IMPERIAL

Machine Construction of G2 Holonomy Manifolds

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The Unreasonable Effectiveness of Toric Geometry

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ESI

Motivation
G2 holonomy manifolds

Berger's Theorem (1955): Suppose g is an irreducible, Riemannian metric on a simply-connected non locally symmetric n-dimensional manifold M. Then exactly one of the following is true:

i) $\mathrm{Hol}(g) = SO(n)$

ii) $n = 2$ with $m \geq 2$, and $\mathrm{Hol}(g) = U(m)$ in $SO(2m)$,

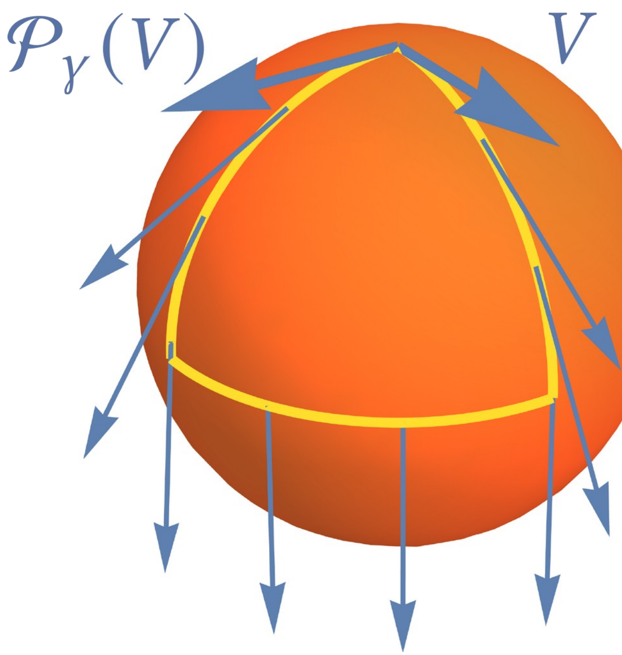
iii) $n = 2m$ with $m \geq 2$, and $\mathrm{Hol}(g) = SU(m)$ in $SO(2m)$,

iv) $n = 4m$ with $m \geq 2$, and $\mathrm{Hol}(g) = Sp(m)$ in $SO(4m)$,

v) $n = 4m$ with $m \geq 2$, and $\mathrm{Hol}(g) = Sp(m)Sp(1)$ in $SO(4m)$,

vi) $n = 7$ with and $Hol(g) = G_2$ in $SO(7)$, or

vii) $n = 8$ with and $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ in $SO(8)$.



Motivation
M-Theory

M-theory compactifications:

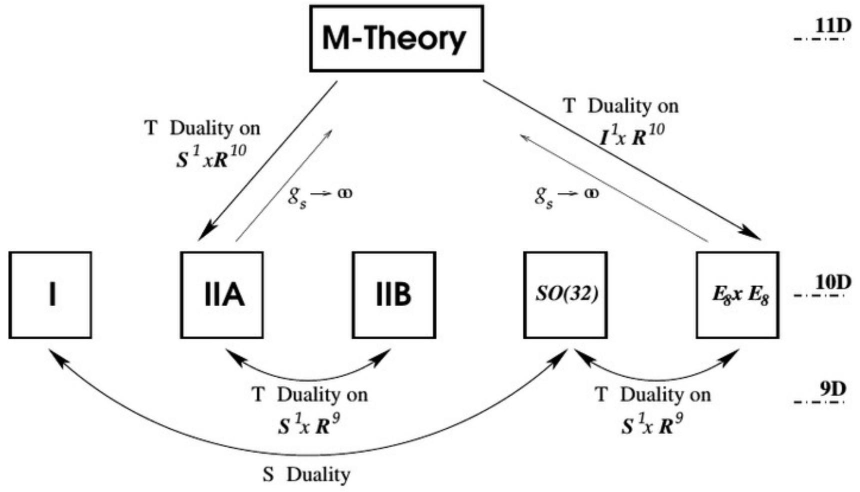
$$M_{11} = \mathbb{R}^{1,3} \times X_7$$
$$\mathrm{Hol}(X_7) = G_2$$

correspond to 4d theories with $\mathcal{N} = 1$ supersymmetry.

