“Why do we need de Sitter space?”

ESI Vienna, October 2020
“Higher structures emerging from renormalisation”

Michał Wrochna
Cergy Paris Université
(Cergy-Pontoise)

Introduction

QFT on curved spacetimes:

- **quantum fields** ϕ propagating on **classical spacetime** (M, g) (fixed or dynamical)
- useful to test low-energy manifestations of Quantum Gravity
- techniques close to non-linear PDEs
  - Epstein-Glaser renormalization emphasizes: singular products of distributions [Brunetti, Fredenhagen], [Hollands, Wald], [Dang,Herscovich], etc.
  - microlocal and global analysis of PDEs important to construct the "vacuum" (Hadamard state $\Rightarrow$ universal **UV behaviour**)
- but **IR aspects** can cause a headache
  - locally covariant perturbative QG [Fredenhagen, Rejzner], but existence of vacua still major open problem (cf. Yang–Mills fields [Gérard, Wrochna])

On the other hand, de Sitter spaces have better IR behaviour and stability properties than Minkowski space.

I. de Sitter space

1. de Sitter is “Minkowski with $\Lambda > 0$”.

de Sitter space is the “maximally symmetric” solution of Einstein equations with positive cosmological constant $\Lambda > 0$:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0.$$

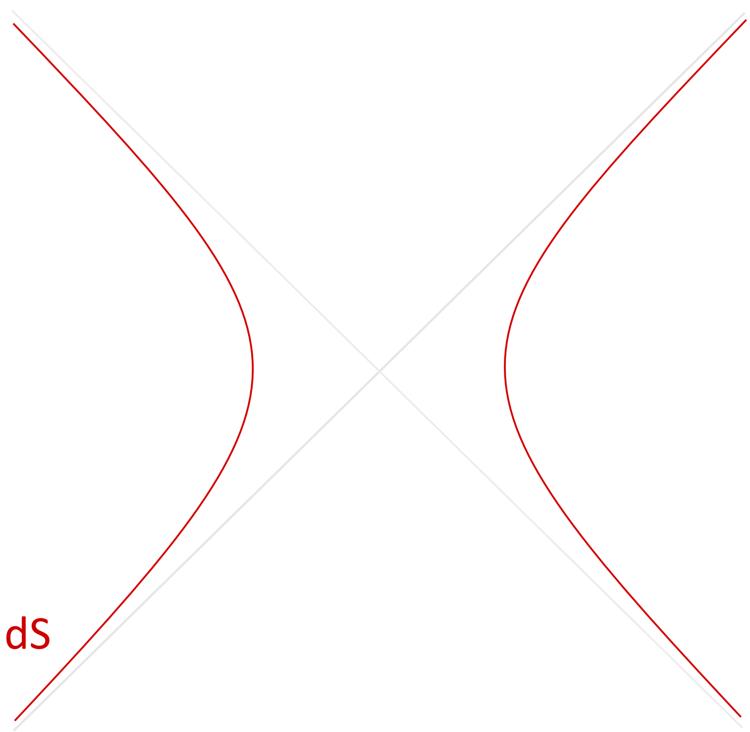
anti-de Sitter space is its analogue with $\Lambda < 0$.

✓ *remark:* limit $\Lambda \to 0$ for quantum fields [Quéva '09]

2. de Sitter is a hyperboloid.

In Minkowski space $\mathbb{R}^{1+d}$, $g = dz_0^2 - (dz_1^2 + \cdots + dz_d^2)$,

