Cancellations in the Wave Trace

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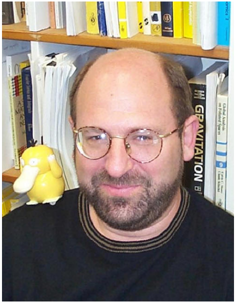
Thank you organizers!




STEVE ZELDITCH

Joint work with Illya Koval and Vadim Kaloshin from IST Austria.

- Laplace Spectrum: Solutions of

$$\begin{cases} -\Delta\psi_j = \lambda_j^2\psi_j \text{ in } \Omega, \\ B\psi_j = 0 \end{cases}$$

- Length Spectrum: Closure of lengths of closed (periodic) geodesics/billiard trajectories + zero.

- Poisson Relation: $\text{SingSupp} \cos t\sqrt{-\Delta} \subset \text{LSP}$.

- Inclusion or Equality? When is Poisson relation a strict inequality?

Motivation

- To what extent do the Laplace spectrum and length spectrum encode the same data?
- Can one translate inverse problems from one area to another? Ex. rigidity phenomena in dynamics / PDE
- Are there limitations to using the wave trace for the inverse spectral problem?
- Bouncing balls? Hyperbolic orbits? Nearly glancing + whispering gallery modes?
- This work is inspired by the work of Steve Zelditch, whose advice was very helpful in the beginning of this project.

Whispering gallary modes (St. Paul's Cathedral, London)

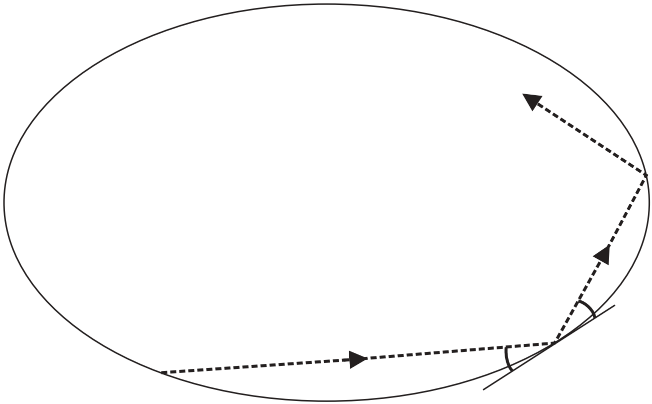
Bouncing ball orbits (NCSU, Raleigh)

Theorem

Let Ω_0 be an ellipse. Then, for a dense set of eccentricities $e \in (0, \infty)$ and for each $m \in \mathbb{N}$, there exist perturbations Ω_ϵ of Ω_0 which fix $2m$ hyperbolic orbits denoted γ_i and γ_j' with corresponding rational rotation numbers p/q, p'/q', $q = q' + 4 \mod 8$, and $\epsilon_n^m(e) \to 0$ as $n \to \infty$ such that that: for some length $L(\epsilon, e) \in LSP(\Omega_\epsilon)$, $w(t) \in C^{m,\alpha}(L - \delta, L + \delta)$ for δ sufficiently small and any $\alpha \in (0, 1)$.

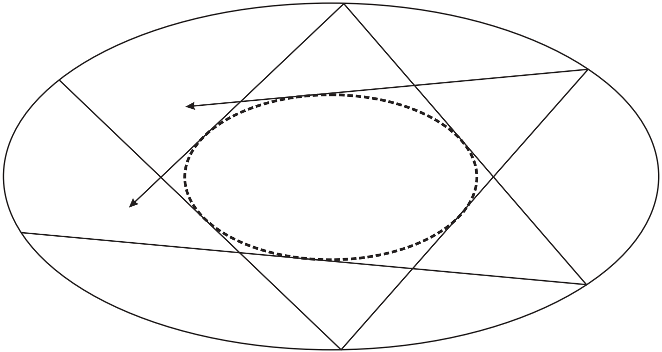
Billiards

- Let Ω_0 be a smooth, convex planar domain.
- $\beta : B^*\partial\Omega_0 \to B^*\partial\Omega_0$



Caustics

A smooth curve $\mathcal{C}$ is called a caustic if any tangent line drawn to $\mathcal{C}$ remains a tangent to $\mathcal{C}$ after reflection at the boundary.

