New Types of Siegel-Weil Formulas

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Outline

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$\theta_\psi(\pi)$-**theta lift** to $O_{2m}(\mathbb{A})$; O_{2m}-corresponding to a quadratic space (V, Q), $dim_F V = 2m$, $Witt(Q) = r \leq m$, χ_V-quadratic character.

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$\theta_\psi(\pi)$-**theta lift** to $O_{2m}(\mathbb{A})$; O_{2m}-corresponding to a quadratic space (V, Q), $dim_F V = 2m$, $Witt(Q) = r \leq m$, χ_V-quadratic character. $\theta_\psi(\pi)$ is spanned by

$$\theta_\psi^\phi(\varphi_\pi)(h) = \int_{[Sp_{2n}]} \theta_\psi^\phi(g, h)\varphi_\pi(g)dg,\ h \in O_{2m}(\mathbb{A}),\ \phi \in \mathcal{S}(V(\mathbb{A})^n).$$

$\theta_\psi^\phi(g, h)$ is a theta series on $\widetilde{Sp}_{4mn}(\mathbb{A})$, restricted to the (image of the) dual pair $Sp_{2n}(\mathbb{A}) \times O_{2m}(\mathbb{A})$,

$$\theta_\psi^\phi(g, h) = \sum_{x \in V(F)^n} \omega_\psi(g, h)\phi(x).$$

To determine the non-vanishing of $\theta_\psi(\pi)$, consider (formally) the **inner product** $(\theta_\psi^{\phi_1}(\varphi_\pi), \theta_\psi^{\phi_2}(\varphi'_\pi))$, $\phi, \phi' \in \mathcal{S}(V(\mathbb{A})^n)$:

$$\int_{[Sp_{2n}\times Sp_{2n}]} \varphi_\pi(g_1)\overline{\varphi'_\pi(g_2)} \left(\int_{[O_{2m}]} \theta_\psi^{\phi_1}(g_1, h)\theta_{\psi^{-1}}^{\bar{\phi}_2}(g_2, h)dh \right) dg_1 dg_2. \tag{1}$$

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Product formula:

$$\theta_\psi^{\phi_1}(g_1, h)\theta_{\psi^{-1}}^{\bar{\phi}_2}(g_2, h) = \theta_\psi^{\phi_1 \otimes \bar{\phi}_2}((g_1, g_2), h). \tag{2}$$

The r.h.s. is a theta series on $\widetilde{Sp}_{8mn}(\mathbb{A})$, restricted to $Sp_{4n}(\mathbb{A}) \times O_{2m}(\mathbb{A})$, and then to $(Sp_{2n}(\mathbb{A}) \times Sp_{2n}(\mathbb{A})) \times O_{2m}(\mathbb{A})$.

Next, interpret the integral

$$I(\Phi, g) = \int_{[O_{2m}]} \theta_\psi^\Phi(g, h) dh, \quad g \in Sp_{4n}(\mathbb{A}),\ \Phi \in \mathcal{S}(V(\mathbb{A})^{2n}).$$

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It is absolutely convergent when $r = 0$, or $2m - r > 2n + 1$. In this range, we have

The **Siegel-Weil formula** (Weil, Kudla-Rallis)

$$I(\Phi, g) = \kappa E(f_{\Phi,s}, g)_{|_{s=m-n-\frac{1}{2}}}.$$

$E(f_{\Phi,s})$- the Eisenstein series on $Sp_{4n}(\mathbb{A})$ attached to $Ind_{Q_{2n}(\mathbb{A})}^{Sp_{4n}(\mathbb{A})}\chi_V|\det\cdot|^s$ and the Siegel-Weil section

$$f_{\Phi,s}(g) = \omega_{\psi}(g, 1)\Phi(0)|a(g)|^{s-m+n+\frac{1}{2}}.$$

Regularization: In the range $2m - r \leq 2n + 1$, $r \geq 1$, Kudla and Rallis found $z \in \mathcal{Z}_{sp_{2n}(F_v)}$, $z' \in \mathcal{Z}_{o_{2m}(F_v)}$, at one archimedean place v, such that

- $\omega_\psi(z)\Phi = \omega_\psi(z')\Phi, \quad \Phi \in \mathcal{S}(V(\mathbb{A})^{2n})$,
- $\theta_\psi^{\omega_\psi(z)\Phi}(g, h)$ is rapidly decreasing in $h \in O_{2m}(F)\backslash O_{2m}(\mathbb{A})$.

