Last time: • $\mathcal{A}^{Z}_{n,k,m} = \tilde{Z}(Gr^{\geq 0}_{k,n}) \subseteq Gr_{k,k+m}$, km-dim'l

• $Z_G := \tilde{Z}(\overline{C_G})$ a tile if km-dim'l & $\tilde{Z}|_{C_G}$ injective

• Tiling: $\mathcal{A}^{Z}_{n,k,m} = \bigcup Z_G$ ← tiles s.t. Z_G°'s p.w. disjoint

§4: Useful tools to study amplituhedra

• The "right" coordinates to use for $\mathcal{A}^{Z}_{n,k,m}$ are NOT the Plücker coords of $Gr_{k,k+m}$

Defn: $i_1,\dots,i_m \in [n]$, $X \in Gr_{k,k+m}$, $Z = \begin{bmatrix} -z_1- \\ \vdots \\ -z_n- \end{bmatrix} \in Mat_{n,k+m}$. The twistor coordinate is

$$\langle i_1 \cdots i_m \rangle_X = \det \begin{bmatrix} X \\ -z_{i_1}- \\ \vdots \\ -z_{i_m}- \end{bmatrix}$$ ← linear in Plücker coords of Y, coeffs are some minors of Z.

• These coords "know abt Z", so are useful for describing $\mathcal{A}^{Z}_{n,k,m}$, inverting $\tilde{Z}$.
e.g. $k=1, m=2$, $A = [a:b:c:0 \cdots :0] \mapsto a z_1 + b z_2 + c z_3 = X$.

$$\langle 23 \rangle_X = \det \begin{bmatrix} X \\ z_2 \\ z_3 \end{bmatrix} = \det \begin{bmatrix} a z_1 \\ z_2 \\ z_3 \end{bmatrix} = a \cdot p_{123}(Z). \rightsquigarrow [\langle 23\rangle_X : -\langle 13 \rangle_X : \langle 12 \rangle_X : 0 \cdots] = A.$$

colsp $Z \subseteq \mathbb{C}^n$ $(k+m)$-dim'l

• More conceptually, have twistor embedding $Gr_{k,k+m} \xrightarrow{\iota_Z} Gr_m(W) \subseteq Gr_{m,n}$.
$X \longmapsto X^{\perp} Z^{T}$

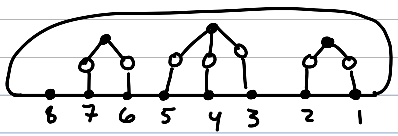
Lem: $\langle i_1 \cdots i_m \rangle = \iota_Z^{*}(p_I)$, so if $\iota_Z(X) = Y$, $\langle i_1 \cdots i_m \rangle_X = p_I(Y)$.

• Sometimes easier to think abt $\tilde{\tilde{Z}} := \iota_Z \circ \tilde{Z}$.

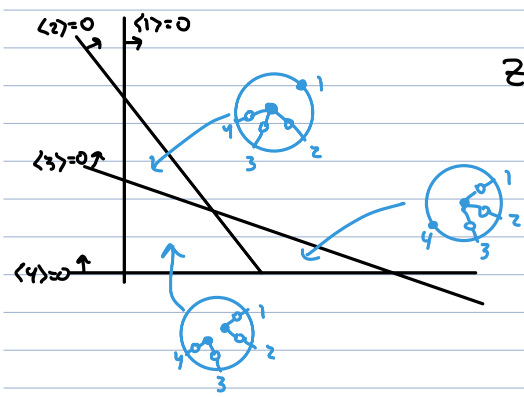
Lem: $A \in Gr^{\geq 0}_{k,n}$, $Y \in Gr_m(W)$. Then $\tilde{\tilde{Z}}(A) = Y \Leftrightarrow Y \perp A$.

§5. m=1 [Karp-Williams]

Thm: {"Smallest" tiles for $\mathcal{A}_{n,k,1}$} $\xleftrightarrow{1-1}$ {Decomp of $[n]$ into $n-k$ disjoint nonempty subintervals}

G:  $\longleftrightarrow$ 12|345|67|8

• $\mathcal{H} := \bigcup_{i=1}^{n} \{\langle i \rangle = 0\} \subseteq Gr_{k,k+1}$. Smallest tiles Z_G are regions of this hyperplane arrangement



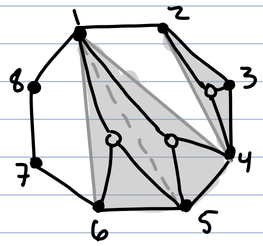
$$Z_G = \{X \in Gr_{k,k+1} : (-1)^{n-i+\#\text{ comp of } G \text{ in } [i+1,n]} \langle i \rangle \geq 0 \ \forall i\}$$

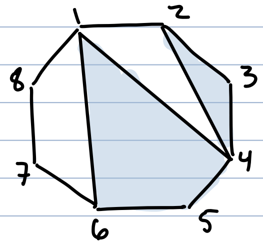
Note: Also have characterization & semi-alg. description of arbitrary tiles (which are certain unions of the smallest tiles), just more annoying to state.

