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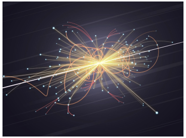
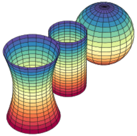
The Batalin-Vilkovisky construction for finite spectral triples

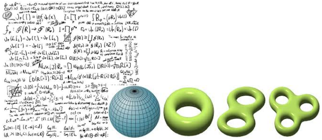
Roberta A. Iseppi

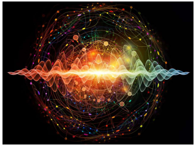
ESI Program: “Higher Structure and Field Theory”

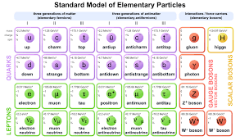
19th August 2022







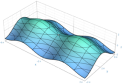




Plan and key concepts

Part 1: noncommutative geometry

- Basic idea: by translating geometrical concepts in algebraic terms, we obtain more general notions, used to describe also *noncommutative* as well as *discrete* situations



Differential geometry $\hookrightarrow$ Noncommutative geometry

- Related to physics: gauge theories from spectral triples

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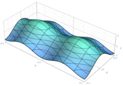

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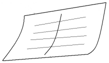

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- How to introduce the ghost fields: the BV extension
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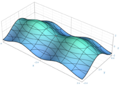

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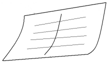


Part 3: The BV construction in the framework of noncommutative geometry

- The notions of BV spectral triple & the total spectral triple
- BV & BRST cohomology as Hochschild complexes

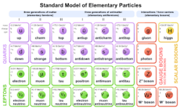
Why noncommutative geometry?

The concept of *quantization* brought two new ideas into the mathematical formalization of physics laws: discreteness & noncommutativity

Gravity:

- curvature of the spacetime
  $\rightsquigarrow$ continuous nature
- framework: Riemannian diff. geometry $\rightsquigarrow$ commutative

Fundamental interactions



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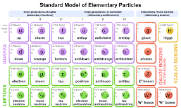

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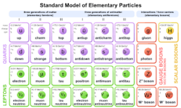

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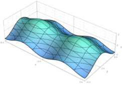
As any field of mathematics, also noncommutative geometry can be presented from different perspectives. One of them looks at it as a sort of extension of classical differential geometry:



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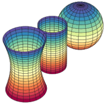
- topological spaces
- points & charts



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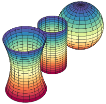

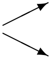

- $\mathcal{C}^*-$algebras $(\mathcal{A}, || \, ||, *)$

finite & infinite dim

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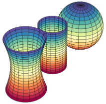

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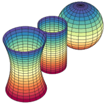

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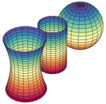

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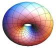

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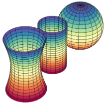

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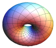

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noncom. mfd $\leftrightsquigarrow$ noncom. spectral tr. $(\mathcal{A}, \mathcal{H}, D)$

From spectral triples to gauge theories

Def: A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of:

- an involutive unital algebra $\mathcal{A}$, faithfully represented as operators on a Hilbert space $\mathcal{H}$, $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$
- a self-adjoint operator $D : \mathcal{H} \to \mathcal{H}$, with a compact resolvent $(\lambda I - D)^{-1}$, for $\lambda \in \mathbb{C}$, such that the commutators $[D, a]$ are bounded operators for each $a \in \mathcal{A}$

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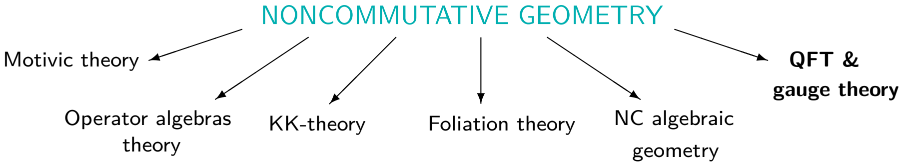

NONCOMMUTATIVE GEOMETRY

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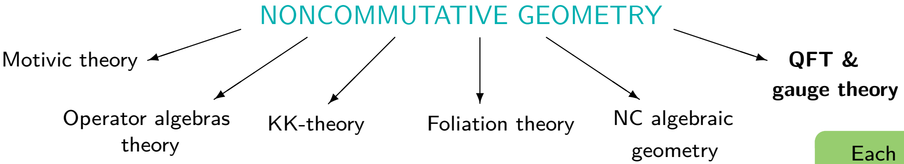

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Def. A gauge theory $(X_0, S_0, \mathcal{G})$ consists of:

- Conf. space $X_0 = \left\{ \varphi = \sum_j a_j[D, b_j] : \varphi^* = \varphi \right\} \rightsquigarrow$ inner fluctuations
- Action func. $S_0[D + \varphi] = Tr(f(D + \varphi)), \quad f \in \mathbb{R}[x] \rightsquigarrow$ spectral action
- $\mathcal{G} = \mathcal{U}(\mathcal{A}) \rightsquigarrow$ gauge group $=$ unitary elements in $\mathcal{A}$

s.t.

$\mathcal{G}$ acts on X_0 through an action $F : \mathcal{G} \times X_0 \to X_0$ and

$$S_0(F(g, \varphi)) = S_0(\varphi) \quad \forall \varphi \in X_0, \forall g \in \mathcal{G}$$

The Standard model as an almost-commut. spectral triple and more

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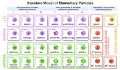
- Chamseddine - Connes [1997]: introduction of the *spectral action principle*
- Chamseddine, Connes, Marcolli [2007]: description of the full Standard Model of particles, with neutrino mixing and minimally coupled to gravity, from purely noncommutative geometrical objects:

$$M \times F$$

M = compact Riem. spin manifold
with canonical spectral triple

$$(\mathcal{C}^\infty(M), L^2(M,S), D_M, J_M, \gamma_M)$$

F= finite noncomm. space
with finite real spectral triple



$$(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \mathbb{C}^{96}, D_{SM}, J_{SM}, \gamma_{SM})$$

Gauge group:
$U(1) \times SU(2) \times SU(3)$

96 particles

Our model: induced by finite spectral triples to describe the particle content

- Chamseddine, Connes, van Suijlekom [2015]: they aim to go *beyond the Standand Model* by relaxing some conditions in the definition of a spectral triple $\rightsquigarrow$ Pati-Salam model