Dualities relate M-theory compactifications to string theory and F-theory compactifications on Calabi-Yau manifolds.

ADE-type codimension-four singularities produce non-abelian gauge fields.

Codimension-seven singularities produce chiral matter fields.



Agenda

G2 Geometry
G2-structure, metric & torsion

Let $(x_1, \ldots, x_7)$ be coordinates on $\mathbb{R}^7$, and define the 3-form φ_0 to be

$$\varphi_0 = dx_{123} + dx_{145} - dx_{167} + dx_{246} - dx_{257} + dx_{347} - dx_{356}$$

where $dx_{ijk} = dx_i \wedge dx_j \wedge dx_k$. The subgroup of $GL(7, \mathbb{R})$ that preserves φ_0 is the exceptional **Lie group $\boldsymbol{G_2}$**.

A **$\boldsymbol{G_2}$-structure** on a 7-manifold M is a 3-form $\varphi \in \Omega^3(M)$ such that, at every point $p \in M$, $\varphi_p = f_p^*(\varphi_o)$ for some frame $f_p: T_pM \to \mathbb{R}^7$.

The G_2-structure φ induces a **Riemannian metric $\boldsymbol{g_\varphi}$** and a volume form vol_φ via the following identity

$$\iota_X \varphi \wedge \iota_Y \varphi \wedge \varphi = -6 g_\varphi(X, Y)\mathrm{vol}_\varphi$$

for any vector fields X and Y on M. The **torsion** of φ is defined as $\nabla\varphi$, where ∇ is the Levi-Civita connection coming from the induced metric g_φ.

A **$\boldsymbol{G_2}$ manifold** is a pair (M, φ), where M is a 7-manifold and φ is a torsion-free G_2-structure on M. In this case $Hol_g(M) \subseteq G_2$.

$$Hol_g(M) = G_2 \Leftrightarrow b_1(M) = 0$$

Twisted Connected Sum Construction
ACyl CYs, S1 fibrations and gluing

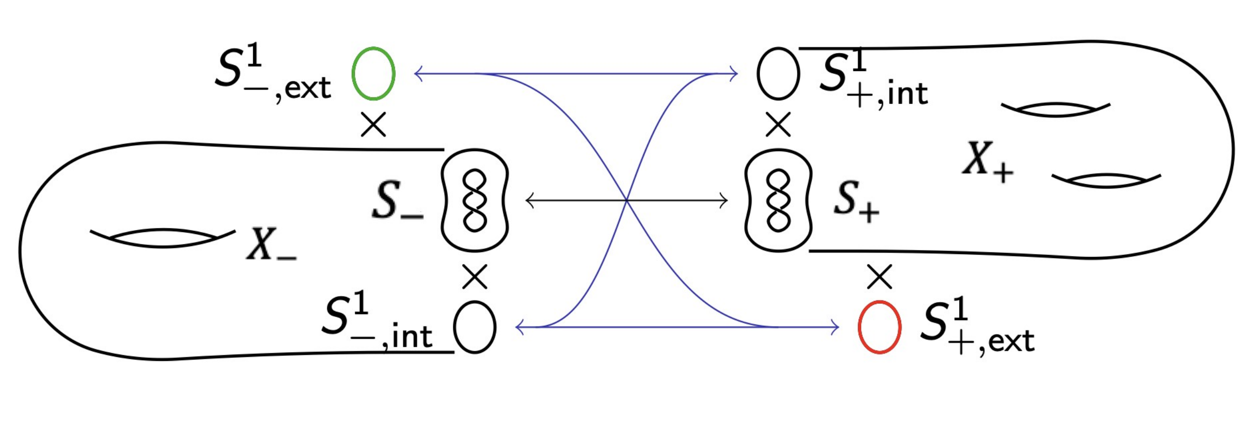
1) Take two **asymptotically cylindrical Calabi-Yau threefolds** X_+, X_-:

$$X_{+,\infty} = \mathbb{R}_{>0} \times S^1_{+,\mathrm{int}} \times S_+$$
$$X_{-,\infty} = \mathbb{R}_{>0} \times S^1_{-,\mathrm{int}} \times S_-$$

where S_+, S_- are K3 surfaces.