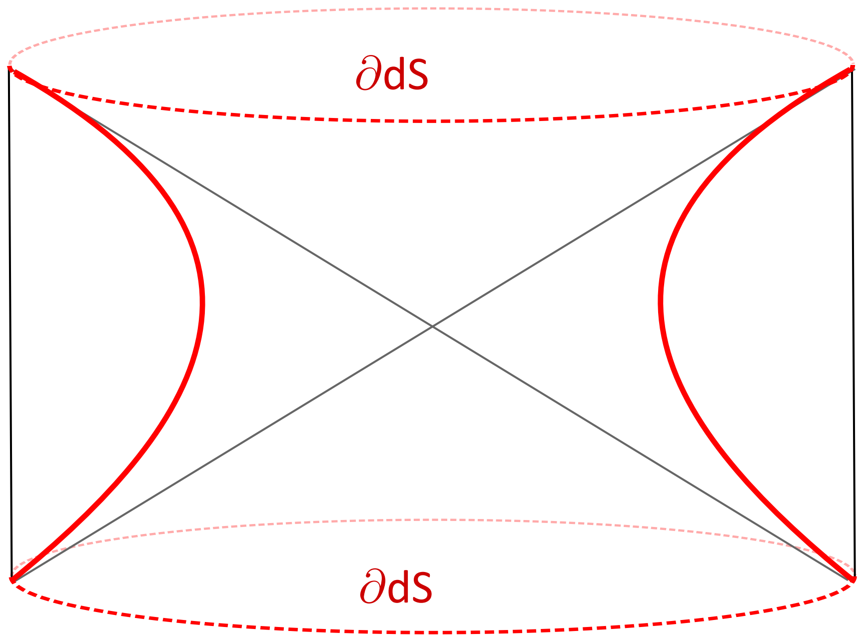
$$\mathsf{dS} = \{z_0^2 - (z_1^2 + \cdots + z_d^2) = -1\}$$



2. de Sitter is a hyperboloid.

In Minkowski space $\mathbb{R}^{1+d}$, $g = dz_0^2 - (dz_1^2 + \cdots + dz_d^2)$,

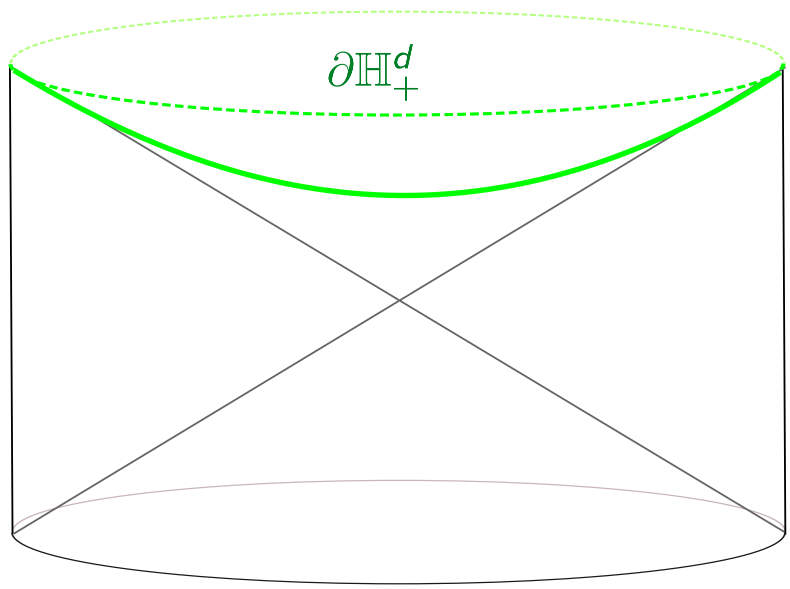
$$\mathrm{dS} = \{z_0^2 - (z_1^2 + \cdots + z_d^2) = -1\}$$



3. hyperbolic space is the Riemannian version.

In Minkowski space $\mathbb{R}^{1,d}$, $g = dz_0^2 - (dz_1^2 + \cdots + dz_d^2)$,

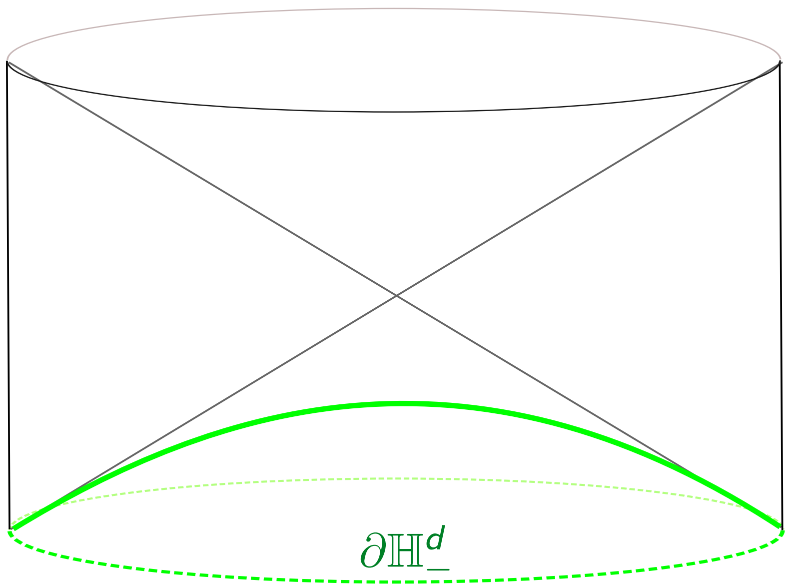
$$\mathbb{H}_+^d = \{z_0^2 - (z_1^2 + \cdots + z_d^2) = 1,\ z_0 > 0\}$$



3. hyperbolic space is the Riemannian version.