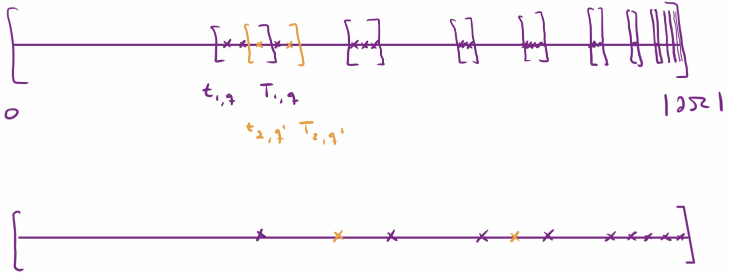


Phase space

- Map $\mathcal{C}$ onto the total phase space $B^*\partial\Omega \approxeq \mathbb{Z}/\ell\mathbb{Z} \times (0, \pi)$ to obtain a smooth closed curve, invariant under the billiard map β.
- Use $\omega(\mathcal{C})$ for the rotation number of invariant curve.
- Lazutkin[73]: In every neighborhood of the boundary there exists a family of convex caustics whose rotation numbers belong to a Cantor set of positive measure.

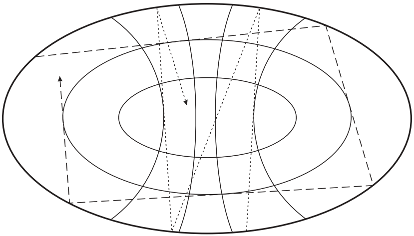
Lengths

- Loop function: $\ell_{p,q}(s) =$ length of rot. number p/q loop emanating from $x(s) \in \partial\Omega$, if it exists.
- Length functional: $\mathcal{L}_q(x_1, \cdots, x_q) = \sum_1^q |x_{i+1} - x_i|$.
- Periodic orbits arise as critical points of $\ell_{p,q}, \mathcal{L}_q$.
- Length spectrum is set of critical values of $\ell_{p,q}, \mathcal{L}_q$.

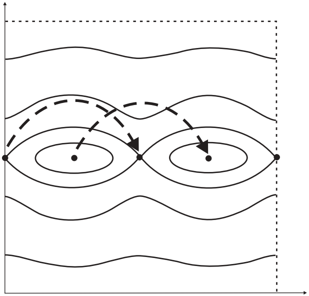


Ellipse : $t_{p,q} = T_{p,q}$

Billiards on the ellipse



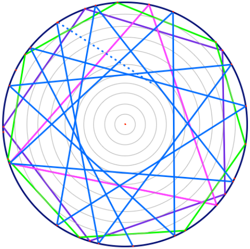
Confocal ellipses and hyperbolas

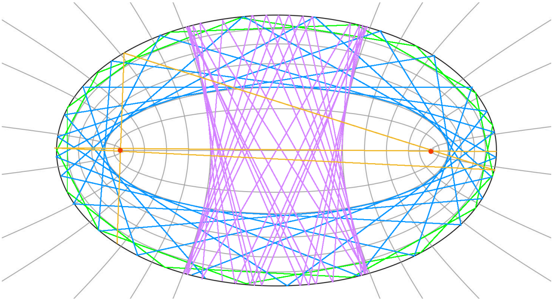


Phase space foliation

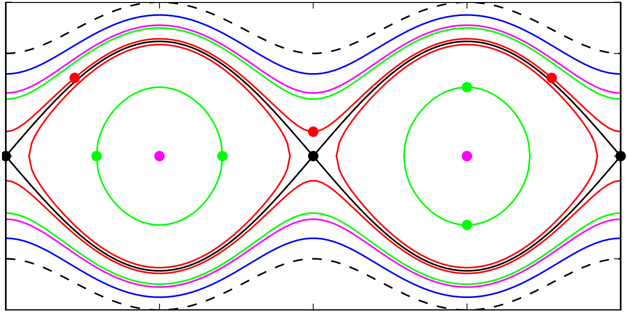
Birkhoff Conjecture: The only integrable strictly convex billiards are ellipses. Integrable means that the union of all convex caustics has a non-empty interior in $\mathbb{R}^2$.