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Our setting

To agree with the notion of gauge theory coming from the framework of noncommutative geometry and in particular with the case of theories induced by finite spectral triple in the following a *gauge theory* (X_0, S_0) will not involve bundles and connections but it will be given by

$$X_0 = \text{field configuration space} \cong \mathbb{A}_{\mathbb{R}}^{n^2} \qquad S_0 : X_0 \to \mathbb{R}, \text{ action functional}, \in \mathcal{O}_{X_0}$$

and $\mathcal{G} \cong U(n)$ a group acting on X_0 through an action $F : \mathcal{G} \times X_0 \to X_0$ such that it holds:

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Induced by finite sp. triples: $(M_n(\mathbb{C}), \mathbb{C}^n, D)$

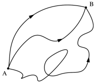
The BV construction: where it was discovered

Context: quantization of a gauge theory (X_0, S_0) via a path integral approach

$$\langle g \rangle = \frac{1}{Z}\int_{X_0} g e^{\frac{i}{\hbar}S_0}[d\mu], \qquad \text{where} \qquad Z := \int_{X_0} e^{\frac{i}{\hbar}S_0}[d\mu].$$

$\langle g \rangle$ ↓ expectation value of a reg. funct. g on X_0

Z ↓ partition function



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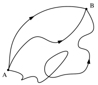

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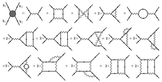

Feynman diagrams

Problem 1: the measure is <u>not well-defined</u> ⇝ perturbative approach with Feynman diagram

$$\int_{X_0} e^{\frac{i}{\hbar}S_0}[d\mu] \underset{\hbar\to 0}{\smile} \sum_{x_0\in\{\text{crit. pts } S_0\}} e^{\frac{i}{\hbar}S_0(x_0)} \;|\det S_0''(x_0)|^{-\frac{1}{2}} \; e^{\frac{\pi i}{4} sign(S_0''(x_0))}(2\pi\hbar)^{\frac{\dim X_0}{2}} \sum_{\Gamma} \frac{\hbar^{-\chi(\Gamma)}}{|Aut(\Gamma)|}\Phi_{\Gamma}.$$

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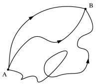

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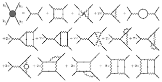


Feynman diagrams

Problem 1: the measure is not well-defined ⇝ perturbative approach with Feynman diagram

$$\int_{X_0} e^{\frac{i}{\hbar}S_0}[d\mu] \underset{\hbar\to 0}{\backsim} \sum_{x_0\in\{\text{crit. pts } S_0\}} e^{\frac{i}{\hbar}S_0(x_0)}\,|\det S_0''(x_0)|^{-\frac{1}{2}}\, e^{\frac{\pi i}{4} sign(S_0''(x_0))}(2\pi\hbar)^{\frac{\dim X_0}{2}} \sum_\Gamma \frac{\hbar^{-\chi(\Gamma)}}{|Aut(\Gamma)|}\Phi_\Gamma.$$

⚠ To apply the perturbative approach the critical points of the action functional S_0 have to be isolated and regular

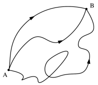
The BV construction: where it was discovered

Context: quantization of a gauge theory (X_0, S_0) via a path integral approach

$$\langle g \rangle = \frac{1}{Z}\int_{X_0} g e^{\frac{i}{\hbar}S_0}[d\mu], \qquad \text{where} \qquad Z := \int_{X_0} e^{\frac{i}{\hbar}S_0}[d\mu].$$

$\downarrow$ expectation value of a reg. funct. g on X_0

$\downarrow$ partition function



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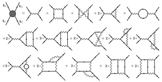

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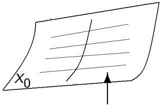
Problem 2: when we consider a gauge invariant action functional, the critical points appear in <u>orbits</u> ⇝ the quantization of gauge theories via a path integral approach is not straightforward



orbits of $\mathcal{G}$

The BV construction: where it was discovered (2)

How to eliminate these redundant symmetries without changing the underlying physical theory?

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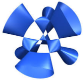

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How to eliminate these redundant symmetries without changing the underlying physical theory?

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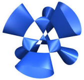

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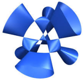

Def. A ghost field φ is characterized by:

- *ghost degree*: $\deg(\varphi) \in \mathbb{Z}$
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where $\epsilon(\varphi) = 0$ is bosonic/real and $\epsilon(\varphi) = 1$ is fermionic/Grassm. s.t. $\deg(\varphi) \equiv \epsilon(\varphi)$ *mod* $\mathbb{Z}/\mathbb{Z}2$

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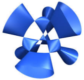

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A bit of history:

- Faddeev - Popov [1967]: in order to construct the perturbative path integral for the Yang-Mills theory, they proposed to eliminate the divergences of the integrand by introducing extra (non-physical) fields

$$(X_0, S_0) \xrightarrow{+\text{ auxiliary odd fields}} (\widetilde{X}, \widetilde{S})$$

where $\widetilde{S}$ has isolated critical points with non-degenerate Hessians.

The introduction of ghost fields

$\rightsquigarrow$ after the implementation of a gauge-fixing process, the stationary phase formula can be applied and the integral computed.

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- Batalin - Vilkovisky [1981/1983]: extended the configuration sp. X_0 by introducing also ghost fields of degree > 1 as well as the corresponding antifields/antighost fields

… and antighost fields

Def. Given a ghost field φ, the corresponding antighost field φ^* is characterized by:

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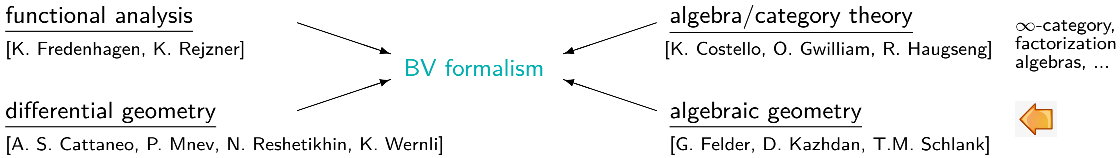

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[3] Each BV-extended theory naturally induces a cohomology complex: the BV-complex.

The BV-complex

Any BV-extended theory $(\widetilde{X}, \widetilde{S})$, with $\{\widetilde{S}, \widetilde{S}\} = 0$, naturally induces a BV cohom. complex with

- Cochain spaces: $\mathcal{C}^i(\widetilde{X}, d_{\tilde{S}}) = [\mathcal{O}_{\widetilde{X}}]^i \cong Sym^i_{\mathcal{O}_{X_0}}(W^*[1] \oplus X_0^*[1] \oplus W)$
- Coboundary op.: $d_{\tilde{S}} := \{\widetilde{S}, -\} : \mathcal{C}^\bullet(\widetilde{X}, d_{\tilde{S}}) \to \mathcal{C}^{\bullet+1}(\widetilde{X}, d_{\tilde{S}})$, $\quad d_{\tilde{S}}^2 = 0$

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➡ The BV construction can be viewed as a cohomological approach to the study of gauge symmetries. Indeed, these cohomology groups capture relevant physical information about the initial gauge theory (X_0, S_0):

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- How can we determine how many and which kind of ghost/antighost fields we have to introduce?
- How can we determine a suitable extended action functional $\tilde{S}$?