In Minkowski space $\mathbb{R}^{1,d}$, $g = dz_0^2 - (dz_1^2 + \cdots + dz_d^2)$,

$$\mathbb{H}^d_- = \{z_0^2 - (z_1^2 + \cdots + z_d^2) = 1,\ z_0 < 0\}$$

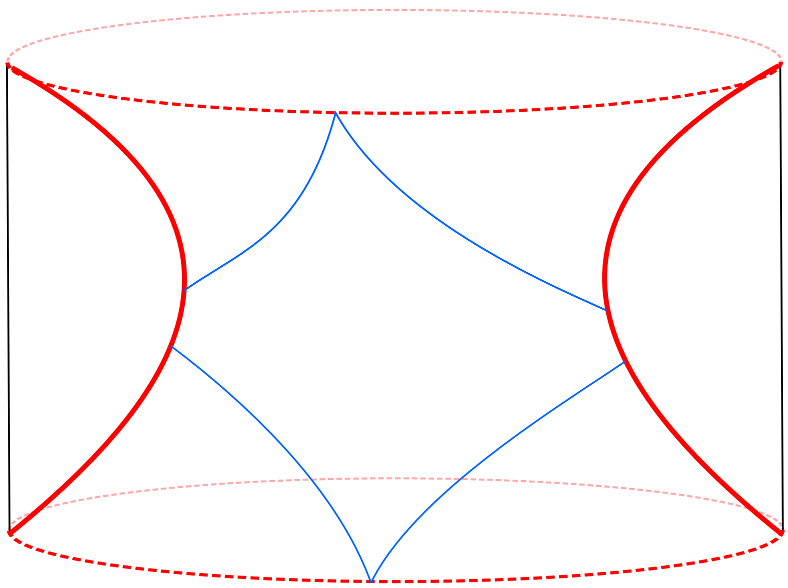


4. de Sitter is *not* static.

Time-orientation and causal structure inherited from Minkowski, but no global time-like vector field.

You can write $g = (1 - r^2)dt^2 - (1 - r^2)^{-1}dr^2 + r^2 d\omega^2$...

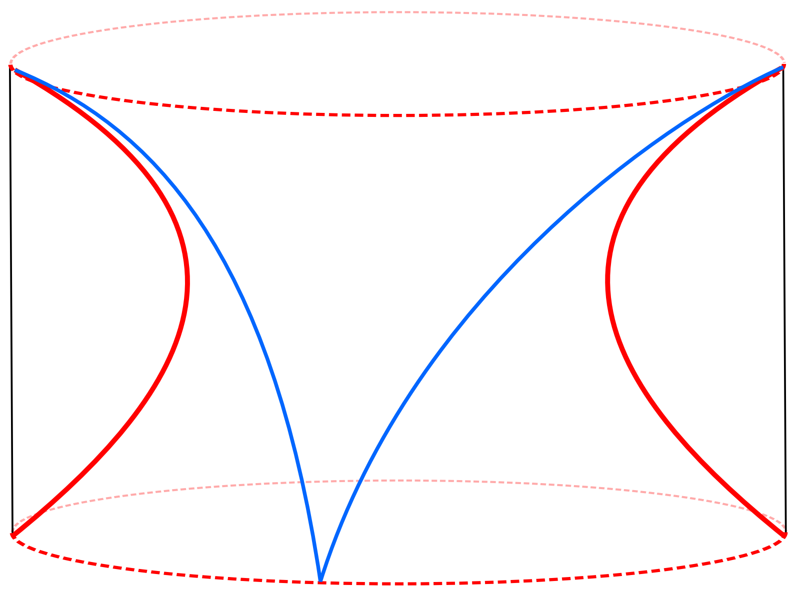
...but not on the whole space.



4. de Sitter is *not* static.

