Bishop-Gromov's inequality: a central tool in Geometry

joint with G. Courtois, S. Gallot and A. Sambusetti

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Sommaire

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Notation: $\mathrm{Ricci} \geq \kappa \iff \forall v \quad \mathrm{Ricci}(v,v) \geq \kappa$

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equality iff $(X, g) = X_\kappa^n$.

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Proof.

1^{st} ineq.: if $P = \{B(x_i,\varepsilon)\}_{i\in I}$ maximal packing, then

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2^{nd} ineq.: Bishop-Gromov's inequality. $\square$

Gromov-Hausdorff topology

Definition

X, Y metric spaces (distances d_X, d_Y), we say that

$$d_{GH}\big((X, d_X), (Y, d_Y)\big) < \varepsilon$$

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$\rightsquigarrow$ GH topology on $\mathcal{M} := \{\text{compact length metric spaces}\}\Big/ \underset{isom.}{\sim}$

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By Bishop-Gromov:

$$\mathrm{Pack}_M(\varepsilon) \leq \frac{b_{-1,n}(D)}{b_{-1,n}(\varepsilon)} = C(n, \varepsilon, D)\ .$$

□

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The proof will be described later, in the context of metric spaces.

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$$0 < \mu\big(B(x,r)\big) < +\infty \ \text{ and } \ \frac{\mu\big(B(x,2r)\big)}{\mu\big(B(x,r)\big)} \leq C \,, \qquad (1)$$

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- Under this "doubling hypothesis", local geometry may be arbitrary.
- $\mathrm{Ricci} \geq -(n-1) \implies C$-doubling at every scale $r_0 > 0$, with $C = \dfrac{b_{-1,n}(4r_0)}{b_{-1,n}(r_0/2)}$.

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$$\dim H_1(X, \mathbf{R}) \leq C^{23\frac{D}{r_0}+40}.$$

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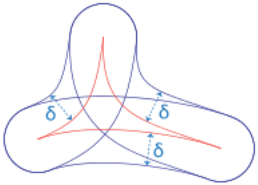

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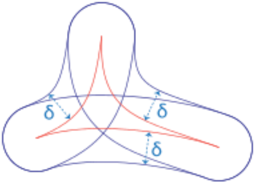

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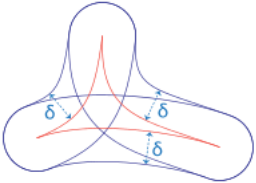

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- "Entropy bounded above" will replace the stronger hypothesis "Ricci curvature bounded below".

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Theorem (BCGS)

Γ *acting geometrically on a* δ*-hyperbolic space* (X, d) *with*

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(True also for any Γ-invariant measure.)
$\rightsquigarrow$ Doubling Property

Idea for the proof

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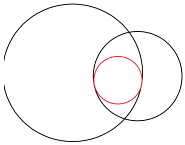
- General trivial property: $B(x, R) \cap B(z, r) \supset B(y, r_1)$, where $r_1 = \frac{1}{2}\big(R + r - d(x, z)\big)$.



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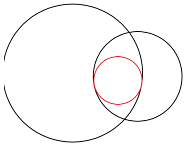

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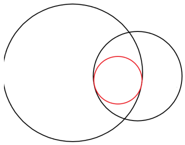
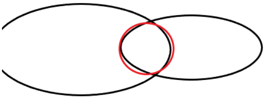


- If X is δ-hyperbolic: $B(x,R) \cap B(z,r) \subset B(y,r_2)$ (red ball), where $r_2 = \frac{1}{2}\left(R + r - d(x,z)\right) + \delta$.



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Set $R_0 = 10(D + \delta)$.

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Theorem

The number of such marked groups (Γ, Σ) *is bounded by* $N'(\delta, H)$.

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(Γ is $\delta'(\delta, D)$-hyp., torsion-free $\rightsquigarrow$ nilpotent $= \mathbf{Z}$).

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A contradiction. Therefore, $\# \left(\overline{G}_0 \cap \Sigma_{2R_0}(x)\right) \leq \nu$ and

$$\#\Sigma_{R_0}(x) \leq \nu^2 .$$

THANKS

A two-years post-doctoral position in Grenoble

Starting September 2024.

- Hilbert geometries on subspaces of projective spaces in real, complex or non Archimedean fields,
- Hilbert geometries and generalisations on real and complex Riemannian manifolds,
- Boundaries of character varieties,
- Non Archimedean representations and actions on Euclidean buildings,
- Group actions in complex hyperbolic geometry, Anosov representations.

Contact : Anne.Parreau@univ-grenoble-alpes.fr