The Batalin-Vilkovisky construction for finite spectral triples

Roberta A. Iseppi

ESI Program: “Higher Structure and Field Theory”

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Plan and key concepts

Part 1: noncommutative geometry

- Basic idea: by translating geometrical concepts in algebraic terms, we obtain more general notions, used to describe also noncommutative as well as discrete situations.

Differential geometry $\rightarrow$ Noncommutative geometry

- Related to physics: gauge theories from spectral triples.
Plan and key concepts

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Part 2: the BV construction
► Motivation: extra symmetries in the context of path integral quantization of gauge theory
► How to introduce the ghost fields: the BV extension
► The BV-BRST cohomology
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Part 3: The BV construction in the framework of noncommutative geometry

▷ The notions of BV spectral triple & the total spectral triple
▷ BV & BRST cohomology as Hochschild complexes
Why noncommutative geometry?

The concept of *quantization* brought two new ideas into the mathematical formalization of physics laws: *discreteness* & *noncommutativity*

**Gravity:**
- curvature of the spacetime $\rightsquigarrow$ continuous nature
- framework: Riemannian diff. geometry $\rightsquigarrow$ commutative

**Fundamental interactions**
- mediated by particle $\rightsquigarrow$ discrete nature
- framework: self-adjoint op. $\rightsquigarrow$ non-commutative
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**Goal:** provide a unifying background \( \iff \) **Idea:** To introduce more general notions, able to capture also *noncommutative* as well as *discrete* situations.
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As any field of mathematics, also noncommutative geometry can be presented from different perspectives. One of them looks at it as a sort of extension of classical differential geometry:

Differential geometry \[\rightarrow\] Noncommutative geometry
How noncommutative geometry?

*Key idea:* to generalize the classical notion of *manifold* by translating the geometrical concept in algebraic terms.
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Topology and differential geometry:

- topological spaces
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Algebra and operator algebra:
- $C^*$-algebras $(\mathcal{A}, ||||, *)$
- finite & infinite dim
- comm. & noncommutative
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### Topology:
- loc. compact
- Hausdorff sp
- $X$

### Algebra:
- $C^*$–alg.
- $\mathcal{A}, || ||, *$

### Example:
- $C_0(X)$

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The BV construction for finite spectral triples

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Noncommutative geometry and QFT
The BV construction and the BV/BRST cohomology
BV construction in NCG
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Metric:
- compact Riem. spin manifold
- canonical spectral triple $(\mathcal{C}_\infty(M), L^2(M, S), D, J, \gamma)$

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- compact Riem. spin manifold
- canonical spectral triple $(C^\infty(M), L^2(M, S), D, J, \gamma)$
- noncom. mfd
- noncom. spectral tr. $(\mathcal{A}, \mathcal{H}, D)$
From spectral triples to gauge theories

Def: A spectral triple \((\mathcal{A}, \mathcal{H}, D)\) consists of:

- an involutive unital algebra \(\mathcal{A}\), faithfully represented as operators on a Hilbert space \(\mathcal{H}\), \(\mathcal{A} \subseteq B(\mathcal{H})\)
- a self-adjoint operator \(D : \mathcal{H} \rightarrow \mathcal{H}\), with a compact resolvent \((\lambda I - D)^{-1}\), for \(\lambda \in \mathbb{C}\), such that the commutators \([D, a]\) are bounded operators for each \(a \in \mathcal{A}\)
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**NONCOMMUTATIVE GEOMETRY**

Motivic theory → Operator algebras theory → KK-theory → Foliation theory → NC algebraic geometry

Def. A **gauge theory** \((X_0, S_0, G)\) consists of:

- Conf. space \(X_0 = \{ \varphi = \sum_j a_j[D, b_j] : \varphi^* = \varphi \} \leadsto \text{inner fluctuations}
- Action func. \(S_0[D + \varphi] = \text{Tr}(f(D + \varphi)), \ f \in \mathbb{R}[x] \leadsto \text{spectral action}
- \(G = \mathcal{U}(\mathcal{A}) \leadsto \text{gauge group} = \text{unitary elements in } \mathcal{A}\)

s.t. \(G\) acts on \(X_0\) through an action \(F: G \times X_0 \rightarrow X_0\) and \(S_0(F(g, \varphi)) = S_0(\varphi) \ \forall \varphi \in X_0, \forall g \in G\)
The Standard model as an almost-commut. spectral triple and more

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Does all of this describe any physically relevant model?

