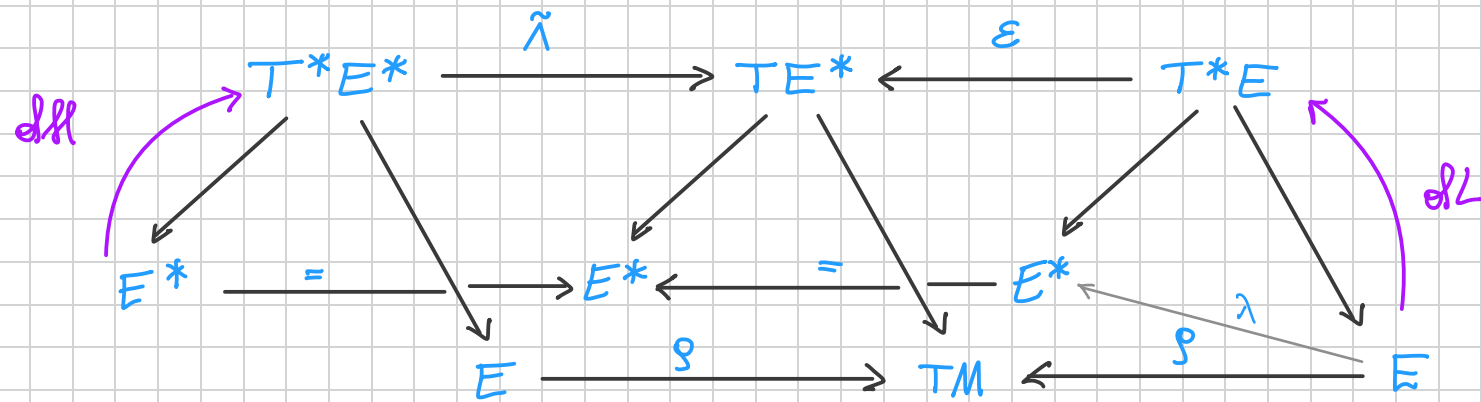


D ITSELF IS A
DOUBLE VECTOR
BUNDLE



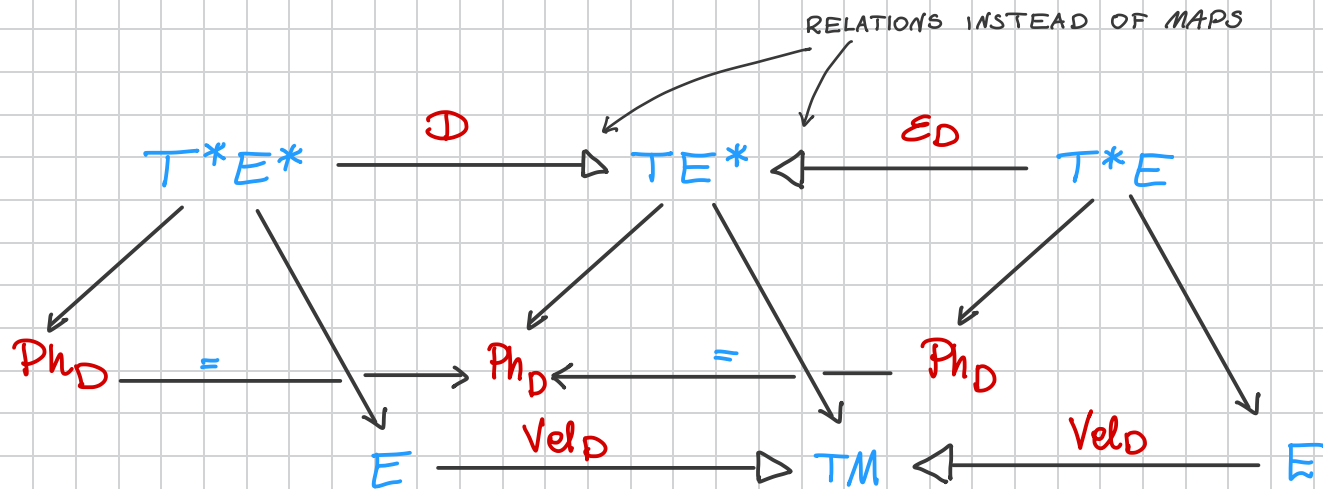
DOUBLE VECTOR
SUBBUNDLE IS A
SUBMANIFOLD
INVARIANT WRT
THE BOTH $(\mathbb{R}_s)$
ACTIONS