Caustics of disks and ellipses





Phase space portrait



Purple dots: bouncing ball orbit on the minor axis.

Black dots: bouncing ball orbit on the major axis.

Green dots: 4-periodic orbit tangent to a confocal hyperbola.

Red dots: 3-periodic orbit tangent to a confocal ellipse.

Wave group

$$U(t) = \begin{pmatrix} \cos t\sqrt{-\Delta} & \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} \\ -\sqrt{-\Delta}\sin t\sqrt{-\Delta} & \cos t\sqrt{-\Delta}, \end{pmatrix} \tag{1}$$

Solves:

$$\begin{cases} (\partial_t^2 - \Delta)u = 0, \\ u\big|_{t=0} = f, \quad \partial_\nu u\big|_{t=0} = g. \end{cases} \tag{2}$$

For $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\langle \operatorname{tr} U(t), \varphi \rangle \equiv \operatorname{tr} \int \varphi(t) U(t)\, dt$$

The Poisson Relation

Anderson-Melrose[77]:

$$\mathrm{SingSupp}\left(\mathrm{Tr}\ \cos t\sqrt{-\Delta_\Omega^B}\right) \subset \{0\} \cup \pm\overline{\mathrm{LSP}(\Omega)},$$

Wave trace asymptotics of γ: $\mathrm{Tr}\cos t\sqrt{-\Delta} \sim$

$$\Re\left\{a_{\gamma,0}(t-L+i0)^{-1} + \sum_{k=0}^{\infty} a_{\gamma,k}(t-L+i0)^k \log(t-L+i0)\right\},$$

Sum over γ, $\mathrm{length}(\gamma) = L$.

Balian-Bloch

Let $\widehat{\rho}$ bump fxn supported near $L \in \mathrm{LSP}(\Omega)$.

$$\int_0^\infty e^{ikt}\widehat{\rho}(t)w(t)dt = \operatorname{tr}\rho * kR(k) \sim$$

$$\sum_{\mathrm{length}(\gamma)=L} \mathcal{D}_\gamma(k) \sum_{j=0}^\infty B_{\gamma,j}k^{-j},$$

where

$$\mathcal{D}_\gamma(k) = \frac{c_0 e^{ikL_\gamma} e^{i\pi \operatorname{sgn} \partial^2 \mathcal{L}/4}}{\sqrt{|\det \partial^2 \mathcal{L}|}}$$

is called the symplectic prefactor.

Potential Theory

Layer potentials:

$$\mathcal{S}\ell(\lambda)f(x) = \int_{\partial\Omega} G_0(\lambda, x, s)f(s)ds, \qquad x \in \mathbb{R}^2 \backslash \partial\Omega$$

$$\mathcal{D}\ell(\lambda)f(x) = \int_{\partial\Omega} \partial_{\nu_s} G_0(\lambda, x, s)f(s)ds, \qquad x \in \mathbb{R}^2 \backslash \partial\Omega$$

Boundary operator:

$$N(\lambda)f(s) = \int_{\partial\Omega} \partial_{\nu_{s'}} G_0(\lambda, s, s')f(s')ds', \qquad s \in \partial\Omega, \tag{3}$$

Dirichlet resolvent:

$$R_D^\Omega(\lambda) = R_0(\lambda) - 2\mathcal{D}\ell(\lambda)(I + N(\lambda))^{-1} r_{\partial\Omega}\mathcal{S}\ell^\dagger(\lambda) \tag{4}$$