Time-orientation and causal structure inherited from Minkowski, but no global time-like vector field.

Cosmological chart:



5. There are *plane waves* on de Sitter.

Plane waves for $-\Delta_{\mathbb{H}^d_\pm} + \sigma^2 + (d-1)^2/4$:

$$\phi_{\mathbb{H}^d_\pm,\xi} = |\xi \cdot z|^{i\nu-(d-1)/2} \text{ restricted to } \mathbb{H}^d_\pm.$$

Plane waves for $-\Box_{\mathrm{dS}^d} + \sigma^2 + (d-1)^2/4$

$$\phi^\pm_{\mathrm{dS},\xi} = (\xi \cdot z \pm i0)^{i\nu-(d-1)/2} \text{ restricted to dS.}$$

6. There is a preferred "vacuum" on de Sitter.

The Bunch–Davies state has two-point functions:

$$\Lambda^{\pm}(x,y) \propto \int \overline{\phi^{\pm}_{\mathrm{dS},\xi}(x)}\phi^{\pm}_{\mathrm{dS},\xi}(y)\, d\mu(\xi).$$

Restricted to the static chart it is *thermal*. Analogue on $\mathbb{H}^d$ is the Schwartz kernel of the *spectral function* of the Laplacian.

Note: Convention such that the analogue on Minkowski is

$$\Lambda^{\pm}(t,\mathrm{x},s,\mathrm{y}) = \frac{e^{\pm i(t-s)\sqrt{-\Delta_{\mathrm{x}}+m^2}}}{\sqrt{-\Delta_{\mathrm{x}}+m^2}}(\mathrm{x},\mathrm{y}).$$

7. You can give *axioms* for QFT on de Sitter.

- ✓ Constructive $P(\phi)_2$ by [Figari, Høegh-Krohn and Nappi '75]
- ✓ Wightman axioms [Bros, Moschella '95]
- ✓ Osterwalder-Schräder theorem and refined aspects of $P(\phi)_2$ *(no infrared problem)* [Barata, Jäkel, Mund '18]
- ✓ Haag-Kastler nets for $P(\phi)_2$ [Jäkel, Mund '19]

8. The large-time behaviour of fields is *universal*.

E.g. for linear classical fields:

If $(-\Box_{\mathsf{dS}} + \nu^2 + (d-1)^2/4)u = 0$ then near ∂dS,

$$u = f^{i\nu-(d-1)/2} a_+ + f^{-i\nu-(d-1)/2} a_-, \quad a_\pm \in C^\infty$$

For quantum fields:

- ✓ The two-point functions $\Lambda^\pm_\omega(x, y)$ of *any* Hadamard state in the GNS-representation of the Bunch-Davies state approaches $\Lambda^\pm(x, y)$ at large time-like separation.
- ✓ This survives for interacting QFTs [Hollands '13], [Marolf, Morrison '11]

9. Data at the boundary at infinity determines the theory.

E.g. for linear classical fields:

If $(-\Box_{\mathrm{dS}} + \nu^2 + (d-1)^2/4)u = 0$ then near $\partial\mathrm{dS}$,