HOW DO WE USE DIRAC ALGEBROIDS IN MECHANICS?



$$\mathcal{Q} = \tilde{\lambda}^*(dH(E))$$

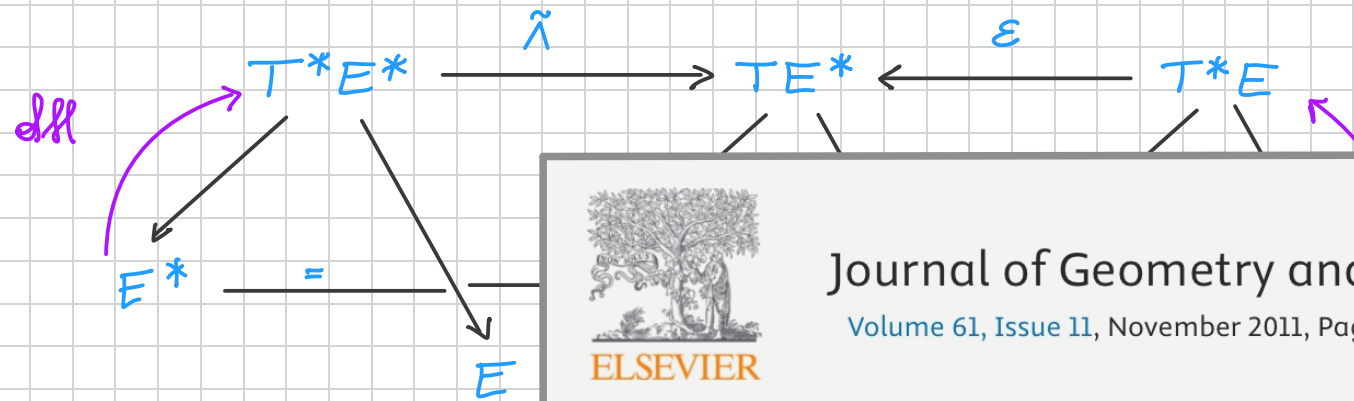
$$\mathcal{Q} = \varepsilon^*(dL(E))$$



$$\mathcal{Q} = \{ v \in TE^* : \exists p \in Ph_D : v + dH(p) \in \mathcal{D} \}$$

$$\mathcal{Q} = \{ v \in T^*E : \exists u \in E : v + dL(u) \in \mathcal{E}_D \}$$

HOW DO WE USE DIRAC ALGEBROIDS IN MECHANICS?