Jump formula:

$$\mathcal{D}\ell(\lambda)f_\pm = \frac{1}{2}(\pm I + N(\lambda))f,$$

Free Green's fxn:

$$G_0(\lambda, z, z') = H_0^{(1)}(\lambda|z - z'|) \tag{5}$$

$H_\nu^{(1)}$ is the Hankel function of the first kind (of order ν)

$$N(\lambda|x(s) - x(s')|) \sim \left(\frac{\lambda \cos^2 \vartheta}{8\pi|x(s) - x(s')|}\right)^{1/2} e^{i\lambda|x(s)-x(s')|+3\pi i/4} \sum_{m=0}^{\infty} \frac{c_m i^m}{\lambda^m |x(s) - x(s')|^m}.$$

Interior-exterior duality

- Let

$$R_{\rho B}^{X} = \int_0^\infty \widehat{\rho}(t) w(t) dt$$

denote smoothed resolvent on domain X with boundary conditons $B = D$ or N.

- Duality:

$$\operatorname{tr}\left(R_{\rho D}^{\Omega}(k) \oplus R_{\rho N}^{\Omega^c}(k) - R_{\rho 0}^{\mathbb{R}^2}(k)\right)$$
$$= \int_{\mathbb{R}} \rho(k-\lambda) \frac{\partial}{\partial \lambda} \log\det\left(1 + N(\lambda)\right) d\lambda$$

$$\int_{\mathbb{R}} \rho(k-\lambda) \frac{\partial}{\partial \lambda} \log \det \left(1 + N(\lambda)\right) d\lambda$$

$$\sim \int_{\mathbb{R}} \rho(k-\lambda) \operatorname{tr} \left(1 + N(\lambda)\right)^{-1} N'(\lambda) d\lambda$$

$$\sim \sum_{M} \frac{(-1)^M}{M+1} \int_{\mathbb{R}} \rho'(k-\lambda) \operatorname{tr} N(\lambda)^{M+1} d\lambda$$

What is $N^{M+1}(\lambda)$? If $x : \partial\Omega \ni s \mapsto \mathbb{R}^2$ a parametrization,

$$N(\lambda) \sim e^{i\lambda|x(s)-x(s')|} a(\lambda|x(s) - x(s')|) \implies$$

$$N^{M+1}(\lambda) \sim \int N(s, s_1) N(s_1, s_2) N(s_1, s_2) \cdots N(s_M, s') ds_1 \cdots ds_M$$

$$\sim \int e^{i\lambda\mathcal{L}(S)} \widetilde{a}(\lambda, S, s, s') dS$$

Recall: $\mathcal{L}(S) = \sum |x_{i+1} - x_i| =$ length functional

- Problem: phase function is not smooth $|x - x'|$
- Regularize: $\chi(\lambda|x(s) - x(s')|)$ a cutoff
- $N = \chi N + (1-\chi)N = N_0 + N_1$: diagonal + off diagonal
- $\chi N \in \Psi^{-1}(\partial\Omega)$,
- $(1 - \chi N)$ a semiclassical $(\hbar = \lambda^{-1})$ FIO quantizing β.
- $N^M = \sum_{\sigma:\mathbb{Z}/M\mathbb{Z}\to\{0,1\}} N_\sigma$ where

  $N_\sigma = N_{\sigma(0)} N_{\sigma(1)} \cdots N_{\sigma(M-1)}$.
- Main term when all $\sigma = 1$.

Stationary phase

For an isolated critical point (periodic orbit),

$$\int_{\mathbb{R}^n} e^{ik\mathcal{L}(S)} a(S) dS \sim (2\pi/k)^{n/2} \frac{e^{ik\mathcal{L}(S_\gamma)} e^{i\pi \mathrm{sgn} \partial^2 \mathcal{L}(S_\gamma)/4}}{|\det \partial^2 \mathcal{L}(S_\gamma)|^{1/2}} \sum_{j=0}^{\infty} k^{-j} L_j a(S_\gamma),$$

where the L_j are differential operators of order $2j$:

$$L_j a(S_\gamma) = \sum_{\nu-\mu=j} \sum_{2\nu \geq 3\mu} i^{-j} 2^{-\nu} \langle \partial^2 \Phi(S_\gamma)^{-1} \partial, \partial \rangle^{\nu} (g^{\mu} a(S_\gamma))/\mu!\nu!.$$

Here, $g(S) = \mathcal{L}(S) - \mathcal{L}(S_\gamma) - \mathcal{L}'(S_\gamma)(S - S_\gamma) - \partial^2 \mathcal{L}(S_\gamma)(S - S_\gamma)^2$.