recursive constr. for tiling — conj reln to tropical Gr. — will discuss results here

§6: m=2 amplituhedron [Bao-He, Lukowski-Parisi-Williams, Parisi-S-Williams]

Thm [PSW]: {Tiles for $\mathcal{A}^{Z}_{n,k,2}$} $\overset{1-1}{\longleftrightarrow}$ {bicolored subdiv. of n-gon w/ area k}

$G_\sigma =$ 

$\longleftrightarrow$ 

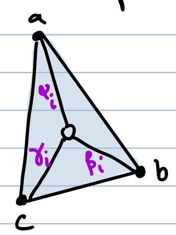
$= \sigma$

area = # black triangles in any triangulation of black polygons

$k=3, n=8$

Why is $\tilde{Z}$ injective on $C_{G_\sigma} =: C_\sigma$?

Best choice of rep. mtx for C_σ:

τ_i 

$$A = \text{rowsp}\begin{array}{c} \cdots\ a\ \cdots\ b\ \cdots\ c\ \cdots \\ \begin{bmatrix} \\ 0 \quad \alpha_i \quad 0 \quad \pm\beta_i \quad 0 \quad \pm\gamma_i \quad 0 \\ \\ \end{bmatrix} \end{array} \leftarrow \text{row } i$$

Max'l minor p_I of $\uparrow$ is D^G_I ✓.

If $X = \tilde{Z}(A)$,
$Y = \tilde{\tilde{Z}}(A)$, then $\alpha_i Y_a \pm \beta_i Y_b \pm \gamma_i Y_c = 0$ since $Y \perp A$. Can check that none of $Y_a, Y_b, Y_c \in \mathbb{R}^2$
$[Y_1 \cdots Y_n] \in Gr_{2,n}$ are parallel, so they satisfy ! lin. reln. up to scaling:
$p_{bc}(Y)\cdot Y_a - p_{ac}(Y)\, Y_b + p_{ab}(Y)\, Y_c = 0$

$\Rightarrow$ row i of A has same span as $[\cdots \langle bc\rangle_X \cdots -\langle ac\rangle_X \cdots \langle ab\rangle_X \cdots]$

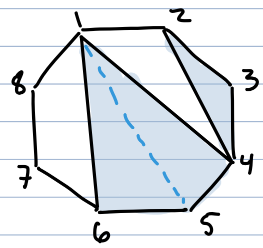
↳ basis of A can be written in terms of twistor coords of $\tilde{Z}(A)$, so if $\tilde{Z}(A) = \tilde{Z}(A')$, have $A = A'$.

Thm [PSW]: σ bic. subdiv. w/ area k.

$$Z^\circ_\sigma = \{X \in Gr_{k,k+2} : \forall \text{ arcs } i \xrightarrow{(i<j)} j \text{ that don't cross } \sigma,\ (-1)^{\text{area}(i\to j)} \langle i j\rangle_Y > 0\}$$

area to left of $i \to j$.

enough to consider just arcs in triangulation of black polygons.

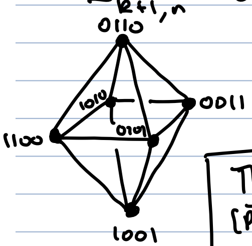
e.g. 

$\langle 23\rangle, \langle 34\rangle, \langle 45\rangle, \langle 56\rangle, \langle 15\rangle > 0.$

$\langle 24\rangle, \langle 14\rangle, \langle 16\rangle < 0$

- For tilings, have recursion [Bao-He]. Also have characterization using the hypersimplex

$\Delta_{k+1,n} = \text{conv}(\text{0-1 vec w/ } k+1 \text{ 1's in } \mathbb{R}^n) \subseteq \{x_1 + \cdots + x_n = k+1\}$



Define polytope $P_\sigma = \left\{ \underline{x} \in \mathbb{R}^n : x_1 + \cdots + x_n = k+1,\ \forall \text{ arcs } i \to j \text{ compat w/ } \sigma,\ \text{area}(i\to j) \le x_i + \cdots + x_{j-1} \le \text{area}(i \to j) + 1 \right\}$

Thm [PSW]: $\{Z_\sigma\}_{\sigma \in \mathcal{E}}$ is a tiling of $\mathcal{A}^Z_{n,k,2}$ $\forall Z$ $\Leftrightarrow$ $\{P_\sigma\}_{\sigma\in\mathcal{E}}$ is a tiling of $\Delta_{k+1,n}$ (i.e. $\bigcup_{\sigma\in\mathcal{E}} P_\sigma = \Delta_{k+1,n}$ & $P^\circ_\sigma \cap P^\circ_{\sigma'} = \phi$)

"<u>Reason</u>:" Facets of Z_σ not in $\partial \mathcal{A}^{z}_{n,k,2}$ $\longleftrightarrow$ Facets of P_σ not in $\partial \Delta_{k+1,n}$

arcs $i-j$ of σ
btw black & white polygon