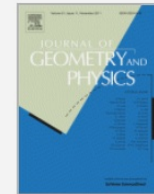


$$\mathcal{D} = \tilde{\Lambda}(dH(E),$$



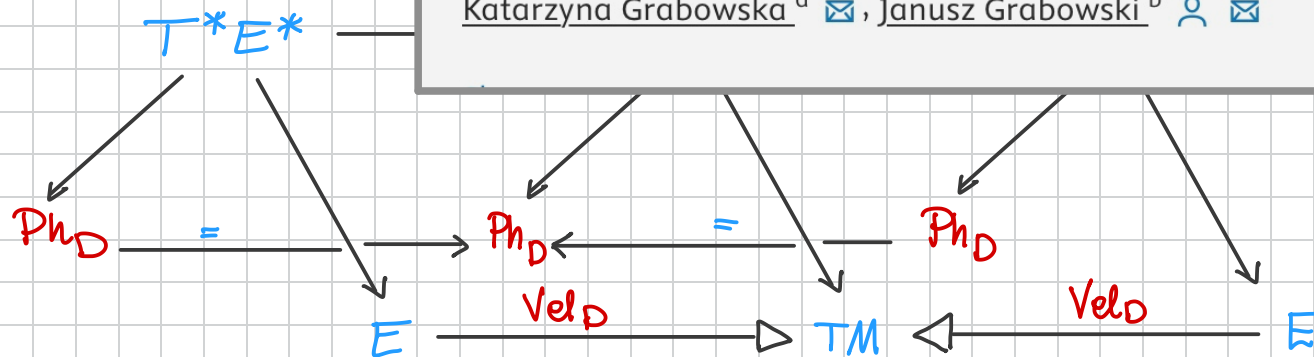
Journal of Geometry and Physics

Volume 61, Issue 11, November 2011, Pages 2233-2253



Dirac algebroids in Lagrangian and Hamiltonian mechanics ☆

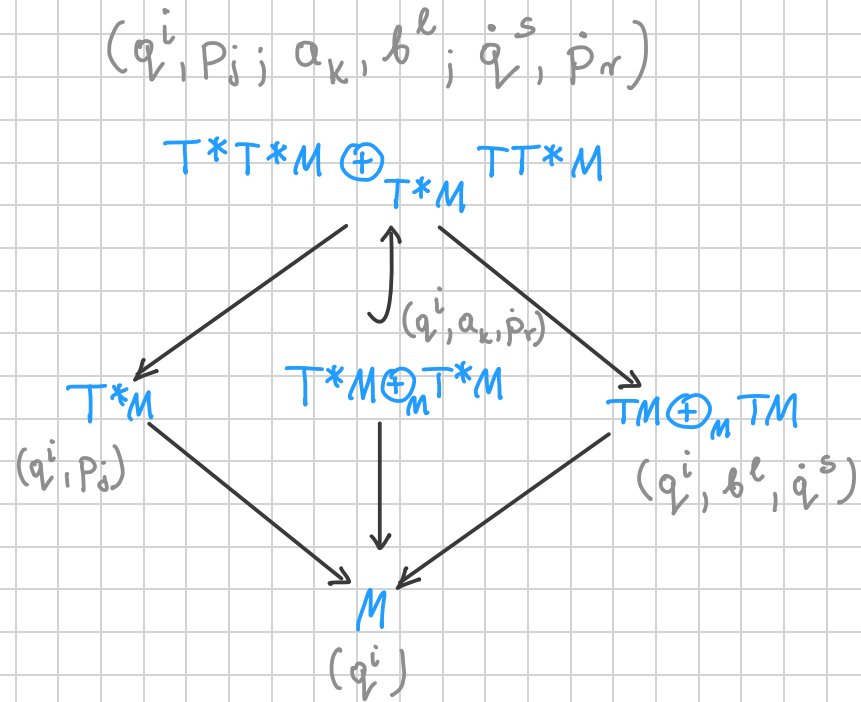
Katarzyna Grabowska^a ✉, Janusz Grabowski^b 👤 ✉



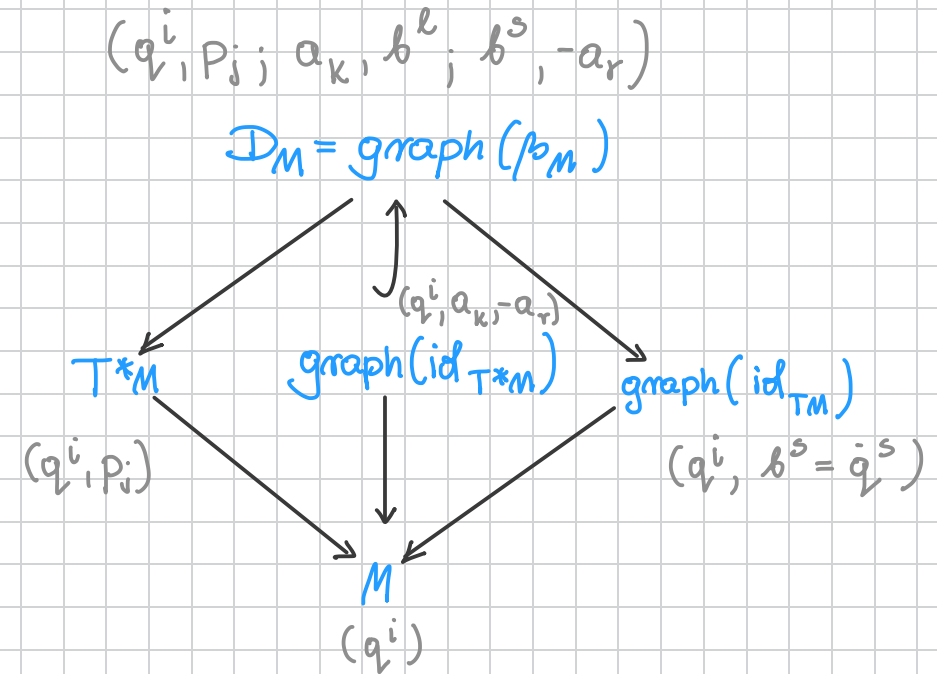
$$\mathcal{D} = \{ v \in TE^* : \exists p \in Ph_D : v + dH(p) \in \mathcal{D} \}$$

$$\mathcal{D} = \{ v \in T^*E : \exists u \in E : v + dL(u) \in \mathcal{E}_D \}$$

