Higher Structures Emerging from Renormalisation

Schrödinger Institute, Vienna

Nov. 15th, 2021; discussion of Gerald Dunne's lecture

The concept of resurgence

• Jean Ecalle (Orsay), since the mid 70's:

Context: divergent series appearing in the formal solutions of complex analytic dynamical systems with continuous or discrete time, at irregular singularities.

• (important!) Precursor: Dingle

Context: asymptotics for linear ODEs, from Physics

• In Quantum mechanics, since the late 70's: A. Voros

• In Quantum Field Theory, early 80's: perturbation series which show features of resurgence:

V. Rivasseau (Φ^4_4, Φ^3_6)...

- M. Berry and followers (C. Howls, A. Olde Daalhuis,...)

Hyperasymptotics

- Explosion of resurgence in Physics (last 10 years): G. Dunne, M. Marino, R. Schiappa, M. Ünsal,...

- Resurgence and low–dim topology: J. Andersen, S. Gukov, M. Kontsevich...

(G. Dunne and coauthors)

$$C^{(0)}(x) = \sum_{n \geqslant 0} C_n \, x^n$$

$$C_n \sim e^{-1} \frac{2^{n+\frac{1}{2}} \Gamma\left(n+\frac{1}{2}\right)}{\sqrt{2\pi}} \left(1 - \frac{\frac{5}{2}}{2\left(n-\frac{1}{2}\right)} - \frac{\frac{43}{8}}{2^2\left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)} + \ldots\right)$$

$$C(x) = \sum_{k=0}^{\infty} \sigma^k \, C^{(k)}(x)$$

$$C^{(1)}(x) \sim \frac{e^{-1}}{\sqrt{2\pi}} \frac{e^{-1/(2x)}}{\sqrt{x}} \left(1 - \frac{5}{2}x - \frac{43}{8}x^2 + \ldots\right)$$

"Resurgent functions **display** at each of their singular points

a behaviour closely related to their behaviour at the origin.

Loosely speaking, these functions resurrect, or surge up - in

a slightly different guise, as it were - at their singularities"

J. Écalle

Resurgence (Ecalle), in a nutshell

• Most divergent series of natural origin, when expressed as formal series in z^{-1}, with z *a suitable variable close to* ∞, are Gevrey–1 (factorial growth), and thus have Borel transforms $\varphi(\zeta) \in \mathbb{C}\{\zeta\}$, where $\mathcal{B}\Big(\sum_{n\geqslant 0} h_n \frac{1}{z^{n+1}}\Big) := \sum_{n\geqslant 0} \frac{h_n}{n!} \zeta^n$.

• $\varphi(\zeta)$ can be analytically continued, with isolated singularities.

• These φ belong to algebras of functions of the ζ variable (wrt to a convolution product), to which there correspond subalgebras of $\mathbb{C}[[z^{-1}]]$, by $\mathcal{B}_z^{-1}$

• There are families of operators Δ_γ on these algebras which measure the nature of the singularity at any given singular point ω, where γ designates a path of analytic continuation that reaches the point ω.

Typically, if the singularity at ω is of logarithmic type (meaning that, for ζ close to ω,

$\varphi(\zeta-\omega) = \log(\zeta-\omega)\psi_\omega(\zeta-\omega), \text{with } \psi_\omega \text{ analytic}$), $\Delta_\gamma(\varphi)(\zeta) := \psi(\zeta)$.

• From the Δ_γ, we can construct families (Δ_ω) of *alien derivations*, namely *singularity–measuring operators which satisfy the Leibniz rule* on algebras $\mathcal{R}$ of resurgent functions; by pullback, we have alien derivations on $\mathcal{B}^{-1}(\mathcal{R})$, denoted by the same symbol.

The operators $\boldsymbol{\Delta}_\omega := e^{-\omega z}\Delta_\omega$ are derivations which commute with ∂_z

$$\partial_z\, y^{\mathrm{nor}} = y^{\mathrm{nor}}$$

$$y(z,\sigma) = \sigma\, e^z$$

$$\partial_z\, y = y + \sum_{m \geqslant -1} b_m(z)\, y^{m+1}$$

$$y(z,\sigma) = \sum_{m \geqslant 0} \sigma^m e^{m\, z}\, y_m(z)$$

- All the y_m are Gevrey–1; their Borel transforms have singularities over $\mathbb{Z}^*$
- We have the following **resurgence relations**:

$$\Delta_n\, y_m = (m-n) A_n\, y_{m-n} \qquad\qquad (-1 \leqslant n \leqslant m, n \neq 0; A_n \in \mathbb{C})$$

In compact form, for the transseries (an instance of the **bridge equation**):

$$\boldsymbol{\Delta}_n\, y(z,\sigma) = \mathbb{A}_n\, y(z,\sigma) = A_n\, \sigma^{n+1} \frac{\partial}{\partial \sigma} y(z,\sigma)$$

Resurgence *stricto sensu*

Classification of tangent to identity analytic diffeomorphisms of $(\mathbb{C},0)$, focussing on the diffeos:

$$F\colon x \longrightarrow x - x^2 + x^3 + h_4\, x^4 + h_5\, x^5 + \ldots$$

We have a formal normal form, expressed in a variable $z \sim \infty$:

$$f^{\mathrm{nor}}(z) = z + 2\pi i$$

$$f(z) = z + 2\pi i + \sum_{n \geqslant 2} c_n\, z^{-n}$$

$$\text{Normalizing transformation } {}^\star f\colon$$

$$f \circ {}^\star f = {}^\star f \circ f^{\mathrm{nor}}$$

- ${}^\star f$ is Gevrey–1; its Borel transform has singularities over $\mathbb{Z}^*$
- We have the resurgence relations ($\forall n \in \mathbb{Z}^*$):

$$\Delta_n {}^\star f(z) = \mathbb{A}_n {}^\star f(z) = A_n\, \partial_z\, {}^\star f(z)$$

Resurgence monomials

$$\left(\mathcal{G}^{(\omega_1,\dots,\omega_r)}\right)_{(\omega_1,\dots,\omega_r)\in\Omega^*}$$

Definition: families of resurgent functions, defined by very explicit recursive formulas, using Ecalle's combinatorics of moulds, which behave well wrt the ordinary derivation and the alien derivations.

Simplest example: *hyperlogarithms*

Question: underlying the nonperturbative structures with Hopf algebras in QFT...?

• Many recent papers by G. Dunne, M. Ünsal, O. Costin, M. Borinsky, M. Marino, R. Schiappa, D. Sauzin, M. Bellon...

• J. Ecalle **Singularités non abordables par la géométrie** (*Singularities which are intractable by geometry*). Ann. Inst. Fourier,Grenoble, 42, 1992, p 73-164.

• J. Ecalle **Twisted Resurgence Monomials and canonical-spherical synthesis of Local Objects.** Proc. of the June 2002 Edinburgh conference on Asymptotics and Analysable Functions, O. Costin ed., World Scient. Publ.