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- How can we determine how many and which kind of ghost/antighost fields we have to introduce?

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[Barnich, Brandt, Henneaux], [Felder, Kazhdan, Schlank]

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Related to the symmetries in the action S_0

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The BV extention, that is, the Koszul-Tate resolution of the Jacobian ideal $J(S_0)$

$$(X_0, S_0) \xrightarrow{\text{BV extension}} \widetilde{X} = \mathcal{F} \oplus \mathcal{F}^*[1]$$

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- alternating parity: real/ Grass.
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The number of lower degree new variables that have to be introduced at any step is determined by the higher order symmetries in S_0

Step 1 for the model: the extended configuration sp. & action (2)

The extended configuration space

We found a finite family of (anti)-ghost generators to extend X_0, which reflects the type of invariance of the action:

$$\widetilde{X} = W^*[1] \oplus X_0^* \oplus X_0 \oplus W \qquad \text{with} \qquad W = \underset{\substack{\text{deg } 1 \\ \sharp\binom{n^2-1}{2}}}{\langle C_{ij}\rangle_{i<j}} \oplus \underset{\substack{\text{deg } 2 \\ \sharp\binom{n^2-1}{3}}}{\langle E_{ijk}\rangle_{i<j<k}} \oplus \cdots \oplus \underset{\substack{\text{deg } n^2-2 \\ \sharp\binom{n^2-1}{n^2-1}}}{\langle E_{1\dots(n^2-1)}\rangle}$$

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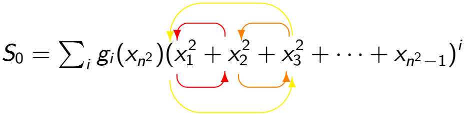

$$S_0 = \sum_i g_i(x_{n^2})(x_1^2 + x_2^2 + x_3^2 + \cdots + x_{n^2-1})^i$$

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$$\widetilde{S} = S_0 + \sum_{p=1}^{n^2-1} S_{-p}^{\,lin} + \sum_{q=1}^{\lceil \frac{p}{2} \rceil} S_{-p}^{\,cor,\ q} \quad \text{with} \quad S_{-p}^{\,lin} = \sum_{i_1<\cdots<i_{p+1}} \Big(\sum_{r=1}^{p+1} (-1)^{i_r} x_{i_r} E^*_{i_1\ldots\hat{i}_r\ldots i_{p+1}} \Big) E_{i_1\ldots i_{p+1}}$$

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- linear in the antifields/ antighost fields
- quadratic in the ghost fields
- exact and finite solution of the classical master equation

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initial th. BV-extended th.

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& \underset{\text{BV complex}}{\mathcal{C}^{\bullet}_{BV}(\widetilde{X}, d_{\tilde{S}})} \arrow[r, phantom, "\cong"] & \underset{\text{total complex}}{\mathcal{C}^{\bullet}_{BV}(X_t, d_{S_t})}
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Step 4 & 5: the gauge fixing and the BRST cohomology

Step 4: given a gauge-fixing fermion $\Psi \in [\mathcal{O}_{X_0 \oplus W}]^{-1}$, we perform the gauge-fixing procedure:

$$X_t|_\Psi := X_t|_{\varphi_i^* = \frac{\partial \Psi}{\partial \varphi_i}}$$

$$S_t|_\Psi = S_0 + S_{BV}(\varphi_i, \varphi_i^* = \frac{\partial \Psi}{\partial \varphi_i})$$

restrict to the Lagrangian submfd. defined by the gauge-fixing conditions $\varphi_i^* = \frac{\partial \Psi}{\partial \varphi_i}$.

The physically relevant quantities do not depend on the gauge-fixing fermion.

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The whole construction

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\underset{\text{initial gauge theory}}{(X_0, S_0)} \arrow[r, "+\text{ gh./anti-gh.}"] & \underset{\text{BV-extended th.}}{(\widetilde{X}, \widetilde{S})} \arrow[r, "+\text{ auxiliary flds}"] \arrow[d] & \underset{\text{total th}}{(X_t, S_t)} \arrow[r, "\text{gauge-fixing}"] \arrow[d] & \underset{\text{gauge-fixed th.}}{(X_t|_\Psi, S_t|_\Psi)} \arrow[d] \\
 & \underset{\text{BV complex}}{\mathcal{C}^\bullet_{BV}(\widetilde{X}, d_{\tilde{S}})} \arrow[r, phantom, "\cong"] & \underset{\text{total complex}}{\mathcal{C}^\bullet_{BV}(X_t, d_{S_t})} & \underset{\text{BRST complex}}{\mathcal{C}^\bullet_{BRST}(X_t, d_{S_t})_\Psi}
\end{tikzcd}$$

Step 4 & 5: the gauge fixing and the BRST cohomology

Step 4: given a gauge-fixing fermion $\Psi \in [\mathcal{O}_{X_0 \oplus W}]^{-1}$, we perform the gauge-fixing procedure:

$$X_t|_\Psi := X_t|_{\varphi_i^* = \frac{\partial \Psi}{\partial \varphi_i}} \qquad\qquad S_t|_\Psi = S_0 + S_{BV}(\varphi_i, \varphi_i^* = \frac{\partial \Psi}{\partial \varphi_i})$$

restrict to the Lagrangian submfd. defined by the gauge-fixing conditions $\varphi_i^* = \frac{\partial \Psi}{\partial \varphi_i}$.

The physically relevant quantities do not depend on the gauge-fixing fermion.