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- Chamseddine - Connes [1997]: introduction of the spectral action principle
- Chamseddine, Connes, Marcolli [2007]: description of the full Standard Model of particles, with neutrino mixing and minimally coupled to gravity, from purely noncommutative geometrical objects:

\[ M = \text{compact Riem. spin manifold with canonical spectral triple} \]
\[ (C^\infty(M), L^2(M, S), D_M, J_M, \gamma_M) \]

\[ F = \text{finite noncomm. space with finite real spectral triple} \]
\[ (\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \mathbb{C}^{96}, D_{SM}, J_{SM}, \gamma_{SM}) \]

Gauge group:
\[ U(1) \times SU(2) \times SU(3) \]

96 particles
Our model: induced by finite spectral triples to describe the particle content

- Chamseddine, Connes, van Suijlekom [2015]: they aim to go *beyond the Standard Model* by relaxing some conditions in the definition of a spectral triple \(\rightsquigarrow\) Pati-Salam model
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*Note:* in this noncommutative geometrical setting, even though the finite case might look as a mathematically simpler but physically less interesting setting to consider, the previous results showed that it is the finite spectral triple term which encodes the particle content.
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**Our setting**

To agree with the notion of gauge theory coming from the framework of noncommutative geometry and in particular with the case of theories induced by finite spectral triple in the following a gauge theory \((X_0, S_0)\) will not involve bundles and connections but it will be given by

\[
X_0 = \text{field configuration space} \cong A_{\mathbb{R}}^{n^2} \quad S_0 : X_0 \to \mathbb{R}, \text{ action functional, } \in \mathcal{O}_{X_0}
\]

and \(G \cong U(n)\) a group acting on \(X_0\) through an action \(F : G \times X_0 \to X_0\) such that it holds:

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Induced by finite sp. triples:

$(M_n(\mathbb{C}), \mathbb{C}^n, D)$
The BV construction: where it was discovered

Context: quantization of a gauge theory \((X_0, S_0)\) via a path integral approach

\[
\langle g \rangle = \frac{1}{Z} \int_{X_0} g e^{i S_0 / \hbar} [d\mu], \quad \text{where} \quad Z := \int_{X_0} e^{i S_0 / \hbar} [d\mu].
\]

- Expectation value of a regular function \(g\) on \(X_0\)
- Partition function
- Path integral
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Problem 1: the measure is not well-defined \(\sim\) perturbative approach with Feynman diagram

\[
\int_{X_0} e^{i\hbar S_0}[d\mu] \sim \sum_{x_0 \in \{\text{crit. pts } S_0\}} e^{i\hbar S_0(x_0)} \left| \det S_0''(x_0) \right|^{-\frac{1}{2}} e^{\frac{\pi i}{4} \text{sign}(S_0''(x_0))} \left(2\pi\hbar\right)^{\text{dim} X_0 / 2} \sum \frac{\hbar^2 - \chi(\Gamma)}{|\text{Aut}(\Gamma)|} \Phi_\Gamma.
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⚠️ To apply the perturbative approach the critical points of the action functional \(S_0\) have to be isolated and regular

**Problem 2:** when we consider a gauge invariant action functional, the critical points appear in orbits \(\leadsto\) the quantization of gauge theories via a path integral approach is not straightforward
The BV construction: where it was discovered (2)

How to eliminate these redundant symmetries without changing the underlying physical theory?
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How to eliminate these redundant symmetries without changing the underlying physical theory?

- take the quotient w.r.t. the action of the group
- \[ \rightsquigarrow \] get orbifolds or even more complicated objects
- add extra auxiliary variables \[ \rightsquigarrow \] ghost fields

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How to eliminate these redundant symmetries without changing the underlying physical theory?

- take the quotient w.r.t. the action of the group → get orbifolds or even more complicated objects
- add extra auxiliary variables → ghost fields

**Def.** A *ghost field* $\varphi$ is characterized by:

- **ghost degree**: $\deg(\varphi) \in \mathbb{Z}$
- **parity**: $\epsilon(\varphi) \in \{0, 1\}$

where $\epsilon(\varphi) = 0$ is bosonic/real and $\epsilon(\varphi) = 1$ is fermionic/Grassm. s.t. $\deg(\varphi) \equiv \epsilon(\varphi) \mod \mathbb{Z}/\mathbb{Z}_2$
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A bit of history:

