# *Deformations of Quantum Field Theories*

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## *Diagrams of Algebras*

An algebraic quantum field theory is in particular a *diagram of algebras*, i.e., a covariant functor

$$\mathcal{A} : \mathsf{C} \to \mathsf{Alg} \ ,$$

where C is a small category, such as Loc.
For $M \in \mathsf{C}$, denote multiplication in the algebra $\mathcal{A}(M)$ by

$$\hat{\mathsf{m}}[M] : \mathcal{A}(M)^{\otimes 2} \to \mathcal{A}(M) \ .$$

For a morphism, $\phi : M \to N$, $\mathcal{A}$ gives a homomorphism $\mathcal{A}[\phi] : \mathcal{A}(M) \to \mathcal{A}(N)$, so

$$\mathcal{A}[\phi] \circ \hat{\mathsf{m}}[M] = \hat{\mathsf{m}}[N] \circ (\mathcal{A}[\phi] \otimes \mathcal{A}[\phi]) \ .$$

The right hand side is **cubic**.

# Deformations

The Hochschild complex $C^\bullet(A, A)$ of an algebra is a differential graded (dg) Lie algebra.

$f \in C^2(A, A)$ is a deformation of $A \iff$
$\hat{\mathrm{m}} + f$ is associative $\iff$
$f$ satisfies the *Maurer-Cartan equation*

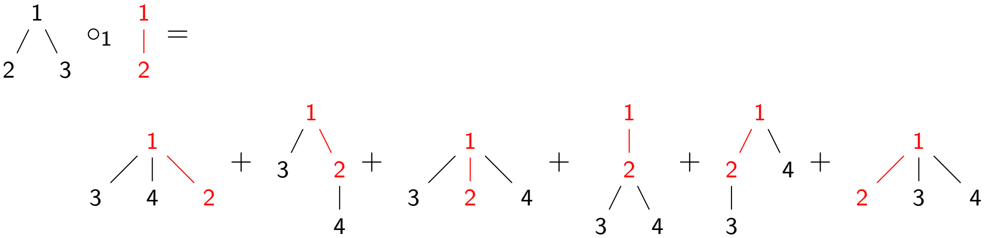
$$0 = \delta f + \tfrac{1}{2}[f, f] \ .$$

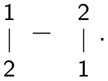
This is quadratic, so it cannot describe deformations of a diagram of algebras.

Kontsevich formality is an $L_\infty$ quasi-isomorphism from the graded Lie algebra of multivector fields to the Hochschild complex.
To generalize this to AQFT, we need an $L_\infty$ structure on the Hochschild bicomplex $C^{\bullet,\bullet}(\mathcal{A}, \mathcal{A})$ of a diagram of algebras.

# *Trees*

The operad *Brace* describes natural operations on the Hochschild complex of a vector space.
The basis of $Brace(n)$ is the set of planar rooted trees with vertices $1, \dots, n$.



The Gerstenhaber bracket is 

# *Words*

The Gerstenhaber-Voronov dg-operad $F_2S$ describes natural operations on the Hochschild complex of an algebra.
The basis of $F_2S(n)$ is the set of words, $W$, such that:

- Each of $1, \ldots, n$ occurs at least once.
- $W \neq \ldots aa \ldots$
- $W \neq \ldots a \ldots b \ldots a \ldots b \ldots$.

$\deg W = \text{length} - n$.
The differential is given by deleting letters, e.g.:

$$\partial 12321 = -2321 + 1321 - 1231 + 1231 \ .$$

Composition: $1232 \circ_2 1232 = 1252343 + 1235343 - 1234543 - 1234353$

# *Quilts*

My dg-operad *Quilt* describes natural operations on the Hochschild bicomplex of a diagram of vector spaces.
The basis of $\mathit{Quilt}(n) \subset F_2S(n) \otimes \mathit{Brace}(n)$ consists of *quilts*.
A quilt $(W, T)$ is a pair of a word and a tree such that:

- If $W = \dots u \dots v \dots$, then $u$ is not below $v$ in $T$.
- If $W = \dots u \dots v \dots u \dots$, then $v$ is to the left of $u$ in $T$.

# $mQuilts$

My dg-operad *mQuilt* describes natural operations on the Hochschild bicomplex, $C^{\bullet,\bullet}(\mathcal{A},\mathcal{A})$, of a diagram of algebras.
*mQuilt* is generated by *Quilt* and $\mathrm{m} \in mQuilt(0)$ of degree $\deg \mathrm{m} = -1$, satisfying several relations. The differential is modified with m.

In the action on $C^{\bullet,\bullet}(\mathcal{A},\mathcal{A})$, m becomes the algebra multiplication $\hat{\mathrm{m}} \in C^{0,2}(\mathcal{A},\mathcal{A})$.

The Hochschild cohomology, $H^{\bullet}(\mathcal{A},\mathcal{A})$ is an $H_{\bullet}(mQuilt)$-algebra. I have constructed a homomorphism $\mathcal{G} : \mathcal{S}Gerst \to H_{\bullet}(mQuilt)$ which makes $H^{\bullet}(\mathcal{A},\mathcal{A})$ a Gerstenhaber algebra.

# $L_\infty$

The dg-operad $L_\infty$ has generators $\ell_n \in L_\infty(n)$ $(\forall n \geq 2)$ with $\deg \ell_n = n-2$.
I define a homomorphism $\mathcal{K} : L_\infty \to \mathit{Quilt}$, by

$$\mathcal{K}(\ell_n) = \sum_{\substack{Q \in \mathrm{Quilt}(n) \\ \deg Q = n-2}} \pm Q \ .$$

This $\mathcal{K}$ makes the Hochschild bicomplex of a diagram of vector spaces into an $L_\infty$-algebra.

The element $\mathrm{m} \in \mathit{mQuilt}(0)$ is formally a Maurer-Cartan solution for $\mathit{Quilt}$, so it modifies $\mathcal{K}$ to another homomorphism,

$$\mathcal{J} : L_\infty \to \mathit{mQuilt} \ ,$$

defined by $\mathcal{J}(\ell_n) = \mathcal{K}(\ell_n) + \mathcal{K}(\ell_{n+1}) \circ_1 \mathrm{m}$.

# Maurer-Cartan

$\mathcal{J} : L_\infty \to mQuilt$ makes $C^{\bullet,\bullet}(\mathcal{A},\mathcal{A})$ into an $L_\infty$-algebra.
This defines a Maurer-Cartan equation.

The *asimplicial subcomplex*

$$C_a^{\bullet,\bullet}(\mathcal{A},\mathcal{A}) = \bigoplus_{\substack{p\geq 0,\\ q\geq 1}} C^{p,q}(\mathcal{A},A)\ ,$$

is an *mQuilt*-subalgebra, so it is also an $L_\infty$-subalgebra.
The Maurer-Cartan equation in $C_a^{\bullet,\bullet}(\mathcal{A},\mathcal{A})$ is cubic. Its solutions are precisely the deformations of $\mathcal{A}$ as a diagram of algebras .

The Maurer-Cartan equation in the full complex $C^{\bullet,\bullet}(\mathcal{A},\mathcal{A})$ is quartic. Its solutions are precisely the deformations of $\mathcal{A}$ as a *skew-diagram of algebras*.

- Hawkins, Eli: "A cohomological perspective on algebraic quantum field theory." Comm. Math. Phys. 360 (2018), no. 1, 439–479. arXiv:1612.05161 [math-ph] MR3795196.
- Hawkins, Eli: "Operations on the Hochschild Bicomplex of a Diagram of Algebras." arXiv:2002.00886 [math.CT]