2) **Fiber $\boldsymbol{X_\pm}$ with an $\boldsymbol{S^1}$**:

$$Y_+ = S^1_{+,\mathrm{ext}} \times X_+$$
$$Y_- = S^1_{-,\mathrm{ext}} \times X_-$$



3) **Glue** Y_+ and Y_- along their cylindrical ends via a diffeomorphism.

Twisted Connected Sum Construction
Building blocks

Definition: A **building block** is a non-singular algebraic threefold Z together with a projective morphism $f: Z \to \mathbb{P}^1$ satisfying the following:

1) The anticanonical class of $-K_Z \in H^2(Z)$ **is primitive**.

2) $S = f^*(\infty)$ is a non-singular **K3 surface** and $S \sim -K_Z$.

3) We have the natural restriction map

$$\rho: H^2(Z, \mathbb{Z}) \to H^2(S, \mathbb{Z}) \cong L$$

where $L = (-E_8)^{\oplus 2} \oplus U^{\oplus 3}$ is the K3 lattice. Denoting the image of ρ by N, we demand that the lattice embedding $N \hookrightarrow L$ **is primitive**, i.e. L/N is torsion-free.

4) $H^3(Z, \mathbb{Z})$ **is torsion-free**.

Corti, Haskins, Nordström, Pacini 2013

Let (Z, S) be a building block as defined above, then $\mathbf{X} = Z \backslash S$ is an asymptotically cylindrical Calabi-Yau threefold.

Matching Criteria
Hyper-Kähler rotations

K3 surfaces have **hyper-Kähler structure** which is defined in terms of a Ricci-flat metric and a triple of complex structures (I, J, K) satisfying the quaternion identities $I^2 = J^2 = K^2 = IJK = -1$ with associated Kähler forms $(\omega_I, \omega_J, \omega_K)$.

In order to get a well-defined G_2 structure on M we must also identify the two K3 surfaces $S_{L/R}$ using a **hyper-Kähler rotation** $\mathfrak{r}: S_+ \to S_-$ that satisfies

$$\mathfrak{r}^*\omega_-^I = \omega_+^I$$
$$\mathfrak{r}^*\omega_-^J = \omega_+^J$$
$$\mathfrak{r}^*\omega_-^K = -\omega_+^K$$

We then define the **diffeomorphism** on the region $t \in (T, T+1)$ as

$$\begin{aligned} F: S^1_{+,\mathrm{ext}} \times \mathbb{R}_{>0} \times S^1_{+,\mathrm{int}} \times S_+ \quad &\to \quad S^1_{-,\mathrm{ext}} \times \mathbb{R}_{>0} \times S^1_{-,\mathrm{int}} \times S_- \\ (\theta', t, \theta, x) \quad &\mapsto \quad \big(\theta, T+1-t, \theta', \mathfrak{r}(x)\big) \end{aligned}$$

It follows that $F^*\varphi_- = \varphi_+$. Using F we can glue Y_+ and Y_- together to form a TCS $M_\mathfrak{r} = Y_+ \cup_F Y_-$.

Matching Criteria
Torelli theorem and matching data

Global Torelli theorem implies: Let $h: H^2(S_-, \mathbb{Z}) \to H^2(S_+, \mathbb{Z})$ be an isometry, extend it to $H^2(S_-;\mathbb{R}) \to H^2(S_+;\mathbb{R})$ and suppose that $h[\omega_-^I] = [\omega_+^J]$, $h[\omega_-^J] = [\omega_+^I]$ and $h[\omega_-^K] = -[\omega_+^K]$, then there is a hyper-Kähler rotation $\mathfrak{r}^* = h$.

