Cancellations in the Wave Trace

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Thank you organizers!
Joint work with Illya Koval and Vadim Kaloshin from IST Austria.
- **Laplace Spectrum:** Solutions of

\[
\begin{cases}
  -\Delta \psi_j = \lambda_j^2 \psi_j & \text{in } \Omega, \\
  B \psi_j = 0
\end{cases}
\]

- **Length Spectrum:** Closure of lengths of closed (periodic) geodesics/billiard trajectories + zero.

- **Poisson Relation:** \( \text{SingSupp} \cos t\sqrt{-\Delta} \subset \text{LSP} \).

- **Inclusion or Equality?** When is Poisson relation a strict inequality?
Motivation

- To what extent do the Laplace spectrum and length spectrum encode the same data?
- Can one translate inverse problems from one area to another? Ex. rigidity phenomena in dynamics / PDE
- Are there limitations to using the wave trace for the inverse spectral problem?
- Bouncing balls? Hyperbolic orbits? Nearly glancing + whispering gallery modes?
- This work is inspired by the work of Steve Zelditch, whose advice was very helpful in the beginning of this project.
Whispering gallery modes (St. Paul’s Cathedral, London)
Bouncing ball orbits (NCSU, Raleigh)
**Theorem**

Let $\Omega_0$ be an ellipse. Then, for a dense set of eccentricities $e \in (0, \infty)$ and for each $m \in \mathbb{N}$, there exist perturbations $\Omega_\epsilon$ of $\Omega_0$ which fix $2m$ hyperbolic orbits denoted $\gamma_i$ and $\gamma'_j$ with corresponding rational rotation numbers $p/q, p'/q'$, $q = q' + 4 \mod 8$, and $\epsilon_n^m(e) \to 0$ as $n \to \infty$ such that: for some length $L(\epsilon, e) \in LSP(\Omega_\epsilon)$, $w(t) \in C^{m,\alpha}(L - \delta, L + \delta)$ for $\delta$ sufficiently small and any $\alpha \in (0, 1)$. 

Let $\Omega_0$ be a smooth, convex planar domain.

$\beta : B^* \partial \Omega_0 \to B^* \partial \Omega_0$
A smooth curve $C$ is called a **caustic** if any tangent line drawn to $C$ remains a tangent to $C$ after reflection at the boundary.
Map $C$ onto the total phase space $B^*\partial\Omega \cong \mathbb{Z}/\ell\mathbb{Z} \times (0, \pi)$ to obtain a smooth closed curve, invariant under the billiard map $\beta$.

Use $\omega(C)$ for the rotation number of invariant curve.

Lazutkin[73]: In every neighborhood of the boundary there exists a family of convex caustics whose rotation numbers belong to a Cantor set of positive measure.
Loop function: $\ell_{p,q}(s) =$ length of rot. number $p/q$ loop emanating from $x(s) \in \partial \Omega$, if it exists.

Length functional: $\mathcal{L}_q(x_1, \cdots, x_q) = \sum_1^q |x_{i+1} - x_i|$.  

Periodic orbits arise as critical points of $\ell_{p,q}, \mathcal{L}_q$.

Length spectrum is set of critical values of $\ell_{p,q}, \mathcal{L}_q$.  


Cancellations in the Wave Trace

Ellipse: $t_{p,q} = T_{p,q}$
Billiards on the ellipse

3.1 Convex billiards

which case the trajectory never intersects the segment between the foci) or the two branches of a confocal hyperbola (where the trajectory always intersects the segment between the foci); see Fig. 3.5. The eccentricity of the corresponding conic section, for example, can be taken as an integral for the elliptical billiard. For proofs and further remarks, the reader may consult [24, 101].

Thus, the confocal ellipses inside an elliptical billiard are convex caustics in accordance with Def. 3.1.5, so the elliptical billiard is foliated by convex caustics (up to the segment between the foci). The branches of the confocal hyperbolae can then be seen as caustics in the more general sense mentioned above.