Step 5: $(X_t|_\Psi, S_t|_\Psi)$ still induces the BRST cohomology:

Cochain sp: $\mathcal{C}^i(X_t|_\Psi, d_{S_t}|_\Psi) = [Sym(X_0 \oplus W_t)]^i$ Coboundary op: $d_{S_t}|_\Psi := \{S_t, -\}|_{\varphi_i^* = \frac{\partial \Psi}{\partial \varphi_i}}$

Physically relevant:

- $H^0(\widetilde{X}, d_{\tilde{S}})$ $= \{$Cl. observable of $(X_0, S_0)\}$
- $H^i(\widetilde{X}, d_{\tilde{S}})$ $= \{$obstruction to quantization$\}$

The whole construction

$$\begin{tikzcd}
(X_0, S_0) \arrow[r, "+\text{ gh./anti-gh.}"] & (\widetilde{X}, \widetilde{S}) \arrow[r, "+\text{ auxiliary flds}"] \arrow[dd] & (X_t, S_t) \arrow[r, "\text{gauge-fixing}"] \arrow[dd] & (X_t|_\Psi, S_t|_\Psi) \arrow[dd] \\
\text{initial gauge theory} & \text{BV-extended th.} & \text{total th} & \text{gauge-fixed th.} \\
 & \mathcal{C}^\bullet_{BV}(\widetilde{X}, d_{\tilde{S}}) \arrow[r, phantom, "\cong"] & \mathcal{C}^\bullet_{BV}(X_t, d_{S_t}) & \mathcal{C}^\bullet_{BRST}(X_t, d_{S_t})_\Psi \\
 & \text{BV complex} & \text{total complex} & \text{BRST complex}
\end{tikzcd}$$

The goals

The discovery of the BRST complex made it evident that the ghost fields are not just a tool to solve the problem of computing path integrals but they encode the structure of the gauge symmetries of the theory.

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Questions and goals:

- Considering the deep connection existing between NCG and gauge theories, can the BV construction be described in terms of spectral triples?
- Can the BRST cohomology be related to other (better understood) cohomological theories?

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Questions and goals:

- Considering the deep connection existing between NCG and gauge theories, can the BV construction be described in terms of spectral triples?
- Can the BRST cohomology be related to other (better understood) cohomological theories?

$$\begin{tikzcd}
\begin{array}{c}(M_n(\mathbb{C}),\mathbb{C}^n,D,f)\\ \text{initial spectral triple}\end{array} \arrow[r] \arrow[dd, leftrightarrow] & \begin{array}{c}?\\ \text{BV spectral triple}\end{array} \arrow[r] \arrow[d] & \begin{array}{c}?\\ \text{total spectral triple}\end{array} \arrow[r] \arrow[d] & \begin{array}{c}?\\ \text{gauge-fixing fermion}\end{array} \arrow[d] & \text{noncommutative geometry} \\
 & \begin{array}{c}\mathcal{C}^{\bullet}_{BV}(\widetilde{X},d_{\tilde{S}}) \cong ?\\ \text{BV complex} \cong ??\end{array} & \begin{array}{c}\mathcal{C}^{\bullet}_{BV}(X_t,d_{S_t}) \cong ?\\ \text{BV tot complex} \cong ??\end{array} & \begin{array}{c}\mathcal{C}^{\bullet}_{BRST}(X_t,d_{S_t})|_{\Psi} \cong ?\\ \text{BRST complex} \cong ??\end{array} & \\
\begin{array}{c}(X_0,S_0)\\ \text{initial gauge theory}\end{array} \arrow[r] & \begin{array}{c}(\widetilde{X},\widetilde{S})\\ \text{extended theory}\end{array} \arrow[r] \arrow[u] & \begin{array}{c}(X_t,S_t)\\ \text{total theory}\end{array} \arrow[r] \arrow[u] & \begin{array}{c}(X_t|_{\Psi}\,,\,S_t|_{\Psi})\\ \text{gauge-fixed theory}\end{array} \arrow[u] & \text{BV construction}
\end{tikzcd}$$

Step 1: the BV construction in NCG (1)

Question 1: Can the BV extended theory $(\widetilde{X}, \widetilde{S})$ be described as a new BV-spectral triple? Can we encode the BV-extension process in the language of NCG?

$$\begin{tikzcd}
(\mathcal{A}_0, \mathcal{H}_0, D_0) \;\&\; f \arrow[r] \arrow[d] & ? \arrow[d] \\
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Problem 1: going from $\mathbb{C}$ to $\mathbb{R}$ $\rightsquigarrow$ real structure

Def. For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, a real structure J is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ such that:

- $J^2 = \pm Id \qquad JD = \pm DJ$
- $[a, Jb^* J^{-1}] = 0, \quad [[D, a], Jb^* J^{-1}] = 0, \quad \forall a, b \in \mathcal{A}$

Then $(\mathcal{A}, \mathcal{H}, D, J)$ is called a (odd) real spectral triple

KO-dim.	1	3	5	7
$J^2 = \pm Id$	1	-1	-1	1
$JD = \pm DJ$	-1	1	-1	1

Step 1: the BV construction in NCG (2)

Problem 2: the appearance of Grassmannian variables in $\widetilde{S}$ $\rightsquigarrow$ two notions of action.

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Spectral action: $S[D+\varphi] = Tr(f(D+\varphi));$

- for f a regular function (good decay, cut off...);
- for φ a self-adjoint element, with $\varphi = \sum_j a_j[D, b_j]$, $a_j, b_j \in \mathcal{A}$

Fermionic action: $S[\psi] = \frac{1}{2}\langle (J)\psi, D\psi \rangle,$

- for $\langle \ , \ \rangle$ the inner product structure on $\mathcal{H}$;
- for $\psi \in \mathcal{H}_f \subseteq \mathcal{H}$ ➡ we can impose a Grassmannian nature to the elements in $\mathcal{H}_f$

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Questions:

- ghost fields: where are they encoded in the initial spectral triple? Which role are they going to play in the BV-spectral triple?
- extended action: how can we determine $\widetilde{S}$ starting from (D_0, f)?

Step 1: the BV construction in NCG (3)

For the model:

$$\mathcal{H}_0 = \mathbb{C}^2 \xrightarrow{+\text{ ghost/anti-ghost fields}} \mathcal{H}_{BV} = \mathcal{Q} \oplus \mathcal{Q}^*[1]$$

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Symmetries of S_0

- 3 indep. ones among pairs of coord. ⇝ 3 ghost in deg 1
- 1 involving all three coordinates ⇝ 1 ghost in deg 2

$$S_0 = \sum_i g_i(x_4)(x_1^2 + x_2^2 + x_3^2)^i$$

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The BV-Hilbert space: $\mathcal{H}_{BV} = \mathcal{Q}^*[1] \oplus \mathcal{Q}$ for $\mathcal{Q} = [M_2(\mathbb{C})]_0 \oplus [M_2(\mathbb{C})]_1 \oplus [M_2(\mathbb{C})]_2$

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The real structure: $J_{BV} : \mathcal{H}_{BV} \to \mathcal{H}_{BV}$ with $J_{BV}(\varphi) = \varphi^\dagger$