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Then they take an Eisenstein series $E(h, \zeta)$ on $O_{2m}(\mathbb{A})$, attached to the maximal parabolic subgroup with Levi part $GL_r \times O_{2(m-r)}$ and $|\det \cdot|^\zeta$. It has a constant residue at $\zeta = m - \frac{r+1}{2}$.

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Consider

$$\mathcal{E}(g, \Phi, \zeta) = \frac{1}{P(\zeta)} \int_{[O_{2m}]} \theta_\psi^{\omega_\psi(z)\Phi}(g, h) E(h, \zeta) dh, \qquad (3)$$

$P(\zeta)$ is the polynomial obtained by the action of z' on $E(h, \zeta)$.

Theorem: $\mathcal{E}(g, \Phi, \zeta)$ is an Eisenstein series on $Sp_{4n}(\mathbb{A})$, attached to the maximal parabolic subgroup with Levi part $GL_r \times Sp_{2(2n-r)}$ and $|\det \cdot|^{\zeta} \otimes \theta_{\psi}^{Sp_{2(2n-r)}}(1_{O(V_{an})})$.

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If $m \leq n$ (then $2m - r \leq 2n + 1$), $P(m - \frac{r+1}{2}) \neq 0$; $\mathcal{E}(g, \Phi, \zeta)$ has a **simple pole** at $\zeta = m - \frac{r+1}{2}$.
If $m > n$ (and $2m - r \leq 2n + 1$), then $P(m - \frac{r+1}{2}) = 0$; $\mathcal{E}(g, \Phi, \zeta)$ has a **double pole** at $\zeta = m - \frac{r+1}{2}$.

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$$\mathcal{E}(g, \Phi, \zeta) = \begin{cases} \frac{B_{-1}(g,\Phi)}{\zeta-(m-\frac{r+1}{2})} + B_0(g, \Phi) + \cdots & m \leq n \\ \frac{B_{-2}(g,\Phi)}{(\zeta-(m-\frac{r+1}{2}))^2} + \frac{B_{-1}(g,\Phi)}{\zeta-(m-\frac{r+1}{2})} + \cdots & m > n \end{cases} \tag{4}$$

The regularized Siegel-Weil formula:

1. When $m \leq n$,

$$2B_{-1}(g,\Phi) = Val_{m-n+\frac{1}{2}} E(f_{\Phi,s},g) = Res_{n-m-\frac{1}{2}} E(f_{\Phi',s},g). \quad (5)$$

$f_{\Phi',s}$ - Siegel-Weil section, $\Phi' \in \mathcal{S}(V'(\mathbb{A})^{2n})$;
$dim_F V' = 4n+2-2m$, V' in the same Witt class of V (complementary quadratic space to V).

2. When $2n+2 \leq 2m \leq 2n+r+1$,

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Using (5), (6), $(\theta_{\psi}^{\phi_1}(\varphi_\pi), \theta_{\psi}^{\phi_2}(\varphi'_\pi))$ can be expressed as

$$Res_{s=|m-n-\frac{1}{2}|} \int_{[Sp_{2n}\times Sp_{2n}]} \varphi_\pi(g_1)\overline{\varphi'}_\pi(g_2) E(f_{\tilde{\Phi},s},(g_1,g_2)) d(g_1,g_2) \quad (7)$$

The integral (7) is the global integral of **the doubling method** of PS-Rallis. It represents the ***L*-function** $L(\pi \times \chi_V, s + \frac{1}{2})$ (up to normalization).

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Theorem (Kudla-Rallis): Let χ be a quadratic character of $F^* \backslash \mathbb{A}^*$. Assume that $L^S(\pi \times \chi, s)$ has a pole at $s = k \geq 1$. Then $k \leq [\frac{n}{2}] + 1$. Let $m = n + k$. Then there is a quadratic space V' of dimension $4n + 2 - 2m = 2n + 2 - 2k$, $\chi_{V'} = \chi$, such that the theta lift of π to $O(V')(\mathbb{A}) = O_{2n+2-2k}(\mathbb{A})$ is non-trivial.

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We want to follow a similar **itinerary**, guided by the **poles of the *L*-functions** for $Sp_{2n}(\mathbb{A}) \times GL_d(\mathbb{A})$, $L(\pi \times \tau, s)$. We now know the **generalized doubling integrals** for $Sp_{2n} \times GL_d$ by Cai, Friedberg, Ginzburg and Kaplan.

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What would be a **new Siegel-Weil formula**?

Outline

The following correspondence is constructed using new "theta kernels". This was done by Ginzburg (IMRN 2003).