(a) Caustics are confocal ellipses and hyperbolae

(b) Phase portrait

Fig. 3.5. The billiard inside an ellipse

The phase portrait of an elliptical billiard is also shown in Fig. 3.5. Although it looks like the phase portrait of the pendulum (Fig. 1.3), the dynamics are quite different. The points (0, 0), (1/2, 0) and its translates do not represent equilibrium points anymore, but belong to the two–periodic orbits corresponding one of the half–axes of the ellipse, and similarly for the other half–axis. Their rotation number is 1/2, which implies that the islands are not fixed, but “wander”: they are mapped onto each other.

Bounding the islands we see separatrices, corresponding to the orbits through the foci. The invariant curve above and below these separatrices represent the orbits not intersecting the segment between the foci (i.e., being tangent to confocal hyperbola).

As an aside, we mention here that a famous conjecture, usually attributed to Birkhoff, states that elliptical billiards are, in fact, the only integrable strictly convex billiards. Integrable means that the union of all convex caustics has a non-empty interior in $\mathbb{R}^2$.
Caustics of disks and ellipses

The segment between the two foci is left out (describing the dynamics explicitly is much more complicated: see for example [13]). In particular, \( \alpha \) stays constant along the orbit and it represents an integral of motion (i.e., it is an exact symplectic map):

\[
y_1 \, dx_1 + y_0 \, dx_0 = dh(x_0, x_1).
\]

Thus, the confocal ellipses inside an elliptical billiards are convex caustics, but they do not foliate the whole domain: the segment between the two foci is left out (describing the dynamics explicitly is much more complicated: see for example [13]). In particular, \( \alpha \) stays constant along the orbit and it represents an integral of motion (i.e., it is an exact symplectic map):

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The frequency map for billiards inside ellipsoids

Figure 1. Phase portrait of the billiard map in $(r, \ell)$ coordinates for $a = 1$ and $b = 4/9$.

The dashed black lines enclose the phase space (10). The black points are the hyperbolic two-periodic points corresponding to the oscillation along the major axis of the ellipse. The black curves are the separatrices of these hyperbolic points. The magenta points denote the elliptic two-periodic points corresponding to the oscillation along the minor axis of the ellipse. The magenta curves are the invariant curves whose rotation number coincides with the frequency of these elliptic points. The invariant curves with rotation numbers $1/6$, $1/4$ and $1/3$ are depicted in blue, green and red, respectively. The red points label a three-periodic trajectory whose caustic is an ellipse. The green points label a four-periodic trajectory whose caustic is a hyperbola.

4.3. Analytical properties of the rotation number

Let $\nu(\ell)$ be the rotation number of the billiard trajectories inside the ellipse $Q$ sharing the nonsingular caustic $Q$. From definition 2 we get that the function $\nu: E[H \to R$ is given by the quotients of elliptic integrals $\nu(\ell) = \nu(a, b) = R_{\min}(b, a) \int_0^\infty \frac{p(s)}{(b s) (a s)} \frac{2 R_{\max}(b, a)}{d s} = R_{\mu} \int_{t_1}^{t_2} \frac{p(t)}{(t_1) (t_2)} \frac{1}{d t}$ (11),

where the parameters $1 < \mu < \mu_{\infty}$ are given by $\mu = (a m)/ (a m_\infty)$ and $\mu = a/ (a m_\infty)$, with $m = \min(b, a)$ and $m_\infty = \max(b, a)$. The second equality follows from the change of variables $t = (a s)/ (a m_\infty)$. The second quotient already appears in [12]. Other equivalent quotients of elliptic integrals were given in [30, 41]. We have drawn the rotation function $\nu(\ell)$ in figure 2, compare with [41, figure 2].

Proposition 8. The function $\nu: E[H \to R$ given in (11) has the following properties.

(i) It is analytic in $\nu = E[H$ and increasing in $E$. 