Let S be a K3 surface and $L = E_8(-1)^2 \oplus U^3$ be the standard 22-dimensional K3 lattice. A **marking** is a lattice isometry $h: H^2(S, \mathbb{Z}) \to L$.

Given a marked K3 surface (S, h) the **period point** is defined as the complex line spanned by the holomorphic 2-form σ under the marking $h(\sigma) \in L \otimes \mathbb{C}$.

Proposition: Let (k_0, k_+, k_-) be an orthonormal triple of positive classes in $L_{\mathbb{R}}$. Let $(S_\pm, I_\pm)$ be complex K3 surfaces with markings $h_\pm: H^2(S_\pm; \mathbb{Z}) \to$ L such that $\langle k_\mp, \pm, k_0 \rangle$ is the period point, and $h_\pm(k_\pm)$ is a Kähler class on $S_\pm$. Then there exists a hyper-Kähler rotation $\mathfrak{r}: S_+ \to S_-$ with

$$\boldsymbol{\mathfrak{r}^* = h_+^{-1} \circ h_-}: H^2(S_-;\mathbb{Z}) \to H^2(S_+;\mathbb{Z})$$

Corti, Haskins, Nordström, Pacini 2013

Matching Criteria

Primitive embeddings and lattice matching conditions

Consider the asymptotically cylindrical Calabi-Yau threefold building blocks $X_\pm = Z_\pm \backslash S_\pm$ where $Z_\pm$ are K3 fibered threefolds and $S_\pm$ are K3 surfaces, and let $N_\pm$ denote the image of the restriction map

$$\rho_\pm : H^2(Z_\pm; \mathbb{Z}) \to H^2(S_\pm; \mathbb{Z}) \cong L$$

The choice of markings $h_\pm : H^2(S_\pm, \mathbb{Z}) \to L$ determines primitive embeddings $N_\pm \hookrightarrow L$ up to the action of $O(L)$. We define the transcendental lattices $\boldsymbol{T_\pm = N_\pm^\perp \subset L}$.

Proposition: Let a $Z_\pm$ be a pair of building blocks with primitive embeddings $N_\pm \hookrightarrow L$ and orthogonal complements $T_\pm = N_\pm^\perp \subset L$. If the following conditions are satisfied, then there exists a hyper-Kahler rotation between S_+ and S_- and the pair can be glued:

1) $N_+ \cap N_-$ is negative definite
2) $T_+ \cap T_-$ has one positive direction
3) $N_\pm \cap T_\mp$ has one positive direction
4) $\mathrm{Int}\big(\mathcal{K}(Z_\pm)|_{S_\pm} \cap N_\pm \cap T_\mp\big) \neq \emptyset$ where $\mathcal{K}(Z_\pm)|_{S_\pm}$ are the restrictions of the Kähler classes of the threefolds $Z_\pm$ to the K3 surfaces $S_\pm$.

Braun, Heyes, Larfors 2026

Matching Criteria
Semi-Fano building blocks

In Kovalev's original he considered asymptotically cylindrical Calabi-Yau threefolds coming from Fano threefolds of which there are only 105 deformation families.

Corti-Haskins-Nordström-Pacini generalised to semi-Fano threefolds of which there are many more deformation families.

Definition: A **weak Fano** threefold is a non-singular projective complex threefold Y such that the $-K_Y$ is
- Nef: $(-K_Y) \cdot C \geq 0$ for every curve $C \subset Y$
- Big: $(-K_Y)^3 > 0$

Definition: Let Y be a weak Fano threefold and

$$\varphi: Y \to X \coloneqq \mathrm{Proj}\, R(Y, -K_Y)$$
$$R(Y, -K_Y) \coloneqq \bigoplus_{n \geq 0} H^0(Y, -nK_Y)$$

If φ is semi-small we call Y a **semi-Fano** threefold.