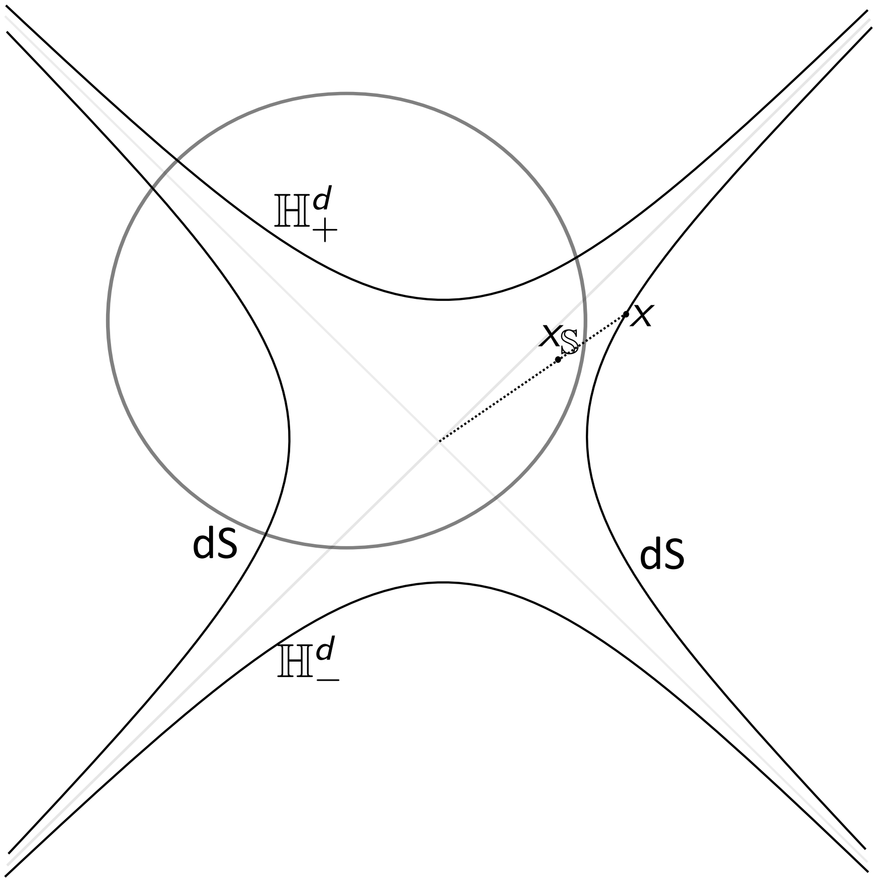
$$u = f^{i\nu-(d-1)/2} a_+ + f^{-i\nu-(d-1)/2} a_-, \qquad a_\pm \in C^\infty$$

and u uniquely determined by (a_+, a_-) restricted to $\partial\mathrm{dS}_-$.
Same for restriction to $\partial\mathrm{dS}_+$.

For quantum fields:

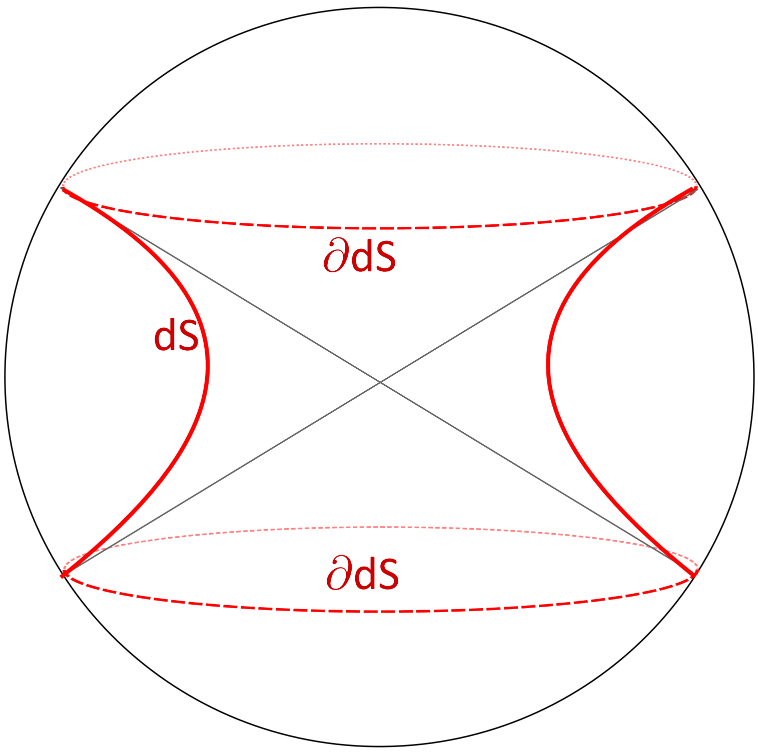
- ? dS/CFT correspondence [Strominger '01]
- ✓ Model provided by "non-linear sigma model with target space $\mathrm{dS} = SO(1,d)/SO(d)$" via $SO(1,d)$-invariant Yang-Baxter operators [Hollands, Lechner '18],