EXAMPLE RELATED TO NONHOLONOMIC CONSTRAINTS : CANONICAL SYMPLECTIC STRUCTURE OF T^*M TOGETHER WITH LINEAR CONSTRAINTS $C \subset TM$ INDUCE A DIRAC ALGEBROID STRUCTURE $\mathcal{D}_C$ ON T^*M WHICH IS DIRAC-LIE IF C IS INTEGRABLE.



$$\mathcal{D}_C = \mathcal{D}_M|_{\Delta} + \Delta^0$$



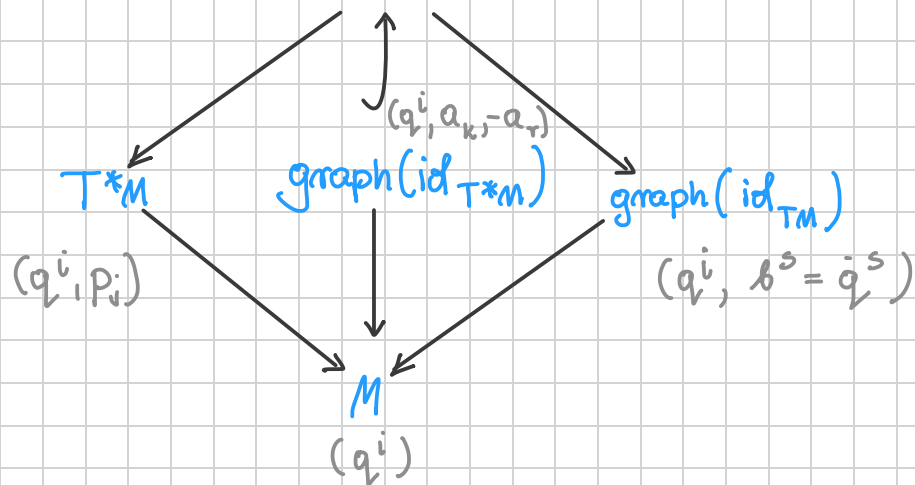
$$C \subset TM$$

$$\Delta = \text{Vel}_{\mathcal{D}_C} = \text{graph}(\text{id}_C) = \{v \oplus v : v \in C\}$$

$$T^*M \oplus_m T^*M \supset \Delta^0 = \{\psi \oplus \psi : \psi + \psi \in C^0\}$$

$$(q_v^i, p_j; a_k, b^l; b^s, -a_r)$$

$$\mathcal{D}_M = \text{graph}(p_M)$$



$$C \subset TM$$

$$\Delta = \text{Vel } \mathcal{D}_C = \text{graph}(\text{id}_C) = \{v \oplus v : v \in C\}$$

$$T^*M \oplus_M T^*M \supset \Delta^0 = \{\varphi \oplus \psi : \varphi + \psi \in C^0\}$$

$$\parallel \mathcal{D}_C = \mathcal{D}_M|_{\Delta} + \Delta^0 \parallel$$

$$\mathcal{D}_M (q_v^i, p_j; a_k, b^l; b^s, -a_r)$$

$$TM \supset C = \langle e_1, \dots, e_k \rangle (e_{k+1}, \dots, e_n)$$

$$(q_v^i, \dot{q}^j) \rightsquigarrow (q_v^i, y^{\alpha}, y^{\alpha})$$

$$C = \{y^{\alpha} = 0\}$$

$$c_{jk}^i = [e_j, e_k]^i = [\partial_m(q^i) \xi_j^m - \partial_k(q^i) \xi_j^m] \xi_j^k \xi_k^m$$

MATRIX OF COORDINATE CHANGE

$$\mathcal{D}_M: (q_v^i, \xi_A, \xi_{\alpha}; a_j, y^{\alpha}, y^{\alpha}; \xi_{\ell}^k y^{\ell}, c_{jk}^i \xi_i y^k - \xi_j^{\ell} a_{\ell})$$

$$\mathcal{D}_C: (q_v^i, \xi_A, \xi_{\alpha}; a_j, y^{\alpha}, 0; \xi_A^k y^{\alpha}, c_{\beta\gamma}^i \xi_i y^{\beta} - \xi_{\beta}^j a_j, \dot{\xi}_{\alpha})$$

↑ ARBITRARY

$$D_C: \left(\underbrace{q^i, \xi_A, \xi_\alpha}_{\text{MOMENTA}}; a_j, \dot{y}^A, 0; g_A^k \dot{y}^k, c_{BD}^i \xi_i \dot{y}^D - g_B^j a_j, \dot{\xi}_\alpha \right)$$