Fano threefolds $\subset$ **semi−Fano threefolds** $\subset$ weak Fano threefolds

Matching Criteria

Orthogonal gluing

Let the primitive embeddings $N_\pm \hookrightarrow L$ be such that $R = N_+ \cap N_-$ is negative definite and we have the primitive inclusions

$$\begin{tikzcd} & N_+ \arrow[dr] & \\ R \arrow[ur] \arrow[dr] & & W \\ & N_- \arrow[ur] & \end{tikzcd}$$

where $W = N_+ + N_-$ is the orthogonal pushout lattice and we can primitively embed W into L. Then it follows that the **matching criteria 1-3 are satisfied!**

Corti, Haskins, Nordström, Pacini 2013

A sufficient condition for the existence of a primitive embedding $W \hookrightarrow L$ is

$$\mathbf{rank}(N_+) + \mathbf{rank}(N_-) \leq \mathbf{11}$$

Nikulin 1979

Tops

Building blocks via projecting tops

Definition: A pair of 4d lattice polytopes $(\Diamond, \Diamond^\circ)$ with $\Diamond \subset M$ and $\Diamond^\circ \in N$ which satisfy

$$\langle \Diamond, \Diamond^\circ \rangle \geq -1$$
$$\langle \Diamond, \nu_e \rangle \geq 0$$
$$\langle m_e, \Diamond^\circ \rangle \geq 0$$

are called a **dual pair of projecting tops**. Here N and M are dual toric lattices and $\nu_e = (0,0,0,-1) \in N$ and $m_e = (0,0,0,1) \in M$.

Braun 2017

For a dual pair of projecting tops $(\Diamond, \Diamond^\circ)$ the fiber polytopes polytopes are:

$$\Delta_F := \{m \in \Diamond | \langle m, \nu_e \rangle = 0\}$$
$$\Delta_F^\circ := \{\nu \in \Diamond^\circ | \langle m_e, \nu \rangle = 0\}$$

Let $\Sigma_n(\Diamond)$ be the normal fan of $\Diamond$ and $\mathbb{P}_{\Sigma_n}$ the corresponding toric variety. Then we define the building block threefold $Z_{(\Diamond,\Diamond^\circ)}$ hypersurface inside $\mathbb{P}_{\Sigma_n}$ by

$$0 = \sum_{m \in \Diamond} c_m z_0^{\langle m, \nu_0 \rangle} \prod_{\nu_i \in \Diamond^\circ, \nu_i \neq \nu_0} z_i^{\langle m, \nu_i \rangle + 1}$$

$\mathbb{P}_{\Delta_F^\circ}$ is semi-Fano if and only if Δ_F° has no interior points to facets.

Tops

N lattices and Hodge numbers

Let $(\Diamond, \Diamond^\circ)$ be a dual pair of projecting tops describing the K3 fibered building block threefold Z and let N denote the image of the restriction map $\rho: H^2(Z,\mathbb{Z}) \to H^2(S,\mathbb{Z})$. We also define $K \equiv \mathrm{Ker}(\rho)/[S]$.

The intersections between divisors on N can be computed from counting lattice points on faces of $(\Delta_F, \Delta_F^\circ)$ and the rank is given by

$$\mathrm{rank}(N) = \ell(\Delta_F^\circ) - 3 + \sum_{vb\ \Theta_F^{\circ[1]}} \ell^*\left(\Theta_F^{\circ[1]}\right)\ell^*\left(\Theta_F^{[1]}\right)$$

Also we have combinatorial formulas for the Hodge numbers of Z:

$$h^{1,1}(Z) = -4 + \sum_{\Theta^{[3]}<\Diamond} 1 + \sum_{\Theta^{[2]}<\Diamond} \ell^*\left(\sigma_n(\Theta^{[2]})\right) + \sum_{\Theta^{[1]}<\Diamond} \left(\ell^*(\Theta^{[1]}) + 1\right)\ell^*\left(\sigma_n(\Theta^{[1]})\right)$$

$$h^{1,2}(Z) = \ell(\Diamond) - \ell(\Delta_F) + \sum_{\Theta^{[2]}<\Diamond} \ell^*(\Theta^{[2]}) \cdot \ell^*\left(\sigma_n(\Theta^{[2]})\right) - \sum_{\Theta^{[1]}<\Diamond} \ell^*(\Theta^{[3]})$$