- Faddeev - Popov [1967]: in order to construct the perturbative path integral for the Yang-Mills theory, they proposed to eliminate the divergences of the integrand by introducing extra (non-physical) fields

\[
(X_0, S_0) \quad \xrightarrow{+ \text{ auxiliary odd fields}} \quad (\tilde{X}, \tilde{S})
\]

where \( \tilde{S} \) has isolated critical points with non-degenerate Hessians.
The introduction of ghost fields

\[ \text{\textbullet} \text{ after the implementation of a gauge-fixing process, the stationary phase formula can be applied and the integral computed.} \]
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  \textit{Note:} with only ghost fields of degree 1:
  \[ \text{BRST cohomology} = \text{Chevalley-Eilenberg cohomology} \]
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▶ Batalin - Vilkovisky [1981/1983]: extended the configuration sp. \( X_0 \) by introducing also ghost fields of degree \( > 1 \) as well as the corresponding antifields/antighost fields
... and antighost fields

Def. Given a ghost field \( \varphi \), the corresponding antighost field \( \varphi^* \) is characterized by:

\[
\deg(\varphi^*) = -\deg(\varphi) - 1 \quad \quad \epsilon(\varphi^*) \equiv \epsilon(\varphi) + 1 \pmod{\mathbb{Z}/2\mathbb{Z}}
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*Initial data:* a gauge theory

- $X_0$: fields configuration space
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- $G$: gauge gr., $F: G \times X_0 \to X_0$
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**BV extended theory**

- $\tilde{X} = X_0 \cup \{\text{ghost/antigh., bosons & fermions}\}$
- $\tilde{S} = S_0 + \text{terms involving ghost/antigh}$
  s.t. 
  
  \[
  \text{there are no more gauge-redundancies}
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---

**BV formalism**

- Functional analysis
  - [K. Fredenhagen, K. Rejzner]
- Differential geometry
  - [A. S. Cattaneo, P. Mnev, N. Reshetikhin, K. Wernli]

**BV formalism**

- Algebra/category theory
  - [K. Costello, O. Gwilliam, R. Haugseng]
- Algebraic geometry
  - [G. Felder, D. Kazhdan, T. M. Schlank]
The key step, in the algebraic geometric approach

\[(X_0, S_0) \xrightarrow{\text{BV extension}} (\tilde{X}, \tilde{S})\]
**The key step, in the algebraic geometric approach**

*Initial data:* a gauge theory

- $X_0$: vector sp $\cong \mathbb{A}_\mathbb{R}^2$
- $S_0 \in \mathcal{O}_{X_0} = \mathbb{R}[x_1, \ldots, x_{n^2}]$
- $g = u(n)$

$(X_0, S_0)$ \xrightarrow{BV\ extension} $(\tilde{X}, \tilde{S})$
The key step, in the algebraic geometric approach

Initial data: a gauge theory

- $X_0$: vector sp $\cong \mathbb{A}^{n^2}_\mathbb{R}$
- $S_0 \in \mathcal{O}_{X_0} = \mathbb{R}[x_1, \ldots, x_{n^2}]$
- $g = u(n)$

$(X_0, S_0)$ \[\xrightarrow{\text{BV extension}}\] $(\tilde{X}, \tilde{S})$

BV extended theory

- $\tilde{X} = \bigoplus_{i \in \mathbb{Z}} [\tilde{X}]^i$, $\mathbb{Z}$-graded super-vect. sp., $\tilde{X} = \mathcal{F} \oplus \mathcal{F}^*[1]$, $[\tilde{X}]^0 = X_0$
- Graded locally free $\mathcal{O}_{X_0}$-mod.
- With hom. comp. of finite rank
- $\tilde{S} \in [\mathcal{O}_{\tilde{X}}]^0$,
  s.t. $\tilde{S}|_{X_0} = S_0$ & $\{\tilde{S}, \tilde{S}\} = 0$
  sol. to the classical master equation
- 1-degree Poisson struc. on $\mathcal{O}_{\tilde{X}}$
  $\{,\}: \mathcal{O}_{\tilde{X}}^n \times \mathcal{O}_{\tilde{X}}^m \to \mathcal{O}_{\tilde{X}}^{n+m+1}$
  $\{\varphi^*_i, \varphi_j\} = \delta_{ij}$
The key step, in the algebraic geometric approach

*Initial data:* a gauge theory

- $X_0$: vector sp $\cong \mathbb{A}^2_{\mathbb{R}}$
- $S_0 \in \mathcal{O}_{X_0} = \mathbb{R}[x_1, \ldots, x_{n^2}]$
- $g = u(n)$