$\frac{\partial H}{\partial q^i}$ $\frac{\partial H}{\partial \xi_A}$ $\frac{\partial H}{\partial \xi_\alpha}$

ARBITRARY

LET US CONSIDER TWO TYPES OF HAMILTONIANS

$$(M, g) \quad TM = C \oplus C^\perp$$

MECHANICAL $H_0(q^i, \xi_A, \xi_\alpha) = \frac{1}{2m} (g^{AB} \xi_A \xi_B + g^{\alpha\beta} \xi_\alpha \xi_\beta) + V(q^i)$

MAGNETIC $H_1(q^i, \xi_A, \xi_\alpha) = \frac{1}{2m} (g^{AB} (\xi_A - A_A) (\xi_B - A_B) + g^{\alpha\beta} (\xi_\alpha - A_\alpha) (\xi_\beta - A_\beta)) + V(q^i)$

EQUATIONS DERIVED FROM D_C

$$\frac{\partial H}{\partial \xi_\alpha} = 0$$

$$\dot{q}^i = g_A^i \frac{\partial H}{\partial \xi_A}$$

$$\dot{\xi}_B = c_{BD}^A \xi_A \frac{\partial H}{\partial \xi_D} + c_{BD}^\alpha \xi_\alpha \frac{\partial H}{\partial \xi_D} - g_D^e \frac{\partial H}{\partial q^e}$$

$$\dot{\xi}_\beta \text{ ARBITRARY}$$

MECHANICAL

$$\frac{\partial H_0}{\partial \xi_\alpha} = \frac{1}{2m} g^{\alpha\beta} \xi_\beta = 0 \Rightarrow \xi_\beta = 0$$

$$\dot{q}^i = g_A^i \frac{\partial H_0}{\partial \xi_A} = \frac{1}{2m} g_A^i g^{AB} \xi_B \rightsquigarrow \text{VELOCITY BELONGS TO } C$$

$$\dot{\xi}_B = c_{BD}^A \xi_A \frac{\partial H_0}{\partial \xi_D} + \cancel{c_{BD}^\alpha \xi_\alpha \frac{\partial H_0}{\partial \xi_D}} - g_D^e \frac{\partial H_0}{\partial q^e}$$

$\xi_\alpha = 0$

+ INTEGRABILITY CONDITION $\dot{\xi}_\beta = 0$

SINCE $\xi_\alpha = 0, \dot{\xi}_\alpha = 0$ EFFECTIVELY THE PHASE EQUATIONS "LIVE ON" C^* . WE HAVE RECOVERED THE PREVIOUS CASE

MAGNETIC

$$\mathcal{H}_1(q^i, \xi_A, \xi_\alpha) = \frac{1}{2m} \left(g^{AB} (\xi_A - A_A) (\xi_B - A_B) \right) + g^{\alpha\beta} (\xi_\alpha - A_\alpha) (\xi_\beta - A_\beta) + V(q^i)$$

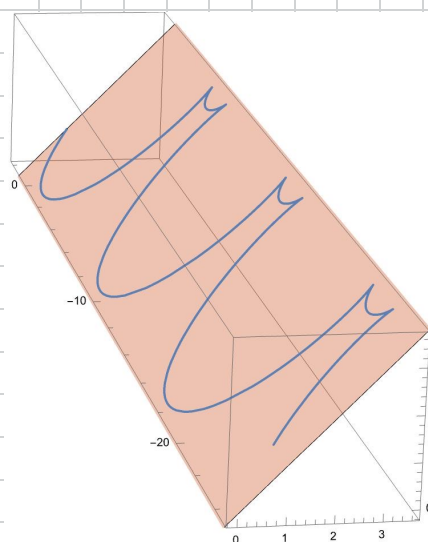
EQUATIONS DERIVED FROM $\mathcal{D}_C$

$$\frac{\partial \mathcal{H}}{\partial \xi_\alpha} = 0$$

$$\dot{q}^i = g_A^i \frac{\partial \mathcal{H}}{\partial \xi_A}$$

$$\xi_B = C_{BD}^A \xi_A \frac{\partial \mathcal{H}}{\partial \xi_D} + C_{BD}^\alpha \xi_\alpha \frac{\partial \mathcal{H}}{\partial \xi_D} - g_{BD}^l \frac{\partial \mathcal{H}}{\partial q^l}$$