Each irreducible, self-dual, automorphic, cuspidal representation τ of $GL_d(\mathbb{A})$ determines a family of such theta kernels, and hence a related Θ_τ-correspondence.

To simplify the exposition, we restrict to τ on $GL_2(\mathbb{A})$, with trivial central character, such that $L(\tau, \frac{1}{2}) \neq 0$.

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Consider an Eisenstein series on $Sp_{4k}(\mathbb{A})$, $E(f_{\Delta(\tau,k),s})$, related to $Ind_{Q_{2k}(\mathbb{A})}^{Sp_{4k}(\mathbb{A})}\Delta(\tau, k)|\det\cdot|^s$, where $\Delta(\tau, k)$ is the Speh representation of $GL_{2k}(\mathbb{A})$ attached to (the parabolic induction from)

$$\tau|\det\cdot|^{\frac{k-1}{2}} \times \tau|\det\cdot|^{\frac{k-3}{2}} \times \cdots \times \tau|\det\cdot|^{\frac{1-k}{2}}.$$

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D. Jiang, B. Liu and L. Zhang determined the positive poles of the corresponding normalized Eisenstein series. These are obtained at $s = \frac{k}{2}, \frac{k}{2} - 1, \frac{k}{2} - 2, ...$

Consider the residual representation at largest pole $s = \frac{k}{2}$.

$$\Theta_{\Delta(\tau,k)} = \{Res_{s=\frac{k}{2}} E(f_{\Delta(\tau,k),s})\}.$$

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$\Theta_{\Delta(\tau,k)}$**-correspondence:** Let π be an irreducible, automorphic, cuspidal representation of $Sp_{2n}(\mathbb{A})$. Define, for $h \in Sp_{4k-2n}(\mathbb{A})$,

$$T_\tau^{4k-2n}(\varphi_\pi, \theta_{\Delta(\tau,k)})(h) = \int_{[Sp_{2n}]} \varphi_\pi(g)\theta_{\Delta(\tau,k)}(g,h)dg.$$

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We get representations $\Theta_{\Delta(\tau,k)}(\pi)$ of $Sp_{4k-2n}(\mathbb{A})$, $k \geq \frac{n}{2}$.
They satisfy the **tower property**: at the first k, such that $\Theta_{\Delta(\tau,k)}(\pi)$ is nontrivial, it is cuspidal.

Theorem:

- Assume that the first occurrence for π is at $\frac{n}{2} \leq k \leq n$. Then π is CAP with respect to

$$Ind_{Q_{2(n-k)}(\mathbb{A})}^{Sp_{2n}(\mathbb{A})} \Delta(\tau, n-k) |\det \cdot|^{\frac{n-k}{2}} \otimes \sigma,$$

  σ- irreducible, cuspidal representation of $Sp_{4k-2n}(\mathbb{A})$.

- If the first occurrence is at $k > n$, then π lifts to a CAP representation with respect to

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Remark: We conjecture that the first occurrence $k \leq \frac{3n}{2}$.
Inner product: We want to test the non-vanishing of $\Theta_{\Delta(\tau,k)}(\pi)$ on $Sp_{4k-2n}(\mathbb{A})$. Consider the inner product

$$(T_{\tau}^{4k-2n}(\varphi_\pi, \theta_{\Delta(\tau,k)}), T_{\tau}^{4k-2n}(\varphi'_\pi, \theta'_{\Delta(\tau,k)})).$$

Formally, this is

$$\int\limits_{[Sp_{2n}\times Sp_{2n}]} \varphi_\pi(g_1)\overline{\varphi'_\pi(g_2)}\left(\int\limits_{[Sp_{4n-2k}]} \theta_{\Delta(\tau,k)}(g_1,h)\overline{\theta'_{\Delta(\tau,k)}(g_2,h)}dh\right) dg_1 dg_2. \tag{8}$$

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We need to make sense out of the *dh*-integral.

First, find an **analog of the product formula** of theta series, as in (2).

Then look for an **analog of the regularized Siegel-Weil formula**, which will relate the inner product to the generalized doubling integrals, representing $L^S(\pi\times\tau, s+\frac{1}{2})$, focusing at $s=n-k+\frac{1}{2}$.

Outline

Assume that $\frac{n}{2} \leq k \leq 2n$. (Eventually, we will be interested in $\frac{n}{2} \leq k \leq n$). We can prove the following approximate analog of the product formula of theta series.