Step 1: the BV construction in NCG (4)

Properties of the action $S_{BV} := \widetilde{S} - S_0$

- it has total degree 0
- it is 0 restricted to X_0
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Given:

$$Ad(x)/Ab(x) : M_2(\mathbb{C}) \to M_2(\mathbb{C}); \quad \phi \mapsto [\alpha(x_i), \phi]_{-/+},$$

The BV operator D_{BV}

$$D_{BV} = \begin{pmatrix} 0 & R \\ R^* & S \end{pmatrix} \quad \text{for} \quad R := \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}Ad(C) & Ab(X) \\ \frac{1}{2}Ad(C) & -\frac{1}{2}Ad(X) & 0 \end{pmatrix}, \quad S := \begin{pmatrix} 0 & Ad(X^*) & Ab(C^*) \\ Ad(X^*) & Ad(C^*) & 0 \\ Ab(C^*) & 0 & 0 \end{pmatrix}$$

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The algebra $\mathcal{A}_{BV}$: given $(\mathcal{H}_{BV}, D_{BV}, J_{BV})$ as defined above, the maximal unital algebra completing them to a spectral triple is

$$\mathcal{A}_{BV} = M_2(\mathbb{C}) = \mathcal{A}_0.$$

Step 1: the BV construction in NCG (5)

Theorem [R.I.]

For a finite spectral triple $(\mathcal{A}_0, \mathcal{H}_0, D_0) = (M_2(\mathbb{C}), \mathbb{C}^2, D_0)$ with induced gauge theory (X_0, S_0), the associated BV-spectral triple is:

$$(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) = (M_2(\mathbb{C}), \mathcal{Q} \oplus \mathcal{Q}^*[1], D_{BV}, J_{BV}))$$

In other words:

$$\widetilde{X} \cong \Omega^1(\mathcal{A}_0) + [\Omega^1(\mathcal{A}_0)]^*[1] + \mathcal{H}_{BV,f} \qquad S_{BV} := \widetilde{S} - S_0 = \frac{1}{2} S_{ferm}$$

$$\boxed{(\mathcal{A}_0, \mathcal{H}_0, D_0) \rightarrow (\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \rightarrow \dots}$$

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The structure of the BV-spectral triple gives a “geometric interpretation” of some physical aspects:

- the parity of fields, antifields, ghost fields and antighost fields is determined by the fact that:

  $$JD_1 = -D_1 J \qquad JD_2 = +D_2 J;$$

  Hence, the operator D_{BV} *partially commutes and partially anticommutes* with the real structure $J_{BV} \rightsquigarrow$ mixed KO-dim

Step 1: the BV construction in NCG (6)

Hence:

$$\boxed{(\mathcal{A}_0, \mathcal{H}_0, D_0) \ \& \ f \quad \xrightarrow{\text{BV construction}} \quad (\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})}$$

The structure of the BV spectral triple is:

- $\mathcal{A}_{BV} \rightsquigarrow$ it simply completes the triple $(\mathcal{H}_{BV}, D_{BV}, J_{BV})$ to a (mixed KO-dim.) real spectral triple;
- $\mathcal{H}_{BV} \rightsquigarrow$ encodes the ghost/antighost sector & it keeps track of the ghost degree;
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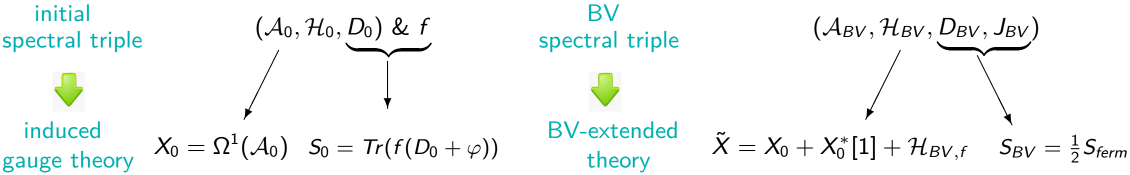

initial spectral triple

induced gauge theory

$$(\mathcal{A}_0,\mathcal{H}_0,\underbrace{D_0)\ \&\ f}$$

$X_0 = \Omega^1(\mathcal{A}_0)$ $\quad S_0 = Tr(f(D_0+\varphi))$

BV spectral triple

BV-extended theory

$$(\mathcal{A}_{BV},\mathcal{H}_{BV},\underbrace{D_{BV},J_{BV})}$$

$\tilde{X} = X_0 + X_0^*[1] + \mathcal{H}_{BV,f}$ $\quad S_{BV} = \frac{1}{2}S_{ferm}$

Step 2: the BV cohomology in NCG

As it happens for the BV-extended theory $(\widetilde{X}, \widetilde{S})$ that naturally induces the BV cohomology, we expect also the BV spectral triple to determine a cohomology complex

$(\widetilde{X}, \widetilde{S}) \longrightarrow \mathcal{C}^{\bullet}_{BV}(\widetilde{X}, d_{\tilde{S}})$ ➡ $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \dashrightarrow$ 

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Idea: to look at cohomology theories naturally appearing in the context of NCG

Riemannian diff. geom	NCG
manifold	spectral triple
differential forms	Hochschild homology
de Rham cohomology	cyclic homology

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Def. For a graded coalgebra $(\mathcal{B}, \Delta)$ and a bi-comodule $(\mathcal{M}, \omega_L, \omega_R)$ over it, the Hochschild complex is given by:

$$\mathcal{C}^q_H(\mathcal{M}, \mathcal{B}) := Hom_{\mathbb{K}}(\mathcal{M}, \mathcal{B}^{\otimes q}) \quad \& \quad d_H : \mathcal{C}^q_H(\mathcal{M}, \mathcal{B}) \to \mathcal{C}^{q+1}_H(\mathcal{M}, \mathcal{B}) \quad \text{with}$$

$$\begin{aligned} d_H(\varphi)|_m &:= (\varphi \otimes Id) \circ \omega_R|_m \\ &\quad + \sum_{i=1}^{q} (-1)^{|-|_1 + \dots + |-|_{i-1} + 1} (Id \otimes \dots \otimes \Delta \otimes \dots \otimes Id) \circ \varphi|_m + (-1)^{q+1} (Id \otimes \varphi) \circ \omega_L|_m \end{aligned}$$

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Try to determine:
$(\mathcal{B}, \Delta)$ & $(\mathcal{M}, \omega_L, \omega_R)$

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Step 2: the BV cohomology in NCG (2)