Purple dots: bouncing ball orbit on the minor axis.
Black dots: bouncing ball orbit on the major axis.
Green dots: 4-periodic orbit tangent to a confocal hyperbola.
Red dots: 3-periodic orbit tangent to a confocal ellipse.
Wave group

\[ U(t) = \begin{pmatrix} \cos t\sqrt{-\Delta} & \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} \\ -\sqrt{-\Delta} \sin t\sqrt{-\Delta} & \cos t\sqrt{-\Delta}, \end{pmatrix} \] (1)

Solves:

\[ \begin{cases} (\partial_t^2 - \Delta) u = 0, \\ u\big|_{t=0} = f, \quad \partial_{\nu} u\big|_{t=0} = g. \end{cases} \] (2)

For \( \varphi \in \mathcal{S}(\mathbb{R}) \),

\[ \langle \text{tr} U(t), \varphi \rangle \equiv \text{tr} \int \varphi(t) U(t) \, dt \]
The Poisson Relation

Anderson-Melrose[77]:

\[
\text{SingSupp} \left( \text{Tr} \cos t \sqrt{-\Delta^B_\Omega} \right) \subset \{0\} \cup \pm \text{LSP}(\Omega),
\]

Wave trace asymptotics of \( \gamma \): \( \text{Tr} \cos t \sqrt{-\Delta} \sim \)

\[
\Re \left\{ a_{\gamma,0} (t - L + i0)^{-1} + \sum_{k=0}^{\infty} a_{\gamma,k} (t - L + i0)^k \log(t - L + i0) \right\},
\]

Sum over \( \gamma \), length(\( \gamma \)) = \( L \).
Let $\hat{\rho}$ bump function supported near $L \in \text{LSP}(\Omega)$.

$$\int_0^\infty e^{ikt}\hat{\rho}(t)w(t)dt = \text{tr} \, \rho \star kR(k) \sim \sum_{\text{length}(\gamma)=L} D_{\gamma}(k) \sum_{j=0}^\infty B_{\gamma,j} k^{-j},$$

where

$$D_{\gamma}(k) = \frac{c_0 e^{ikL_{\gamma}} e^{i\pi \text{sgn}\partial^2 \mathcal{L}/4}}{\sqrt{|\det \partial^2 \mathcal{L}|}}$$

is called the symplectic prefactor.
Potential Theory

Layer potentials:
\[ S\ell(\lambda)f(x) = \int_{\partial\Omega} G_0(\lambda, x, s)f(s)ds, \quad x \in \mathbb{R}^2 \setminus \partial\Omega \]
\[ D\ell(\lambda)f(x) = \int_{\partial\Omega} \partial_{\nu_s} G_0(\lambda, x, s)f(s)ds, \quad x \in \mathbb{R}^2 \setminus \partial\Omega \]

Boundary operator:
\[ N(\lambda)f(s) = \int_{\partial\Omega} \partial_{\nu_{s'}} G_0(\lambda, s, s')f(s')ds', \quad s \in \partial\Omega, \quad (3) \]

Dirichlet resolvent:
\[ R_{DQ}^\partial(\lambda) = R_0(\lambda) - 2D\ell(\lambda)(I + N(\lambda))^{-1}r_{\partial\Omega}S\ell(\lambda) \quad (4) \]

Jump formula:
\[ D\ell(\lambda)f_\pm = \frac{1}{2}(\pm I + N(\lambda))f, \]
Free Green’s fn:

\[ G_0(\lambda, z, z') = H_0^{(1)}(\lambda|z - z'|) \]  

(5)

\( H_\nu^{(1)} \) is the Hankel function of the first kind (of order \( \nu \))

\[
N(\lambda|x(s) - x(s')|) \sim \left( \frac{\lambda \cos^2 \vartheta}{8\pi|x(s) - x(s')|} \right)^{1/2} e^{i\lambda|x(s) - x(s')|} + 3\pi i/4 \sum_{m=0}^{\infty} \frac{c_m i^m}{\lambda^m|x(s) - x(s')|^m}.
\]
Let

\[ R_{\rho B}^X = \int_0^\infty \hat{\rho}(t)w(t)dt \]

denote smoothed resolvent on domain \( X \) with boundary conditions \( B = D \) or \( N \).