Perturbations

- Idea: perturb away from the ellipse, where $\partial^2 \mathcal{L}$ is degenerate and negative semidefinite. Keep track of maximal Hessians.
- $\mathrm{sgn}\mathcal{L} = q - 1$ and $\det \mathcal{L} = 0$.
- If orbit rotation numbers p/q and p'/q', with $q \equiv q' + 4 \mod 8$,

  $e^{i\pi(q-1)/4} =$ *minus* $e^{i\pi(q'-1)/4}$.
- Choose $p/q, p'/q'$ caustics with same length and a bunch of orbits to make cancellations

- If perturbation of size δ, $\partial^2\mathcal{L}^{-1} = O(\delta^{-1})$.
- Then $\partial^2\mathcal{L}^{-1}_{\epsilon,\delta}(S_\gamma) \sim V^{-1}(S_\gamma)\delta^{-1}M(S_\gamma)$, where $M \in C^\infty(\partial\Omega^q)$ consisting of minors of $\partial^2\mathcal{L}_0$ and V is the $(q-1)\times(q-1)$ determinant of $\partial^2\mathcal{L}$ upon quotienting out the degenerate direction.
- Furthermore, $M = O(\|\kappa_\Omega\|_{C^0})$ and is uniformly bounded in δ.
- Can show that maximal Hessian terms contribute

$$a_0(S_\gamma)(h^{lm})^{3j}(\partial^3_{ijk}\mathcal{L})^{2j}.$$

- Multiscale perturbation: fix δ and then introduce ε for each orbit.

- Choose $p/q, p'/q'$ caustics in the ellipse such that lengths are both L and $q \equiv q' + 4 \mod 8$.
- Choose m p/q orbits and m p'/q' orbits with disjoint vertices. U a small nbhd of vertices.
- There exists an arbitrarily small deformation μ_1 outside U such that for every smooth deformation μ_2 inside U, there are no other orbits of length L assuming deformation tangent only at reflection points

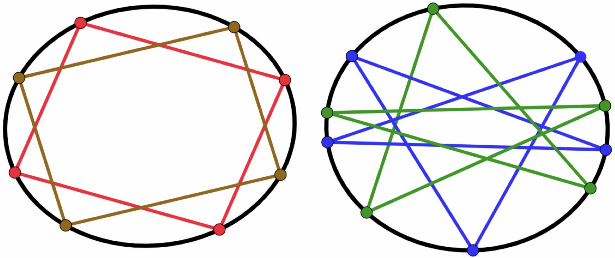
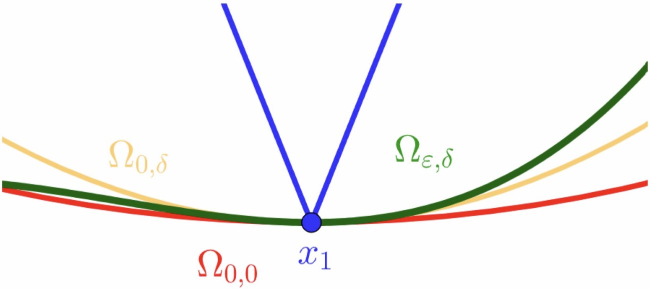


FIGURE 1. To cancel all the B_j for $j \leq m$, we need m orbits of both types.