The bi-comodule $\mathcal{M}$

$$\mathcal{M} := \langle \Omega^1(\mathcal{A}_0) \rangle \cong \mathcal{O}_{X_0}, \qquad \text{for}$$

$$\Omega^1(\mathcal{A}_0) = \left\{ \varphi = \sum\nolimits_i a_j [D_0, b_j] : \varphi^* = \varphi, a_j, b_j \in \mathcal{A}_0 \right\}$$

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Left/Right module structure:

$$\omega_L(g) := \frac{1}{8} \sum_k \frac{\partial g}{\partial x_k^0} \left\{ \langle J_{BV}(-), D_{BV}(-) \rangle , x_k^0 \right\}$$

$$= -\omega_R(g)$$

The coalgebra $\mathcal{B}$

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$$\text{for} \quad \mathcal{Q}_f = [i\mathfrak{su}(2)]_0 \oplus [i\mathfrak{su}(2)]_1 \oplus [\mathfrak{u}(1)]_2,$$

Step 2: the BV cohomology in NCG (2)

The bi-comodule $\mathcal{M}$

$$\mathcal{M} := \langle \Omega^1(\mathcal{A}_0) \rangle \cong \mathcal{O}_{X_0}, \qquad \text{for}$$

$$\Omega^1(\mathcal{A}_0) = \left\{ \varphi = \sum_i a_j [D_0, b_j] : \varphi^* = \varphi, a_j, b_j \in \mathcal{A}_0 \right\}$$

Left/Right module structure:

$$\omega_L(g) := \frac{1}{8} \sum_k \frac{\partial g}{\partial x_k^0} \left\{ \langle J_{BV}(-), D_{BV}(-) \rangle, x_k^0 \right\}$$

$$= -\omega_R(g)$$

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Coproduct:

$$\Delta(\alpha) := \frac{1}{4} \sum_{r \neq 0, k} \left\{ \langle J_{BV}(-), D_{BV}(-) \rangle, a_k^r x_k^r \right\}$$

where $\{-,-\}$ is the antibracket structure induced by the pairing fields/anti-fields and x_k^r is a basis of $\mathcal{H}_{BV,f}$.

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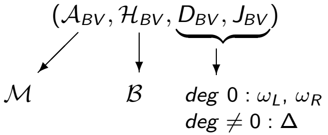
BV-spectral triple

Hochschild complex



$$(\mathcal{A}_{BV}, \mathcal{H}_{BV}, \underbrace{D_{BV}, J_{BV}})$$

$\mathcal{M}$ $\quad$ $\mathcal{B}$

deg $0 : \omega_L, \omega_R$

deg $\neq 0 : \Delta$

$$\boxed{(\mathcal{A}_0, \mathcal{H}_0, D_0) \to (\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \to (\mathcal{C}_H^\bullet(\mathcal{M}, \mathcal{B}), d_H) \to \ldots}$$

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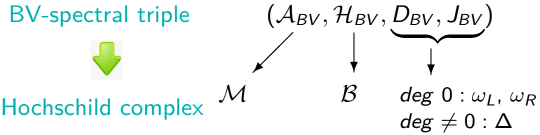

BV-spectral triple

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where $\{-,-\}$ is the antibracket structure induced by the pairing fields/anti-fields and x_k^r is a basis of $\mathcal{H}_{BV,f}$.

Theorem: [R.I.]

Given a BV spectral triple $(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$, let $(\mathcal{B}, \mathcal{M})$ be defined as above. Then it holds that:

$$(\mathcal{C}_H^\bullet(\mathcal{M}, \mathcal{B}), d_H) \cong (\mathcal{C}_{BV,f}^\bullet(\widetilde{X}, d_{\tilde{S}}), d_{\tilde{S},f}).$$

Step 3: auxiliary fields

$$(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \xrightarrow{+\text{ auxiliary fields/anti-fields}} (\mathcal{A}_t, \mathcal{H}_t, D_t, J_t)$$

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- $\mathcal{A}_t = \mathcal{A}_{BV} = \mathcal{A}_0 = M_2(\mathbb{C})$ $\rightsquigarrow$ no changes for the algebra
- $\mathcal{H}_t := \mathcal{H}_{BV} \oplus \mathcal{H}_{aux}$, $\mathcal{H}_{BV} = [M_2(\mathbb{C})]^{\oplus 6}$, $\mathcal{H}_{aux} = [M_2(\mathbb{C})]^{\oplus 12}$ where $\mathcal{H}_{aux,f} = \mathcal{R}_f \oplus \mathcal{R}_f^*[1]$

  for $\mathcal{R}_f := [\mathfrak{u}(1)]_{-2} \oplus [\mathfrak{u}(1) \oplus \mathfrak{su}(2)]_{-1} \oplus [i\mathfrak{su}(2) \oplus \mathfrak{u}(1)]_0 \oplus [\mathfrak{u}(1)]_1$.
- D_t

$$D_t = \begin{pmatrix} D_{BV} & 0 \\ 0 & D_{aux} \end{pmatrix} \quad \text{for} \quad D_{aux} = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$$

where
$$T := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \cdot Id \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Id & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i \cdot Id & 0 & 0 & 0 & 0 \end{pmatrix}$$

↓

determined by the pairing
between contractible pairs

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$$D_t = \begin{pmatrix} D_{BV} & 0 \\ 0 & D_{aux} \end{pmatrix} \quad \text{for} \quad D_{aux} = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix} \quad \text{where} \quad T := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i \cdot Id \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Id & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i \cdot Id & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\downarrow$

determined by the pairing between contractible pairs

Properties:

- The total spectral triple has the same structure as the BV spectral triple:

$$X_t = X_0 + X_0^*[1] + \mathcal{H}_{t,f} \qquad S_t = S_0 + \tfrac{1}{2}\langle J_t\varphi, D_t\varphi\rangle$$

- Completely determined by the *level of reducibility* of the theory, that is, by the maximal degree in $\mathcal{H}_{BV}$
- The induced Hochschild/BV complex is s.t. $\mathcal{C}^{\bullet}_{BV,t}(X_t, d_{S_t}) \cong C^{\bullet}_{H}(\mathcal{M}_t, \mathcal{B}_t)$ quasi isom. $\mathcal{C}^{\bullet}_{BV}(\widetilde{X}, d_{\tilde{S}}) \cong C^{\bullet}_{H}(\mathcal{M}, \mathcal{B})$