Braun 2017

And the rank of the K lattice is given as

$$\mathrm{rank}(K) = h^{1,1}(Z) - \mathrm{rank}(N) - 1$$

Dataset

Classification algorithm

Theorem: The F-minimal projecting tops for a given fiber polytope Δ_F, are of the form

$$\Diamond_\nu = \{(\Delta_F, 0) \cup (\nu, -1)\}_{conv}$$

where a projecting top is called F-minimal if it contains no other projecting top as a subpolytope.

Require: A reflexive fibre polytope Δ_F°

Compute the dual fibre polytope Δ_F.

Determine all F-minimal projecting tops $\{\Diamond_a\}$ over Δ_F.

Compute the dual tops $\{\Diamond_a^\circ\}$, which are the F-maximal tops over Δ_F°.

for all F-maximal tops $\Diamond_a^\circ = \mathrm{Conv}\big\{(\nu_F, h_a(\nu_F)) \mid \nu_F \in \Delta_F^\circ \cap N_F\big\}$ **do**

  Determine the maximal integral height

$$h_a^{\max} = \max\{\ h_a(\nu_F)\ \}\ .$$

  for all $h(\nu_F) \leq h_a^{\max}$ **do**

    Construct

$$\Diamond_h^\circ = \mathrm{Conv}\big\{(\nu_F, h(\nu_F)) \mid \nu_F \in \Delta_F^\circ \cap N_F\big\}.$$

    Add $\Diamond_h^\circ$ to the list $\mathcal{T}(\Delta_F^\circ)$.

  end for

end for

Remove equivalent tops from $\mathcal{T}(\Delta_F^\circ)$.

return $\mathcal{T}(\Delta_F^\circ)$, the resulting set of inequivalent projecting tops.

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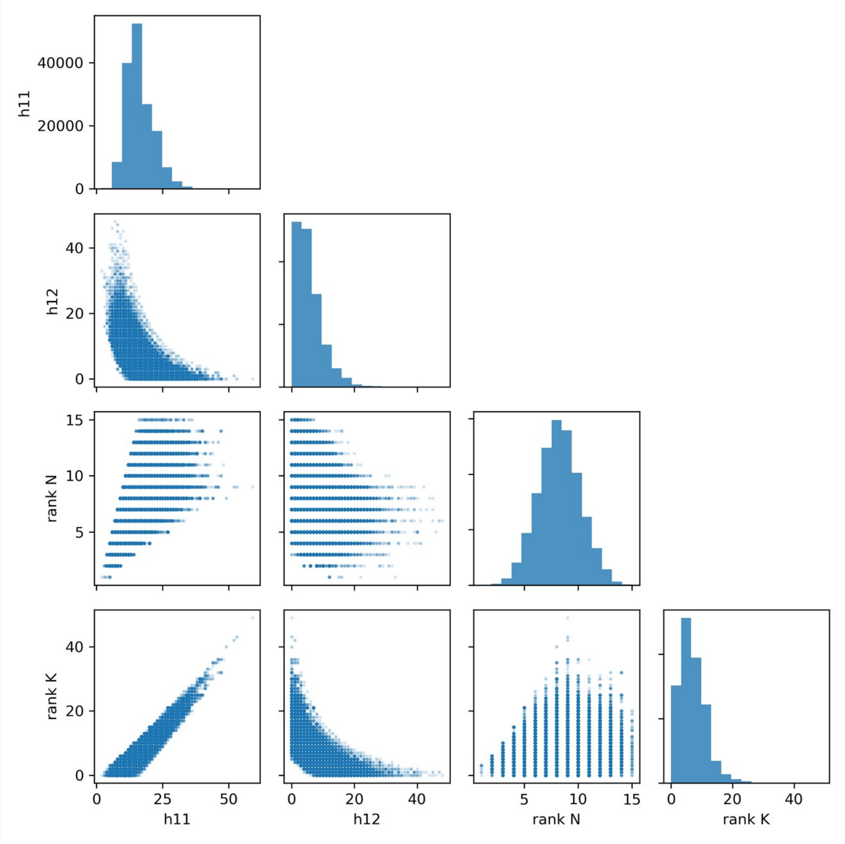
Dataset