ξ_B ARBITRARY



MAGNETIC

$$\frac{\partial \mathcal{H}_1}{\partial \xi_\alpha} = \frac{1}{2m} g^{\alpha\beta} (\xi_\beta - A_\beta) \Rightarrow \xi_\beta = A_\beta$$

INTEGRABILITY CONDITION $\dot{\xi}_\beta = \frac{\partial A_\beta}{\partial q^i} \dot{q}^i$

$$\dot{q}^i = g_A^i \frac{\partial \mathcal{H}_1}{\partial \xi_A} = \frac{1}{2m} g_A^i g^{AB} (\xi_B - A_B) \leftarrow \text{THE VELOCITY BELONGS TO } C$$

$$\xi_B = C_{BD}^A \xi_A \frac{\partial \mathcal{H}_1}{\partial \xi_D} + C_{BD}^\alpha \xi_\alpha \frac{\partial \mathcal{H}_1}{\partial \xi_D} - g_{BD}^l \frac{\partial \mathcal{H}_1}{\partial q^l}$$

$$C_{BD}^\alpha A_\alpha \frac{\partial \mathcal{H}_1}{\partial \xi_D}$$

SINCE WE HAVE THE CONDITION $\xi_\alpha = A_\alpha$ OUR EFFECTIVE PHASE SPACE IS NOT C^* BUT AN AFFINE SUBBUNDLE OF T^*M BEING $C^* + A^\perp$

AN ALTERNATIVE APPROACH VIA ALMOST POISSON BRACKET HAS BEEN PROPOSED THE BRACKET IS DEFINED ON C^* AND (PART OF) THE MAGNETIC POTENTIAL ENTERS THE BRACKET

MAGNETIC

$$\mathcal{H}_1(q^i, \xi_A, \xi_\alpha) = \frac{1}{2m} \left(g^{AB} (\xi_A - A_A) (\xi_B - A_B) + g^{\alpha\beta} (\xi_\alpha - A_\alpha) (\xi_\beta - A_\beta) \right) + V(q^i)$$

EQUATIONS DERIVED FROM $\mathcal{D}_C$

$$\frac{\partial \mathcal{H}}{\partial \xi_\alpha} = 0$$

$$\dot{q}^i = g_A^i \frac{\partial \mathcal{H}}{\partial \xi_A}$$

$$\dot{\xi}_B = C_{BD}^A \xi_A \frac{\partial \mathcal{H}}{\partial \xi_D} + C_{BD}^\alpha \xi_\alpha \frac{\partial \mathcal{H}}{\partial \xi_D} - g_{BD}^l \frac{\partial \mathcal{H}}{\partial q^l}$$

 $\dot{\xi}_B$ ARBITRARY

MAGNETIC

$$\frac{\partial \mathcal{H}_1}{\partial \xi_\alpha} = \frac{1}{2m} g^{\alpha\beta} (\xi_\beta - A_\beta) \Rightarrow \xi_\beta = A_\beta$$

INTEGRABILITY CONDITION $\dot{\xi}_\beta = \frac{\partial A_\beta}{\partial q^i} \dot{q}^i$

$$\dot{q}^i = g_A^i \frac{\partial \mathcal{H}_1}{\partial \xi_A} = \frac{1}{2m} g_A^i g^{AB} (\xi_B - A_B) \leftarrow \text{THE VELOCITY BELONGS TO } C$$

$$\dot{\xi}_B = C_{BD}^A \xi_A \frac{\partial \mathcal{H}_1}{\partial \xi_D} + C_{BD}^\alpha \xi_\alpha \frac{\partial \mathcal{H}_1}{\partial \xi_D} - g_{BD}^l \frac{\partial \mathcal{H}_1}{\partial q^l}$$

$$C_{BD}^\alpha A_\alpha \frac{\partial \mathcal{H}_1}{\partial \xi_D}$$

SINCE WE HAVE THE CONDITION $\xi_\alpha = A_\alpha$ OUR
EFFECTIVE PHASE SPACE IS NOT C^* BUT
AN AFFINE SUBBUNDLE OF T^*M BEING $C^* + A^\perp$

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AN ALTERNATIVE APPROACH VIA ALMOST POISSON BRACKET HAS BEEN PROPOSED
THE BRACKET IS DEFINED ON C^* AND (PART OF) THE MAGNETIC POTENTIAL
ENTERS THE BRACKET



THANK YOU!