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Theorem (formal): Given $\theta_{\Delta(\tau,2n+k)} \in \Theta_{\Delta(\tau,2n+k)}$, $\theta_{\Delta(\tau,k)} \in \Theta_{\Delta(\tau,k)}$, there exist $\theta^i_{\Delta(\tau,k)} \in \Theta_{\Delta(\tau,k)}$, $i = 1, ..., N$, such that

$$\int\limits_{[U_{2n}]} \int\limits_{[Sp_{4k}]} \theta_{\Delta(\tau,2n+k)}(x, u \cdot t(g_1, g_2))\theta_{\Delta(\tau,k)}(x)\psi^{-1}_{U_{2n}}(u)dxdu = \tag{9}$$

$$= \sum_{i=1}^{N} \int_{[Sp_{4k-2n}]} \theta_{\Delta(\tau,k)}(g_2, h)\overline{\theta^i_{\Delta(\tau,k)}(g_1, h)}dh.$$

Assume that $\frac{n}{2} \leq k \leq 2n$. (Eventually, we will be interested in $\frac{n}{2} \leq k \leq n$). We can prove the following approximate analog of the product formula of theta series.

Theorem (formal): Given $\theta_{\Delta(\tau,2n+k)} \in \Theta_{\Delta(\tau,2n+k)}$, $\theta_{\Delta(\tau,k)} \in \Theta_{\Delta(\tau,k)}$, there exist $\theta^i_{\Delta(\tau,k)} \in \Theta_{\Delta(\tau,k)}$, $i = 1, ..., N$, such that

$$\int_{[U_{2n}]} \int_{[Sp_{4k}]} \theta_{\Delta(\tau,2n+k)}(x, u \cdot t(g_1, g_2))\theta_{\Delta(\tau,k)}(x)\psi^{-1}_{U_{2n}}(u)dxdu = \tag{9}$$

$$= \sum_{i=1}^{N} \int_{[Sp_{4k-2n}]} \theta_{\Delta(\tau,k)}(g_2, h)\overline{\theta^i_{\Delta(\tau,k)}(g_1, h)}dh.$$

$$U_{2n}(\mathbb{A}) : u = \begin{pmatrix} I_{2n} & y_1 & y_2 & y_3 & * \\ & I_n & & & y'_3 \\ & & I_{2n} & & y'_2 \\ & & & I_n & y'_1 \\ & & & & I_{2n} \end{pmatrix} \in Sp_{8n}(\mathbb{A});$$

$$\psi_{U_{2n}}(u) = \psi\left(tr\left((y_1, y_2, y_3)\begin{pmatrix} I_n & 0 \\ 0 & 0 \\ 0 & I_n \end{pmatrix}\right)\right);$$

For $(g_1, g_2) \in Sp_{2n}(\mathbb{A}) \times Sp_{2n}(\mathbb{A})$,

$$t(g_1, g_2) = \begin{pmatrix} g_1 & & & & \\ & a_1 & & b_1 & \\ & & g_2 & & \\ & c_1 & & d_1 & \\ & & & & g_1^* \end{pmatrix} \in Sp_{8n}(\mathbb{A}),\ g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}.$$

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The relation (9) is still at the formal level. Our **starting point** is the *dx* integral inside (9). Consider, for $h \in Sp_{8n}(\mathbb{A})$,

$$I(\theta_{\Delta(\tau,2n+k)}, \theta_{\Delta(\tau,k)}, h) = \int_{[Sp_{4k}]} \theta_{\Delta(\tau,2n+k)}(g, h)\theta_{\Delta(\tau,k)}(g)dg. \tag{10}$$

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We regularize this integral.

Ichino obtained the regularization of the Kudla-Rallis integrals using convolution against spherical functions on one member of the dual pair at one finite place. In the same spirit, we take a finite place v, where τ_v is unramified, $\theta_{\Delta(\tau,2n+k)}$ is $Sp_{4k}(\mathcal{O}_v) \times Sp_{8n}(\mathcal{O}_v)$-right invariant, and $\theta_{\Delta(\tau,k)}$ is $Sp_{4k}(\mathcal{O}_v)$-right invariant.