Tops, fibers, N lattices and Hodge numbers

Dataset of **156,478 inequivalent projecting tops** corresponding to semi-Fano building blocks:

$$\mathcal{D} = \left\{\Diamond_i, \Diamond_i^\circ, \Delta_{F,i}, \Delta_{F,i}^\circ, I_{N_i}, \mathrm{rank}(K_i), h_i^{1,1}, h_i^{1,2}\right\}_{i=1,\ldots,156478}$$



Dataset

Primitive embeddings

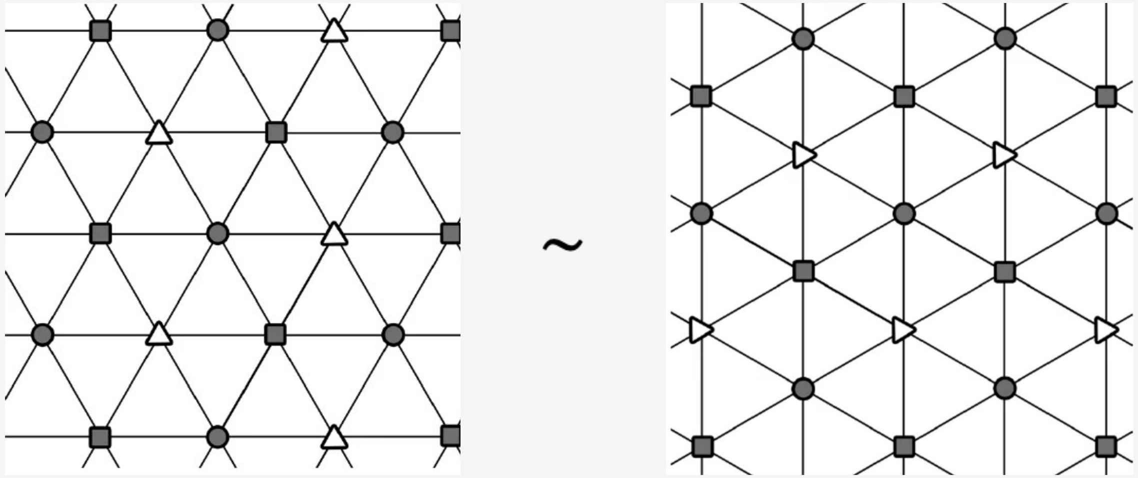
For each building block in our dataset

$$\{\lozenge, \lozenge^\circ, \Delta_F, \Delta_F^\circ, I_N, \mathrm{rank}(K), h^{1,1}, h^{1,2}\}$$

we compute the primitive embedding

$$N \hookrightarrow L = (-E_8)^{\oplus 2} \oplus U^{\oplus 3}$$

We use the primitive_embeddings Oscar function by Müller and Brandhorst (2025), based on results by Nikulin (1979). The output **embeddings are defined up to actions of $O(L)$.**