Duality:

\[
\text{tr} \left( R_{\rho D}^\Omega (k) \oplus R_{\rho N}^\Omega^c (k) - R_{\rho 0}^{\mathbb{R}^2} (k) \right) \\
= \int_\mathbb{R} \rho(k - \lambda) \frac{\partial}{\partial \lambda} \log \det (1 + N(\lambda)) \, d\lambda
\]
\[ \int_{\mathbb{R}} \rho(k - \lambda) \frac{\partial}{\partial \lambda} \log \det (1 + N(\lambda)) \, d\lambda \]

\[ \sim \int_{\mathbb{R}} \rho(k - \lambda) \text{tr} \left(1 + N(\lambda)\right)^{-1} N'(\lambda) d\lambda \]

\[ \sim \sum_{M} \frac{(-1)^M}{M + 1} \int_{\mathbb{R}} \rho'(k - \lambda) \text{tr} N(\lambda)^{M+1} \, d\lambda \]
What is $N^{M+1}(\lambda)$? If $x : \partial \Omega \ni s \mapsto \mathbb{R}^2$ a parametrization,

$$N(\lambda) \sim e^{i\lambda|x(s) - x(s')|} a(\lambda|x(s) - x(s')|) \implies$$

$$N^{M+1}(\lambda) \sim \int N(s, s_1)N(s_1, s_2)N(s_1, s_2) \cdots N(s_M, s') ds_1 \cdots ds_M$$

$$\sim \int e^{i\lambda\mathcal{L}(S)} \tilde{a}(\lambda, S, s, s') dS$$

Recall: $\mathcal{L}(S) = \sum |x_{i+1} - x_i| = \text{length functional}$
Problem: phase function is not smooth $|x - x'|$

Regularize: $\chi(\lambda|x(s) - x(s')|)$ a cutoff

$N = \chi N + (1 - \chi)N = N_0 + N_1$: diagonal + off diagonal

$\chi N \in \Psi^{-1}(\partial \Omega)$,

$(1 - \chi N)$ a semiclassical ($\hbar = \lambda^{-1}$) FIO quantizing $\beta$.

$N^M = \sum_{\sigma: \mathbb{Z}/M\mathbb{Z} \to \{0, 1\}} N_\sigma$ where

$N_\sigma = N_{\sigma(0)} N_{\sigma(1)} \cdots N_{\sigma(M-1)}$.

Main term when all $\sigma = 1$. 
For an isolated critical point (periodic orbit),

\[
\int_{\mathbb{R}^n} e^{ik\mathcal{L}(S)} a(S) dS \sim (2\pi/k)^{n/2} \frac{e^{ik\mathcal{L}(S_\gamma)} e^{i\pi \text{sgn} \partial^2 \mathcal{L}(S_\gamma) / 4}}{|\det \partial^2 \mathcal{L}(S_\gamma)|^{1/2}} \sum_{j=0}^{\infty} k^{-j} L_j a(S_\gamma),
\]

where the \( L_j \) are differential operators of order \( 2j \):

\[
L_j a(S_\gamma) = \sum_{\nu - \mu = j} \sum_{2\nu \geq 3\mu} i^{-j} 2^{-\nu} \langle \partial^2 \Phi(S_\gamma)^{-1} \partial, \partial \rangle^\nu (g^\mu a(S_\gamma)) / \mu! \nu!.
\]

Here, \( g(S) = \mathcal{L}(S) - \mathcal{L}(S_\gamma) - \mathcal{L}'(S_\gamma)(S - S_\gamma) - \partial^2 \mathcal{L}(S_\gamma)(S - S_\gamma)^2 \).
Perturbations

- Idea: perturb away from the ellipse, where $\partial^2 L$ is degenerate and negative semidefinite. Keep track of maximal Hessians.

- $\text{sgn} L = q - 1$ and $\det L = 0$.

- If orbit rotation numbers $p/q$ and $p'/q'$, with $q \equiv q' + 4 \mod 8$,

  $$e^{i\pi(q-1)/4} = \text{minus} \ e^{i\pi(q'-1)/4}.$$

- Choose $p/q, p'/q'$ caustics with same length and a bunch of orbits to make cancellations
• If perturbation of size \( \delta \), \( \partial^2 \mathcal{L}^{-1} = O(\delta^{-1}) \).