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Theorem: There is an algebra homomorphism $\eta : \mathcal{H}(Sp_{8n}(F_v)//Sp_{8n}(\mathcal{O}_v)) \to \mathcal{H}(Sp_{4k}(F_v)//Sp_{4k}(\mathcal{O}_v)$, s.t.

$$(1 \otimes \xi_v) * \theta_{\Delta(\tau,2n+k)} = (\eta_{\xi_v} \otimes 1) * \theta_{\Delta(\tau,2n+k)}.$$

There is $\xi_v^0 \in \mathcal{H}(Sp_{8n}(F_v)//Sp_{8n}(\mathcal{O}_v))$, such that the function $g \mapsto (1 \otimes \xi_v^0) * \theta_{\Delta(\tau,2n+k)}(g,h)$ is rapidly decreasing in $Sp_{4k}(F)\backslash Sp_{4k}(\mathbb{A})$, uniformly in h inside bounded sets of a Siegel domain of $Sp_{8n}(F)\backslash Sp_{8n}(\mathbb{A})$.

We regularize the integral (10) by

$$I_{reg}(\theta_{\Delta(\tau,2n+k)}, \theta_{\Delta(\tau,k)}, h) =$$

$$c^{-1} \int_{[Sp_{4k}]} (1 \otimes \xi_v^0) * \theta_{\Delta(\tau,2n+k)}(g,h)\theta_{\Delta(\tau,k)}(g)dg. \quad (11)$$

The constant c is obtained from the action of $\eta_{\xi_v^0}$ on $\theta_{\Delta(\tau,k)}$. It is nonzero when $k \leq n$ (and then $4k - 2n \leq 2n$). It is zero when $n < k \leq 2n$.

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More generally, write in (11),

$$\theta_{\Delta(\tau,k)} = Res_{\zeta=\frac{k}{2}} E(\varphi_{\Delta(\tau,k),\zeta}).$$

Define, for $h \in Sp_{8n}(\mathbb{A})$,

$\mathcal{E}(\theta_{\Delta(\tau,2n+k)}, \varphi_{\Delta(\tau,k),\zeta}, h) =$

$$= \frac{1}{P(q_v^{-\zeta})} \int_{[Sp_{4k}]} (1 \otimes \xi_v^0) * \theta_{\Delta(\tau,2n+k)}(g,h) E(\varphi_{\Delta(\tau,k),\zeta}, g) dg, \tag{12}$$

where $P(q_v^{-\zeta})$ is the polynomial in $q_v^{\pm\zeta}$ obtained by the action of $\eta_{\xi_v^0}$ on $E(\varphi_{\Delta(\tau,k),\zeta})$.

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Theorem: $\mathcal{E}(\theta_{\Delta(\tau,2n+k)}, \varphi_{\Delta(\tau,k),\zeta}, h)$ is an Eisenstein series on $Sp_{8n}(\mathbb{A})$, attached to

$$Ind_{Q_{2k}(\mathbb{A})}^{Sp_{8n}(\mathbb{A})} \Delta(\tau,k)|\det\cdot|^{\zeta} \otimes \Theta_{\Delta(\tau,2n-k)}.$$

Outline

The polynomial $P(q_v^{-\zeta})$ in (12) satisfies

$$P(q_v^{-\frac{k}{2}}) \neq 0, \textit{for } k \leq n; \; P(q_v^{-\frac{k}{2}}) = 0, \textit{for } n < k \leq 2n.$$

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Thus,

$\mathcal{E}(\theta_{\Delta(\tau,2n+k)}, \varphi_{\Delta(\tau,k),\zeta}, h) =$

$$\begin{cases} \frac{B_{-1}(\theta_{\Delta(\tau,2n+k)},\theta_{\Delta(\tau,k)},h)}{\zeta-\frac{k}{2}} + \cdots, & k \leq n \\ \frac{B_{-2}(\theta_{\Delta(\tau,2n+k)},\theta_{\Delta(\tau,k)},h)}{(\zeta-\frac{k}{2})^2} + \cdots, & n < k \leq 2n. \end{cases} \tag{13}$$

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$$\mathcal{E}(\theta_{\Delta(\tau,2n+k)}, \varphi_{\Delta(\tau,k),\zeta}, h) = \begin{cases} \frac{B_{-1}(\theta_{\Delta(\tau,2n+k)}, \theta_{\Delta(\tau,k)}, h)}{\zeta - \frac{k}{2}} + \cdots, & k \leq n \\ \frac{B_{-2}(\theta_{\Delta(\tau,2n+k)}, \theta_{\Delta(\tau,k)}, h)}{(\zeta - \frac{k}{2})^2} + \cdots, & n < k \leq 2n. \end{cases} \quad (13)$$