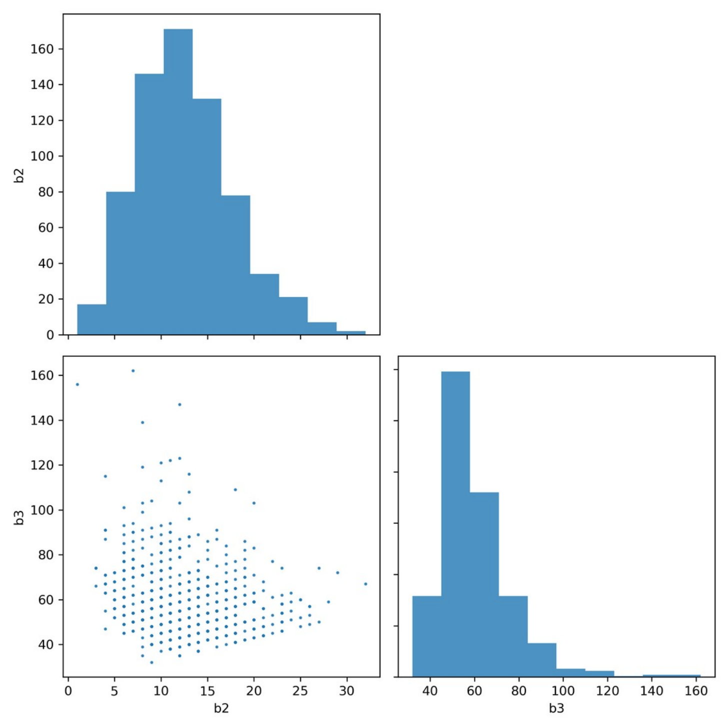
Dataset
Matching pairs

The total number of unordered pairs of building blocks is approximately

$$\frac{156478(156478-1)}{2} \approx 1.2\times 10^{10}$$

We sample 100,000 pairs of tops from our dataset and check matching.

	Matching	Non-matching	Total
Orthogonal	66	4342	4408
Non-orthogonal	622	94970	95592
Total	688	99312	100000



Betti numbers for the TCS G2 7-manifold:

$$b_2(M_7) = \mathrm{rank}(N_+ \cap N_-) + \mathrm{rank}(K_+) + \mathrm{rank}(K_-)$$
$$b_3(M_7) = 23 + \mathrm{rank}(N_+ \cap N_-) + \mathrm{rank}(N_- \cap T_+) + \mathrm{rank}(N_+ \cap T_-) - \mathrm{rank}(N_+) - \mathrm{rank}(N_-) + \mathrm{rank}(K_+) + \mathrm{rank}(K_-) + 2h^{1,2} + 2h^{1,2}$$

Experiments (in progress)
Problem set up

States: elements M of $O(L)$:

$$O(L) = \{M \in GL(22, \mathbb{Z}) | M^T G M = G\}$$

where G is the Gram matrix of L

Fitness:

$$f(M) = -w_1|\mathrm{pos}(N_+ \cap N'_-)| - w_2|\mathrm{pos}(T_+ \cap T'_-) - 1| - w_3|\mathrm{pos}(N_+ \cap T'_-) - 1| - w_4|\mathrm{pos}(N'_- \cap T_+) - 1| - w_5\mathcal{K}_\pm - w_6\mathcal{K}_\mp$$

where

- $N'_- = M^T N\_M$ and similarly for T'_-.
- $\mathrm{pos}(A)$ is the number of positive eigenvalues of the Gram matrix of A
- $\mathcal{K}_{\pm/\mp} = \begin{cases} 0 & \text{if the restricted Kahler cone of } L \text{ has non empty intersection in } N_\pm \cap T'_\mp \\ 1 & \text{otherwise} \end{cases}$

Goal: find M which maximises fitness $f(M)$

Conclusions

Done and to do

What we know:

- Compact G2 holonomy manifolds are interesting but hard to construct.
- We can create building blocks for TCS using tops.

What we have done:

- Built a complete dataset of projecting tops defining building blocks.
- Define lattice matching conditions between two building blocks given in terms of lattice conditions.
- Built a useable SageMath package for building and studying TCS manifolds from pairs of tops.

What we are working on:

- Design search algorithm to find automorphisms of K3 lattice that transform primitive embeddings to produce matching pairs of building blocks.
- Run large scale checks for matching pairs.

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Thank you

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