10. For linear fields this is explained by *conformal extension*.



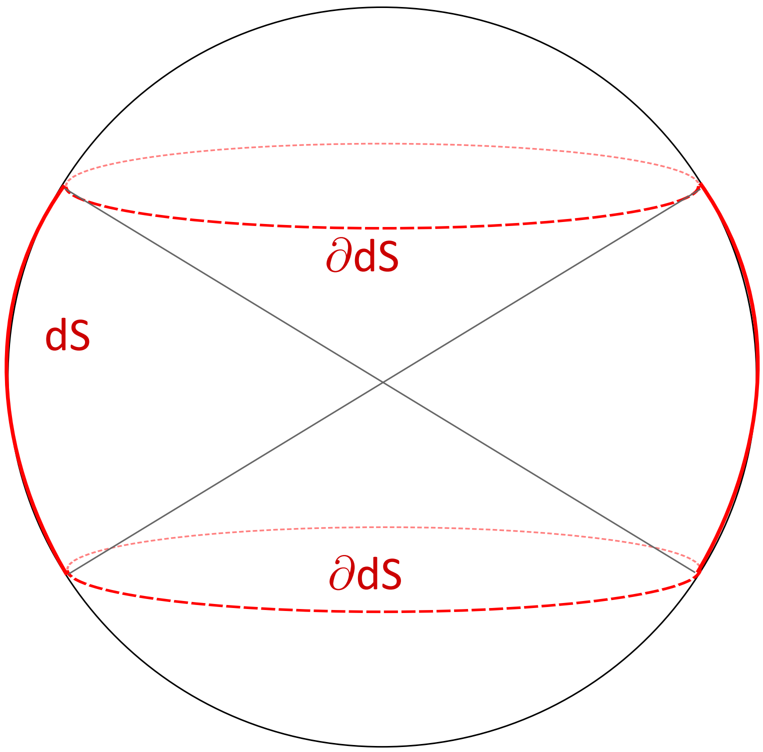
Compactification to sphere $x \mapsto x_{\mathbb{S}} \in \mathbb{S}^d$ $(x \in \mathsf{dS})$.

Alternatively: compactify ambient Minkowski space, and use sphere at infinity.



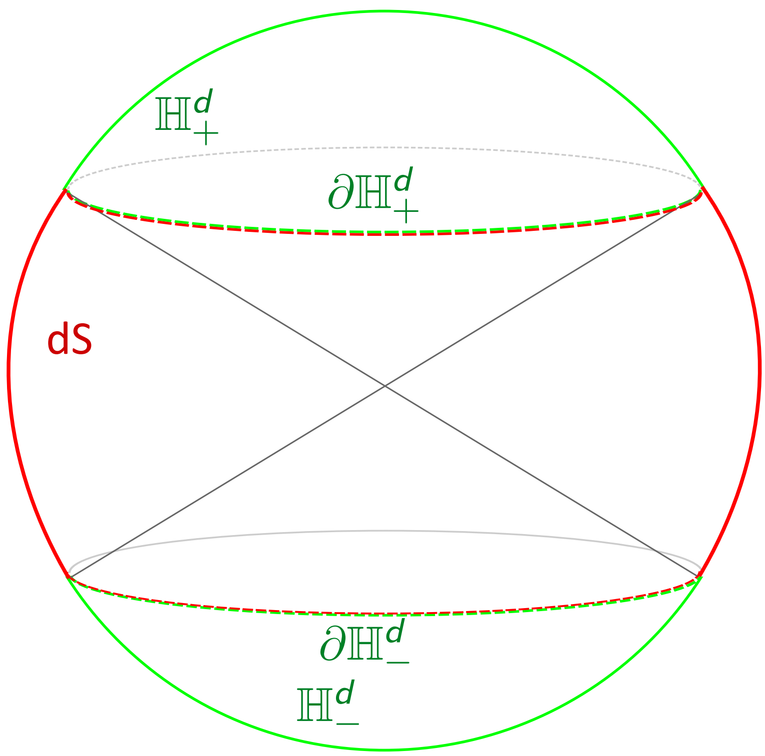
Alternatively: compactify ambient Minkowski space, and use sphere at infinity.

- Identification dS $\subset \mathbb{S}^d$: coord. $y_{\mathbb{S}} = f y_{\mathrm{dS}}$,
  $f = \left| \frac{z_0^2 - (z_1^2 + \cdots + z_d^2)}{z_0^2 + z_1^2 + \cdots + z_d^2} \right|^{\frac{1}{2}}$.