Step 4 & 5: gauge-fixing and BRST cohomology

$$\begin{tikzcd}
(\mathcal{A}_0, \mathcal{H}_0, D_0) \arrow[r, "+\text{ ghost fields}"] \arrow[d, "\text{(bosonic) induced th.}"] & (\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \arrow[rr, "+\text{ auxiliary flds}"] \arrow[d, "\text{(fermionic) induced th.}"] \arrow[ddr] & & (\mathcal{A}_t, \mathcal{H}_t, D_t, J_t) \arrow[d, "\text{(fermionic) induced th.}"] \arrow[ddr] & \\
(X_0, S_0) \arrow[r, "+\text{ ghost fields}"] & (\widetilde{X}, \widetilde{S}) \arrow[rr, "+\text{ auxiliary fields}"] \arrow[dd] & & (X_t, S_t) \arrow[dd] & \\
 & & (\mathcal{M}, \mathcal{B}) \arrow[d] & & (\mathcal{M}_t, \mathcal{B}_t) \arrow[d] \\
 & \mathcal{C}^\bullet_f(\widetilde{X}, d_{\widetilde{S}}) & \cong \mathcal{C}^\bullet_H(\mathcal{M}, \mathcal{B}) & \mathcal{C}^\bullet_f(X_t, d_{S_t}) & \cong \mathcal{C}^\bullet_H(\mathcal{M}_t, \mathcal{B}_t)
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 & & (\mathcal{M}, \mathcal{B}) \arrow[d] & & (\mathcal{M}_t, \mathcal{B}_t) \arrow[d] \\
 & \mathcal{C}_f^\bullet(\widetilde{X}, d_{\widetilde{S}}) \arrow[r, phantom, "\cong"] & \mathcal{C}_H^\bullet(\mathcal{M}, \mathcal{B}) & \mathcal{C}_f^\bullet(X_t, d_{S_t}) \arrow[r, phantom, "\cong"] & \mathcal{C}_H^\bullet(\mathcal{M}_t, \mathcal{B}_t)
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The construction
is all coherent
and consistent

Step 4 & 5: gauge-fixing and BRST cohomology

$$\begin{tikzcd}
(\mathcal{A}_0, \mathcal{H}_0, D_0) \arrow[r, "+\text{ ghost fields}"] \arrow[d, "\substack{\text{(bosonic)}\\ \text{induced th.}}"] & (\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \arrow[rr, "+\text{ auxiliary flds}"] \arrow[d, "\substack{\text{(fermionic)}\\ \text{induced th.}}"] \arrow[ddr] & & (\mathcal{A}_t, \mathcal{H}_t, D_t, J_t) \arrow[d, "\substack{\text{(fermionic)}\\ \text{induced th.}}"] \arrow[ddr] & \\
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The construction
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What it happens if we apply the gauge-fixing procedure?

What can we say about the BRST cohomology complex?

Step 4 & 5: gauge-fixing and BRST cohomology

$$\begin{tikzcd}
(\mathcal{A}_0, \mathcal{H}_0, D_0) \arrow[r, "+\text{ ghost fields}"] \arrow[d, "\substack{\text{(bosonic)}\\ \text{induced th.}}"] & (\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \arrow[rr, "+\text{ auxiliary flds}"] \arrow[d, "\substack{\text{(fermionic)}\\ \text{induced th.}}"] \arrow[ddr] & & (\mathcal{A}_t, \mathcal{H}_t, D_t, J_t) \arrow[d, "\substack{\text{(fermionic)}\\ \text{induced th.}}"] \arrow[ddr] & \\
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The construction
is all coherent
and consistent

What it happens if we apply the gauge-fixing procedure?

What can we say about the BRST cohomology complex?

Theorem: [R.I.]

Given a total spectral triple $(\mathcal{A}_t, \mathcal{H}_t, D_t, J_t)$ and a gauge-fixing fermion $\Psi \in [\mathcal{O}_{\mathcal{Q}_f \oplus \mathcal{R}_f}]^{-1}$, it holds that:

$$(\mathcal{C}^\bullet_f(X_t, d_{S_t})|_\Psi), d_{S_t}|_\Psi) = (\mathcal{C}^\bullet_H(\mathcal{M}_\Psi, \mathcal{B}_\Psi), d_H), \qquad \text{where}$$

$$\mathcal{B}_\Psi := \Omega^1(\mathcal{A}_0) \otimes \left[\mathcal{Q}_f \oplus \left\{\varphi_i^* = \frac{\partial \Psi}{\partial \varphi_j}\right\}\right] \oplus \mathcal{R}_f \oplus \left\{\chi_i^* = \frac{\partial \Psi}{\partial \chi_j}\right\}\Big] \text{ and } \mathcal{M}_\Psi := \left\langle \Omega^1(\mathcal{A}_0) \right\rangle$$

Where we are:

Noncommutative geometry

$$\begin{tikzcd}
\underset{\text{initial spectral triple}}{(M_n(\mathbb{C}), \mathbb{C}^n, D, f)} \arrow[r] \arrow[dd, leftrightarrow] & \underset{\text{BV spectral triple}}{(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})} \arrow[r] \arrow[d] & \underset{\text{total spectral triple}}{(\mathcal{A}_t, \mathcal{H}_t, D_t, J_t)} \arrow[r] \arrow[d] & \underset{\text{gauge-fixing fermion}}{\Psi \in \mathcal{F}^{-1}_{\mathcal{O}_{X_0}}(\mathcal{H}_{BV,f} \oplus \mathcal{H}_{aux,f})} \arrow[d] \\
 & \underset{\text{BV/ Hochschild complex}}{\mathcal{C}^\bullet_H(\mathcal{M}, \mathcal{B}) \cong \mathcal{C}^\bullet_{BV}(\widetilde{X}, d_{\tilde{S}})} & \underset{\text{BV tot / Hochschild complex}}{\mathcal{C}^\bullet_H(\mathcal{M}, \mathcal{B}_t) \cong \mathcal{C}^\bullet_{BV}(X_t, d_{S_t})} & \underset{\text{BRST/ Hochschild complex}}{\mathcal{C}^\bullet_H(\mathcal{M}, \mathcal{B}_\Psi) \cong \mathcal{C}^\bullet_{BRST}(X_t, d_{S_t})|_\Psi} \\
\underset{\text{initial gauge theory}}{(X_0, S_0)} \arrow[r] & \underset{\text{extended theory}}{(\widetilde{X}, \widetilde{S})} \arrow[r] \arrow[u] & \underset{\text{total theory}}{(X_t, S_t)} \arrow[r] \arrow[u] & \underset{\text{gauge-fixed theory}}{(X_t, S_t)|_\Psi} \arrow[u]
\end{tikzcd}$$