- First match symplectic prefactors:

$$\mathcal{D}_\gamma(k) = \frac{c_0 e^{ikL_\gamma} e^{i\pi \operatorname{sgn} \partial^2 \mathcal{L}/4}}{\sqrt{|\det \partial^2 \mathcal{L}|}}$$

- Then match terms of form

$$\frac{C(j)\delta^{-3j}}{4^q q} \left(\prod_{i=1}^{q} \frac{\cos\theta_i}{|x_i(S) - x_{i+1}(S)|^{1/2}} \right) \times \left(\partial^3 \ell_{p,q}(s_i(\gamma))\right)^{2j} V^{-3j}(S) M^{3j}(S) + O\left(\frac{\delta^{-3j+1}}{k}\right).$$

- Use adapted action angle coordinates to switch from length fxnl to loop fxn
- Need to regularize integral, keeping track of dependence on curvature, to show $|\sigma| \geq 1$ terms don't contribute to maximal Hessians
- Each orbit has a vector $u_i = (B_{1,\gamma_i}), \cdots, B_{m,\gamma_i}$ for $1 \leq 1 \leq 2m$.
- $u_= v_i + w_i$, v_i highest order terms, s_i remainders.

- Equations are homogeneous: renormalise so that

$$v_i = \left(\varepsilon_i^2, \varepsilon_i^4, \cdots, \varepsilon_i^{2m}\right)$$
$$w_i = \left(O\left(\frac{\delta}{\varepsilon^{2m}}\right), \cdots, O\left(\frac{\delta}{\varepsilon}\right)\right)$$

- Can find a solution $\sum v_i = 0$ by matching curvatures.
- Locally near solution for highest order terms, Vandermonde determinant:

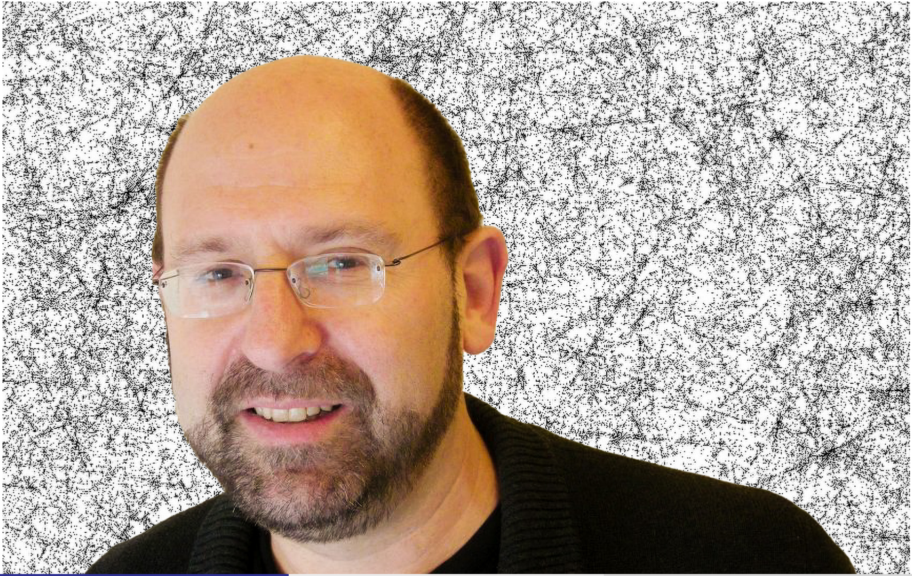
$$\frac{\partial \sum v_i}{\partial \varepsilon} \neq 0, \tag{6}$$

- Map $\varepsilon \mapsto v$ a submersion, so there exists a nearby solution with remainders.

Future directions

- C^∞?
- How many lengths can be canceled simultaneously?
- Can probably be done for closed manifolds too, eg. surface of revolution with Maslov = Morse index, Liouville metrics on $\mathbb{T}^2$, etc...
- When is one guaranteed a singularity at a given length?
- Noncoincidence condition of Marvizi-Melrose near boundary?
- Robin boundary conditions on an ellipse? (Guillemin-Melrose)
- Obstacle scattering? Casimir Energy?

In memory of Steve Zelditch



Thank you for your attention!

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