$\{f = 0\} = \partial\mathrm{dS}$

- Identification dS $\subset \mathbb{S}^d$: coord. $y_{\mathbb{S}} = f y_{\mathrm{dS}}$,
  $f = \left| \frac{z_0^2 - (z_1^2 + \cdots + z_d^2)}{z_0^2 + z_1^2 + \cdots + z_d^2} \right|^{\frac{1}{2}}$.
- Identification $\mathbb{H}^d_{\pm} \subset \mathbb{S}^d$: coord. $y_{\mathbb{S}} = f y_{\mathbb{H}}$



$$\mathbb{S}^d = \mathbb{H}^d_+ \cup \mathrm{dS} \cup \mathbb{H}^d_-$$

$$\{f = 0\} = \partial\mathrm{dS} = \partial\mathbb{H}^d_+ \cup \partial\mathbb{H}^d_-$$

- Identification dS $\subset \mathbb{S}^d$: coord. $y_{\mathbb{S}} = f y_{\mathrm{dS}}$,
  $f = \left| \frac{z_0^2 - (z_1^2 + \cdots + z_d^2)}{z_0^2 + z_1^2 + \cdots + z_d^2} \right|^{\frac{1}{2}}$.
- Identification $\mathbb{H}^d_\pm \subset \mathbb{S}^d$: coord. $y_{\mathbb{S}} = f y_{\mathbb{H}}$
- $\mathbb{S}^d = \mathbb{H}^d_+ \cup \mathrm{dS} \cup \mathbb{H}^d_-$; $\quad \{f = 0\} = \partial\mathrm{dS} = \partial\mathbb{H}^d_+ \cup \partial\mathbb{H}^d_-$
- **Plane waves**

$$\phi_\xi^\pm = (\xi \cdot z \pm i0)^{i\nu - (d-1)/2} \restriction_{\mathbb{S}^d} = \begin{cases} f^{i\nu-(d-1)/2}\phi^\pm_{\mathrm{dS},\xi} & \text{on dS} \\ f^{i\nu-(d-1)/2}\phi_{\mathbb{H},\xi} & \text{on } \mathbb{H}^d_+ \end{cases}$$

  Setting $v := -f^2$ on dS, $v := f^2$ on $\mathbb{H}^d_\pm$, $\phi^\pm_\xi \sim (v \pm i0)^{-i\nu}$!
- These are solutions of $P\phi = 0$, $P = 4v\partial_v^2 + \ldots \in \mathrm{Diff}(\mathbb{S}^d)$

$$P = \begin{cases} f^{i\nu-(d-1)/2-2}(\Box_{\mathrm{dS}} - (\frac{d-1}{2})^2 - \nu^2) f^{-i\nu+(d-1)/2} & \text{on dS,} \\ f^{i\nu-(d-1)/2-2}(-\Delta_{\mathbb{H}_\pm} + \nu^2 - \frac{(n-1)^2}{4}) f^{-i\nu+(d-1)/2} & \text{on } \mathbb{H}^d_\pm, \end{cases} \qquad ,$$

II. asymptotically de Sitter spaces

1. Asymptotically de Sitter spaces: a large class.

Even **asymptotically de Sitter** space:

$$g = \frac{df^2 - h(f^2, y, dy)}{f^2} \tag{1}$$

near the space-like boundary $\{f = 0\} = S_+ \cup S_-$, where $h(f^2, y, dy)$ is smooth.

- ✓ Stability of *de Sitter* space as solution of Einstein equations [Friedrichs '86]
  - ✓ *Minkowski*: [Christodoulou, Klainerman '93]
- ✓ Stability of *Kerr-de Sitter* [Hintz, Vasy '18]‘
  - ? *Kerr*: only linearized setting now [Häfner, Hintz, Vasy '20]

2. Conformal extensions survive.

Gluing under Guillarmou's evenness condition:

- $g = df^2 - h(f^2, y, dy)$ in $v < 0$ (f^2 times as. dS metric) ,
  $g = df^2 + h_\pm(f^2, y, dy)$ in $v > 0$ (f^2 times as. $\mathbb{H}^d_\pm$ metric)
  (close to conformal boundary
  $\{v = 0\} = \{f = 0\} =: S_+ \cup S_-$).
- good properties under *non-trapping* assumption.