BV construction & BRST cohomology

Where we are:

Noncommutative geometry

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 & \underset{\text{BV/ Hochschild complex}}{\mathcal{C}^\bullet_H(\mathcal{M}, \mathcal{B}) \cong \mathcal{C}^\bullet_{BV}(\widetilde{X}, d_{\tilde{S}})} & \underset{\text{BV tot / Hochschild complex}}{\mathcal{C}^\bullet_H(\mathcal{M}, \mathcal{B}_t) \cong \mathcal{C}^\bullet_{BV}(X_t, d_{S_t})} & \underset{\text{BRST/ Hochschild complex}}{\mathcal{C}^\bullet_H(\mathcal{M}, \mathcal{B}_\Psi) \cong \mathcal{C}^\bullet_{BRST}(X_t, d_{S_t})|_\Psi} \\
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BV construction & BRST cohomology

Results and reached goals:

Where we are:

Noncommutative geometry

$$\begin{tikzcd}
\begin{array}{c}(M_n(\mathbb{C}), \mathbb{C}^n, D, f) \\ \text{initial spectral triple}\end{array} \arrow[r] \arrow[ddd, leftrightarrow] & \begin{array}{c}(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \\ \text{BV spectral triple}\end{array} \arrow[r] \arrow[d] & \begin{array}{c}(\mathcal{A}_t, \mathcal{H}_t, D_t, J_t) \\ \text{total spectral triple}\end{array} \arrow[r] \arrow[d] & \begin{array}{c}\Psi \in \mathcal{F}^{-1}_{\mathcal{O}_{X_0}}(\mathcal{H}_{BV,f} \oplus \mathcal{H}_{aux,f}) \\ \text{gauge-fixing fermion}\end{array} \arrow[d] \\
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 & & & \\
\begin{array}{c}(X_0, S_0) \\ \text{initial gauge theory}\end{array} \arrow[r] & \begin{array}{c}(\widetilde{X}, \widetilde{S}) \\ \text{extended theory}\end{array} \arrow[r] \arrow[uu] & \begin{array}{c}(X_t, S_t) \\ \text{total theory}\end{array} \arrow[r] \arrow[uu] & \begin{array}{c}(X_t, S_t)|_{\Psi} \\ \text{gauge-fixed theory}\end{array} \arrow[uu]
\end{tikzcd}$$

BV construction & BRST cohomology

Results and reached goals:

- Determined how the information about the BV extended theory can be extracted from the initial spectral triple

Where we are:

Noncommutative geometry

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\end{tikzcd}$$

BV construction & BRST cohomology

Results and reached goals:

- Determined how the information about the BV extended theory can be extracted from the initial spectral triple
- Established the (noncom.) geometrical role played by ghost/anti-ghost fields in the BV spectral triple

Where we are:

Noncommutative geometry

$$\begin{tikzcd}
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\end{tikzcd}$$

BV construction & BRST cohomology

Results and reached goals:

- Determined how the information about the BV extended theory can be extracted from the initial spectral triple
- Established the (noncom.) geometrical role played by ghost/anti-ghost fields in the BV spectral triple
- Discovered the relation existing between BV/BRST cohomology and Hochschild cohomology

Where we are:

Noncommutative geometry

$$\begin{tikzcd}
\begin{array}{c}(M_n(\mathbb{C}), \mathbb{C}^n, D, f)\\ \text{initial spectral triple}\end{array} \arrow[r] \arrow[dd, leftrightarrow] & \begin{array}{c}(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})\\ \text{BV spectral triple}\end{array} \arrow[r] \arrow[d] & \begin{array}{c}(\mathcal{A}_t, \mathcal{H}_t, D_t, J_t)\\ \text{total spectral triple}\end{array} \arrow[r] \arrow[d] & \begin{array}{c}\Psi \in \mathcal{F}^{-1}_{\mathcal{O}_{X_0}}(\mathcal{H}_{BV,f} \oplus \mathcal{H}_{aux,f})\\ \text{gauge-fixing fermion}\end{array} \arrow[d] \\
 & \begin{array}{c}\mathcal{C}^\bullet_H(\mathcal{M}, \mathcal{B}) \cong \mathcal{C}^\bullet_{BV}(\widetilde{X}, d_{\tilde{S}})\\ \text{BV/ Hochschild complex}\end{array} & \begin{array}{c}\mathcal{C}^\bullet_H(\mathcal{M}, \mathcal{B}_t) \cong \mathcal{C}^\bullet_{BV}(X_t, d_{S_t})\\ \text{BV tot / Hochschild complex}\end{array} & \begin{array}{c}\mathcal{C}^\bullet_H(\mathcal{M}, \mathcal{B}_\Psi) \cong \mathcal{C}^\bullet_{BRST}(X_t, d_{S_t})|_\Psi\\ \text{BRST/ Hochschild complex}\end{array} \\
\begin{array}{c}(X_0, S_0)\\ \text{initial gauge theory}\end{array} \arrow[r] & \begin{array}{c}(\widetilde{X}, \widetilde{S})\\ \text{extended theory}\end{array} \arrow[r] \arrow[u] & \begin{array}{c}(X_t, S_t)\\ \text{total theory}\end{array} \arrow[r] \arrow[u] & \begin{array}{c}(X_t, S_t)|_\Psi\\ \text{gauge-fixed theory}\end{array} \arrow[u]
\end{tikzcd}$$

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The construction is consistent and coherent

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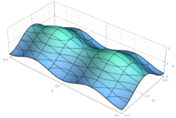
- To extend the construction to the case of 4-dim. spacetime, that is, to consider a spectral triple over $\mathcal{C}^\infty(M)$



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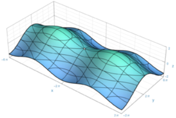


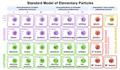
$$\begin{tikzcd} & M \times F \arrow[dl] \arrow[dr] & \\ \text{M = compact Riem. spin manifold} & & \text{F= finite noncomm. space} \end{tikzcd}$$

M = compact Riem. spin manifold with canonical spectral triple

$$(\mathcal{C}^\infty(M), L^2(M,S), D_M, J_M, \gamma_M)$$

F= finite noncomm. space with finite real spectral triple

$$(\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \mathbb{C}^{96}, D_{SM}, J_{SM}, \gamma_{SM})$$



$$S_0 = \underbrace{Tr(f(D_M/\Lambda))}_{\text{spectral action}} + \underbrace{\langle J_{SM}(\psi), D_{SM}\psi\rangle}_{\text{fermionic action}}$$