Conjecture (regularized Siegel-Weil formula):

$$B_{-1}(\theta_{\Delta(\tau,2n+k)}, \theta_{\Delta(\tau,k)}, h) = Value_{s=k-n} E^*(f_{\Delta(\tau,2n),s}, h) =$$

$$= Res_{s=n-k} E^*(f'_{\Delta(\tau,2n),s}, h), \quad \textit{for } k \leq n.$$

For $n < k \leq 2n$,

$$B_{-2}(\theta_{\Delta(\tau,2n+k)}, \theta_{\Delta(\tau,k)}, h) = Res_{s=k-n} E^*(f_{\Delta(\tau,2n),s}, h).$$

We can write an explicit section $f_{\Delta(\tau,2n),s}$ in terms of $(1 \otimes \xi_v^0) * \theta_{\Delta(\tau,2n+k)}$ and $\theta_{\Delta(\tau,k)}$ (analog of the Weil-Siegel section $f_{\Phi,s}$).
$f'_{\Delta(\tau,2n),s}$ is the image of $f_{\Delta(\tau,2n),s}$ under the intertwining operator. (Work in progress).

Outline

Assume that $L^S(\pi \times \tau, s)$ has its **largest pole** at $s = n - k + \frac{1}{2}$, $n > k$. We know that, necessarily, $n - k \leq \frac{n}{2}$, i.e. $\frac{n}{2} \leq k$ (G.S.).

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Using the generalized doubling integrals representing $L^S(\pi \times \tau, s)$, there are data, such that

$$Res_{s=n-k} \int_{[Sp_{2n}]} \varphi_\pi(g) \int_{[U_{2n}]} E(f_{\Delta(\tau,2n),s}, u\cdot(I_{2n}, g))\psi^{-1}_{U_{2n}}(u) du dg \neq 0.$$

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Using the regularized Siegel-Weil formula, for some choice of data,

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Then, from (12), (9),

$$\int_{[Sp_{2n}]} \varphi_\pi(g) \int_{[Sp_{4k-2n}]} \theta_{\Delta(\tau,k)}(g, h)\overline{\theta'_{\Delta(\tau,k)}(I_{2n}, h)}dhdg \neq 0.$$

Hence

$$T_{\tau}^{4k-2n}(\varphi_{\pi}, \theta_{\Delta(\tau,k)})(h) = \int_{[Sp_{2n}]} \varphi_{\pi}(g)\theta_{\Delta(\tau,k)}(g,h)dg \neq 0. \quad (14)$$

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Thus, $\Theta_{\Delta(\tau,k)}(\pi)$, the $\Theta_{\Delta(\tau,k)}$-lift of π to $Sp_{4k-2n}(\mathbb{A})$, is nontrivial. Since $s = n - k + \frac{1}{2}$ is the largest pole of $L^S(\pi \times \tau, s)$, one can show that $\Theta_{\Delta(\tau,k)}(\pi)$ is cuspidal. This will prove

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Theorem: Assume that $L^S(\pi \times \tau, s)$ has its **largest pole** at $s = \ell + \frac{1}{2}$, $1 \leq \ell \leq \frac{n}{2}$. Then $\Theta_{\Delta(\tau,n-\ell)}(\pi)$ is cuspidal. Let σ be an irreducible summand (on $Sp_{2n-4\ell}(\mathbb{A})$). Then π is CAP with respect to

$$Ind_{Q_{2\ell}(\mathbb{A})}^{Sp_{2n}(\mathbb{A})} \Delta(\tau,\ell)|\det\cdot|^{\frac{\ell}{2}} \otimes \sigma.$$

Hence the functorial lift of π to $GL_{2n+1}(\mathbb{A})$ has the form

$$\Delta(\tau, 2\ell) \boxplus \cdots$$

Note the case $\ell = \frac{n}{2}$, n- even, i.e. $n - k = \frac{n}{2}$, and so $k = \frac{n}{2}$. Then (14) is identically zero. Indeed,

$$T^0_\tau(\varphi_\pi, \theta_{\Delta(\tau,\frac{n}{2})})(h) = \int_{[Sp_{2n}]} \varphi_\pi(g)\theta_{\Delta(\tau,\frac{n}{2})}(g)dg \equiv 0.$$

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Corollary: Let τ be an irreducible, cuspidal representation of $GL_2(\mathbb{A})$, with trivial central character, such that $L(\tau, \frac{1}{2}) \neq 0$. Then there is no CAP representation of $Sp_{2n}(\mathbb{A})$ (n even) with respect to

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Remark: When $L(\tau, \frac{1}{2}) = 0$, such CAP representations exist (Piatetski-Shapiro, Ikeda).

Happy Birthday, Gordan!