The **conformally extended** operator [Vasy '13]

$$P = \begin{cases} f^{i\nu-(d-1)/2-2}(\Box_{f^2g} - (\frac{d-1}{2})^2 - \nu^2)f^{-i\nu+(d-1)/2} \text{ on } \{v < 0\}, \\ f^{i\nu-(d-1)/2-2}(-\Delta_{f^2g} + \nu^2 - \frac{(n-1)^2}{4})f^{-i\nu+(d-1)/2} \text{ on } \{v > 0\}, \end{cases},$$

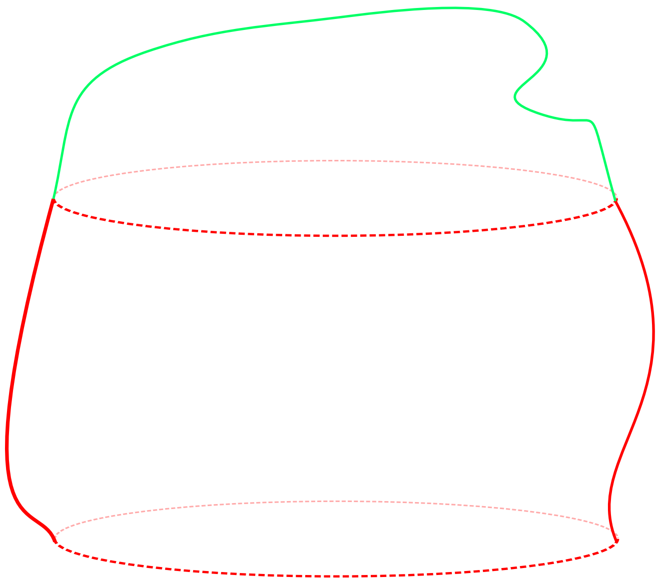
- Solutions of $Pu = 0$, $\mathrm{WF}(u) \subset N^*\{v = 0\}\}$ have asymptotics:

  $$u = (v + i0)^{-i\nu}a^+ + (v - i0)^{-i\nu}a^- + a, \quad a^+, a^-, a \in C^\infty(M).$$

- P has inverse P^{-1} on anisotropic Sobolev spaces! [Vasy '13]

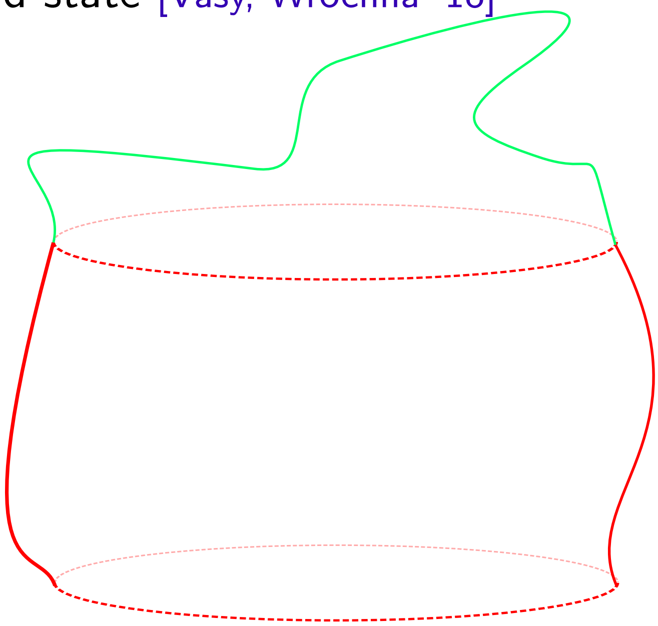
3. Geometries ↔ vacua.

To each conformal extension of a de Sitter space, we can associate a unique Hadamard state [Vasy, Wrochna '18]



3. Geometries $\leftrightarrow$ vacua.

To each conformal extension of a de Sitter space, we can associate a unique Hadamard state [Vasy, Wrochna '18]



A couple of questions:

- Hadamard states for linearized gravity on de Sitter spaces?
- Stability of de Sitter in semi-classical Gravity?
- Do no-hair theorems survive on general de Sitter spaces?
- Is $\Box_g$ essentially self-adjoint? If so, spectral action principle (see Viet's talk)?
- Non-linear wave equations on de Sitter as SPDEs?

A couple of questions:

- Hadamard states for linearized gravity on de Sitter spaces?
- Stability of de Sitter in semi-classical Gravity?
- Do no-hair theorems survive on general de Sitter spaces?
- Is $\Box_g$ essentially self-adjoint? If so, spectral action principle (see Viet's talk)?
- Non-linear wave equations on de Sitter as SPDEs?

Thank you for your attention!