• Then \( \partial^2 \mathcal{L}_{\epsilon,\delta}^{-1}(S_\gamma) \sim V^{-1}(S_\gamma)\delta^{-1} M(S_\gamma) \), where \( M \in C^\infty(\partial\Omega^q) \) consisting of minors of \( \partial^2 \mathcal{L}_0 \) and \( V \) is the \((q - 1) \times (q - 1)\) determinant of \( \partial^2 \mathcal{L} \) upon quotienting out the degenerate direction.

• Furthermore, \( M = O(\|\kappa_\Omega\|_{C^0}) \) and is uniformly bounded in \( \delta \).

• Can show that maximal Hessian terms contribute

\[
a_0(S_\gamma)(h^{lm})^{3j}(\partial^3_{ijk} \mathcal{L})^{2j}.
\]

• Multiscale perturbation: fix \( \delta \) and then introduce \( \epsilon \) for each orbit.
Choose \( p/q, p'/q' \) caustics in the ellipse such that lengths are both \( L \) and \( q \equiv q' + 4 \mod 8 \).

Choose \( m \ p/q \) orbits and \( m \ p'/q' \) orbits with disjoint vertices. \( U \) a small nbhd of vertices.

There exists an arbitrarily small deformation \( \mu_1 \) outside \( U \) such that for every smooth deformation \( \mu_2 \) inside \( U \), there are no other orbits of length \( L \) assuming deformation tangent only at reflection points
Figure 1. To cancel all the $B_j$ for $j \leq m$, we need $m$ orbits of both types.
First match symplectic prefactors:

\[ \mathcal{D}_\gamma(k) = \frac{c_0 e^{ikL\gamma} e^{i\pi \text{sgn}\partial^2 L/4}}{\sqrt{|\det \partial^2 L|}} \]

Then match terms of form

\[ \frac{C(j)\delta^{-3j}}{4^q q} \left( \prod_{i=1}^{q} \frac{\cos \theta_i}{|x_i(S) - x_{i+1}(S)|^{1/2}} \right) \times \]

\[ (\partial^3 \ell_{p,q}(s_i(\gamma)))^{2j} V^{-3j}(S) M^{3j}(S) + O \left( \frac{\delta^{-3j+1}}{k} \right). \]
Use adapted action angle coordinates to switch from length \( f \) to loop \( f \).

Need to regularize integral, keeping track of dependence on curvature, to show \( |\sigma| \geq 1 \) terms don’t contribute to maximal Hessians.

Each orbit has a vector \( u_i = (B_1, \gamma_i), \ldots, B_m, \gamma_i \) for \( 1 \leq 1 \leq 2m \).

\( u = v_i + w_i \), \( v_i \) highest order terms, \( s_i \) remainders.
Equations are homogeneous: renormalise so that

\[ v_i = (\varepsilon_i^2, \varepsilon_i^4, \cdots, \varepsilon_i^{2m}) \]

\[ w_i = \left( O \left( \frac{\delta}{\varepsilon^{2m}} \right), \cdots, O \left( \frac{\delta}{\varepsilon} \right) \right) \]

Can find a solution \( \sum v_i = 0 \) by matching curvatures.

Locally near solution for highest order terms, Vandermonde determinant:

\[ \frac{\partial \sum v_i}{\partial \varepsilon} \neq 0, \quad (6) \]

Map \( \varepsilon \mapsto v \) a submersion, so there exists a nearby solution with remainders.
Future directions

- $C^\infty$?
- How many lengths can be canceled simultaneously?
- Can probably be done for closed manifolds too, eg. surface of revolution with Maslov = Morse index, Liouville metrics on $\mathbb{T}^2$, etc...
- When is one guaranteed a singularity at a given length?
- Noncoincidence condition of Marvizi-Melrose near boundary?
- Robin boundary conditions on an ellipse? (Guillemin-Melrose)
- Obstacle scattering? Casimir Energy?
In memory of Steve Zelditch
Thank you for your attention!