On ‘Topological recursion, discrete surfaces and cohomological field theories’ by Elba Garcia-Failde

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Introduction

- Elba's talk reflects progress in topological recursion, a highly active research area.
- TR is a central part of a larger picture. It is a picture of amazing connections between mathematical fields.

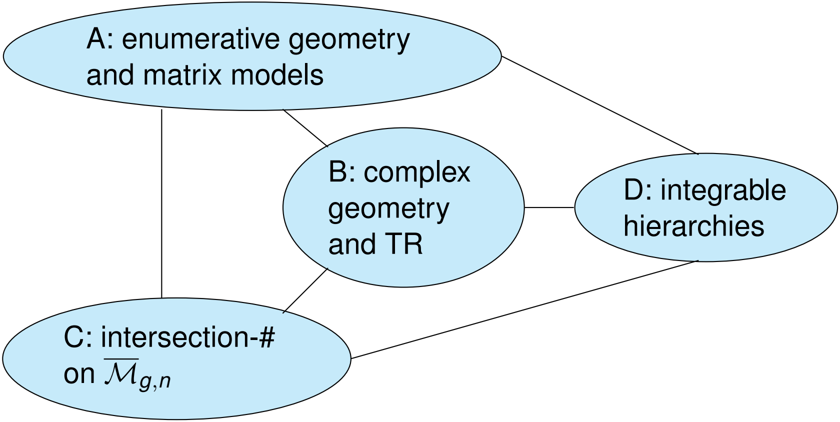
For a quick orientation, read the well-written introduction of:

'Topological recursion for generalised Kontsevich graphs and r-spin intersection numbers'
by R. Belliard, S. Charbonnier, B. Eynard & E. Garcia-Failde

(experts should read everything)

Connections

The article emphasises surprising connections between four large areas of mathematics.



The talk fills this picture for a large family of examples

Some history

- As far as I know, the first example was the Hermitean 1-matrix model (A) which in 1986 was understood to be related to the KdV hierarchy (D).
- In 1990, Witten conjectured that there is a relation (D)-(C) between the KdV hierarchy and intersection numbers.
- Proved by Kontsevich in 1991 by showing that (C) naturally gives rise to a new matrix model (A) [which is deep] from which one gets to (D).
- After experience with more examples, in particular the Hermitean 2-matrix model, Eynard and Orantin in 2007 formulated (B) as a universal framework.

On TR

- One organises initial data into as spectral curve $(x:\Sigma\to\Sigma_0, y, \omega_{0,2})$ where $x:\Sigma\to\Sigma_0$ is a ramified covering of Riemann surfaces, $\omega_{0,1}=ydx$ a meromorphic one-form on Σ and $\omega_{0,2}$ the Bergmann kernel on Σ.
- From these data a family $\omega_{g,n}$ of meromorphic differentials on Σ^n is constructed recursively in the Euler characteristic.
- After suitable variable transform, intersection numbers of κ- and ψ-classes can be read off the $\omega_{g,n}$.

Today we know very many examples where these structures apply. Elba lists several ones in her talk.

The r-spin family of spectral curves

For me it is easiest to start with the matrix model (A).

Given *any* polynomial V of degree $r \geq 3$ and a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_N)$, one studies a measure

$$d\mu = \frac{1}{Z} \int_{H_N} \exp(-N\alpha^{r+1} \mathrm{Tr}(\tilde{V})), \quad \tilde{V} := V(M) - V(\Lambda) - (M - \Lambda) V'(\Lambda).$$

- In a new variable $\tilde{M} = M - \Lambda$ the polynomial has absent constant and linear term.
- It thus gives rise to ribbon graphs with assigned weights:
  - weight $1/([\tilde{M}^{\otimes 2}]\tilde{V})$ to edges
  - weights $[\tilde{M}^{\otimes v}]\tilde{V}$ to vertices of valency $v = 3...r$.

The Kontsevich model results for $V(M) = \frac{1}{3}M^3$.

Various generating functions

- $F_{g,n}$ for generalised Kontsevich graphs/maps
- $W_{g,n}$ for ciliated generalised Kontsevich graphs/maps
- $U_{g,n}$ for square ciliated generalised Kontsevich graphs/maps

There are various Tutte/loop/Dyson-Schwinger equations between these generating functions and a some auxiliary functions $H_{g,n}$ and $P_{g,n}$.

- The auxiliary functions admit a very elegant solution.
- The linear and quadratic loop equations (the heart of TR) then come for free.

A personal remark: This method to prove the linear and quadratic loop equations is great. It even works in an example we are working on [noticed by Alex Hock].

What else?

- Discussion of degenerate cases (higher-order TR)
- r-spin intersection numbers: Consider

$$\langle \tau_{d_1,j_1} \dots \tau_{d_n,j_n} \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} c_W(j_1, \dots, j_n) \psi_1^{d_1} \cdots \psi_n^{d_n}$$

  where c_W is Witten's r-spin class. Then there is an explicit formula giving $\langle \tilde{M}_{i_1 i_1} \dots \tilde{M}_{i_n i_n} \rangle_c$ in terms of $\langle \tau_{d_1,j_1} \dots \tau_{d_n,j_n} \rangle_g$.
- integrability (r-KdV) due to [Faber-Shadrin-Zvonkine, 2010]
- Outlook to
  - free probability
  - large-genus asymptotics and resurgence
  - x-y symmetry of the spectral curve