*BV extension* $(X_0, S_0) \rightarrow (\tilde{X}, \tilde{S})$

*BV extended theory* 

- $\tilde{X} = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}$-graded super-vect. sp., $\tilde{X} = \mathcal{F} \oplus \mathcal{F}^*[1]$, $[\tilde{X}]^0 = X_0$
- graded locally free $\mathcal{O}_{X_0}$-mod. with hom. comp. of finite rank
- $\tilde{S} \in [\mathcal{O}_{\tilde{X}}]^0$, s.t. $\tilde{S}|_{X_0} = S_0$ \& $\{\tilde{S}, \tilde{S}\} = 0$ sol. to the classical master equation

1-degree Poisson strut. on $\mathcal{O}_{\tilde{X}}$

$\{\varphi_i^*, \varphi_j\} = \delta_{ij}$

**Note:**

[1] While $\mathcal{F}$ accounts for the ghost field sector, $\mathcal{F}^*[1]$ describes the anti-ghost content for each ghost field introduced we also include the corresponding anti-ghost field.
The key step, in the algebraic geometric approach

**Initial data:** a gauge theory

- \( X_0 \): vector sp \( \cong \mathbb{A}^n_{\mathbb{R}} \)
- \( S_0 \in \mathcal{O}_{X_0} = \mathbb{R}[x_1, \ldots, x_n] \)
- \( g = u(n) \)

\[ (X_0, S_0) \xrightarrow{\text{BV extension}} (\tilde{X}, \tilde{S}) \]

**BV extended theory**

- \( \tilde{X} = \bigoplus_{i \in \mathbb{Z}} [\tilde{X}]^i \), \( \mathbb{Z} \)-graded super-vect. sp., \( \tilde{X} = F \oplus F^*[1], \quad [\tilde{X}]^0 = X_0 \)
  
  graded locally free \( \mathcal{O}_{X_0} \)-mod.
  
  with hom. comp. of finite rank

- \( \tilde{S} \in [\mathcal{O}_{\tilde{X}}]^0 \), s.t. \( \tilde{S}|_{X_0} = S_0 \) & \( \{ \tilde{S}, \tilde{S} \} = 0 \) sol. to the classical master equation

\[ \{ , \} : \mathcal{O}^n_{\tilde{X}} \times \mathcal{O}^m_{\tilde{X}} \to \mathcal{O}^{n+m+1}_{\tilde{X}} \quad \{ \varphi_i^* \varphi_j \} = \delta_{ij} \]

**Note:**

[1] While \( F \) accounts for the ghost field sector, \( F^*[1] \) describes the anti-ghost content \( \rightsquigarrow \) for each ghost field introduced we also include the corresponding anti-ghost field.

[2] In degree 0 we have only the initial (physical) fields. If we restrict to \( X_0 \), we get back the initial (physically relevant) theory.
The key step, in the algebraic geometric approach

**Initial data:** a gauge theory

- $X_0$: vector sp $\cong \mathbb{A}^n_\mathbb{R}$
- $S_0 \in \mathcal{O}_{X_0} = \mathbb{R}[x_1, \ldots, x_n]$
- $g = u(n)$

**BV extension**

$$(X_0, S_0) \xrightarrow{\text{BV extension}} (\tilde{X}, \tilde{S})$$

**BV extended theory**

- $\tilde{X} = \bigoplus_{i \in \mathbb{Z}} [\tilde{X}]^i$, $\mathbb{Z}$-graded super-vect. sp., $\tilde{X} = \mathcal{F} \oplus \mathcal{F}^*[1]$, $[\tilde{X}]^0 = X_0$
- $\tilde{S} \in [\mathcal{O}_{\tilde{X}}]^0$, s.t. $\tilde{S}|_{X_0} = S_0$ & $\{\tilde{S}, \tilde{S}\} = 0$ sol. to the classical master equation

Note:

[1] While $\mathcal{F}$ accounts for the ghost field sector, $\mathcal{F}^*[1]$ describes the anti-ghost content for each ghost field introduced we also include the corresponding anti-ghost field.

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The BV-complex

Any BV-extended theory \((\tilde{X}, \tilde{S})\), with \(\{\tilde{S}, \tilde{S}\} = 0\), naturally induces a **BV cohom. complex** with

- **Cochain spaces**: \(C^i(\tilde{X}, d_{\tilde{S}}) = \big[\mathcal{O}_{\tilde{X}}\big]^i \cong \text{Sym}^i_{\mathcal{O}_{\tilde{X}_0}}(W^*[1] \oplus X_0^*[1] \oplus W)\)

- **Coboundary op.**: \(d_{\tilde{S}} := \{\tilde{S}, -\} : C^\bullet(\tilde{X}, d_{\tilde{S}}) \to C^{\bullet+1}(\tilde{X}, d_{\tilde{S}}), \quad d_{\tilde{S}}^2 = 0\)
The BV-complex

Any BV-extended theory \((\tilde{X}, \tilde{S})\), with \(\{\tilde{S}, \tilde{S}\} = 0\), naturally induces a \textbf{BV cohom. complex} with

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The BV construction can be viewed as a cohomological approach to the study of gauge symmetries. Indeed, these cohomology groups capture relevant physical information about the initial gauge theory \((X_0, S_0)\):

\[
H^0_{BV}(\tilde{X}, d_{\tilde{S}}) = \{\text{classical observables}\}
\]
Any BV-extended theory \((\tilde{X}, \tilde{S})\), with \(\{\tilde{S}, \tilde{S}\} = 0\), naturally induces a BV cohom. complex with

- **Cochain spaces:**
  \[
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  \]

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  \[
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How can we determine how many and which kind of ghost/antighost fields we have to introduce?

How can we determine a suitable extended action functional \(\tilde{S}\)?
Any BV-extended theory \((\tilde{X}, \tilde{S})\), with \({\tilde{S}, \tilde{S}} = 0\), naturally induces a BV cohom. complex with

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- **Coboundary op.:** \(d_{\tilde{S}} := \{\tilde{S}, -\} : C^\bullet(\tilde{X}, d_{\tilde{S}}) \rightarrow C^{\bullet+1}(\tilde{X}, d_{\tilde{S}}), \quad d_{\tilde{S}}^2 = 0\)

The BV construction can be viewed as a cohomological approach to the study of gauge symmetries. Indeed, these cohomology groups capture relevant physical information about the initial gauge theory \((X_0, S_0)\):

\[H^0_{BV}(\tilde{X}, d_{\tilde{S}}) = \{\text{classical observables}\}\]

- How can we determine how many and which kind of ghost/antighost fields we have to introduce?
  - \(\leadsto\) compute the Koszul-Tate resolution of the Jacobian ideal \(J(S_0)\)
- How can we determine a suitable extended action functional \(\tilde{S}\)?
  - \(\leadsto\) approximation procedure

[Barnich, Brandt, Henneaux], [Felder, Kazhdan, Schlank]
The BV-complex

Any BV-extended theory \((\tilde{X}, \tilde{S})\), with \(\{\tilde{S}, \tilde{S}\} = 0\), naturally induces a BV cohom. complex with

- Cochain spaces: \(C^i(\tilde{X}, d_{\tilde{S}}) = [O_{\tilde{X}}]^i \cong \text{Sym}^i_{O_{X_0}} (W^*[1] \oplus X_0^*[1] \oplus W)\)
- Coboundary op.: \(d_{\tilde{S}} := \{\tilde{S}, -\} : C^*(\tilde{X}, d_{\tilde{S}}) \to C^{*+1}(\tilde{X}, d_{\tilde{S}}), \quad d_{\tilde{S}}^2 = 0\)

The BV construction can be viewed as a cohomological approach to the study of gauge symmetries. Indeed, these cohomology groups capture relevant physical information about the initial gauge theory \((X_0, S_0)\):

\[H^0_{BV}(\tilde{X}, d_{\tilde{S}}) = \{\text{classical observables}\}\]

How can we determine how many and which kind of ghost/antighost fields we have to introduce?

\(\sim\) compute the Koszul-Tate resolution of the Jacobian ideal \(J(S_0)\)

How can we determine a suitable extended action functional \(\tilde{S}\)?

\(\sim\) approximation procedure

[Barnich, Brandt, Henneaux], [Felder, Kazhdan, Schlank]

Related to the symmetries in the action \(S_0\)
The BV extension, that is, the Koszul-Tate resolution of the Jacobian ideal $J(S_0)$

$$(X_0, S_0) \xrightarrow{\text{BV extension}} \tilde{X} = \mathcal{F} \oplus \mathcal{F}^* [1]$$
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To determine the (anti)-ghost fields content:
compute the Koszul-Tate resolution of the Jacobian ideal $J(S_0) = \langle \partial_1 S_0, \ldots, \partial_{n^2} S_0 \rangle$ over the ring $\mathcal{O}_{X_0}$
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To determine the (anti)-ghost fields content:
compute the Koszul-Tate resolution of the Jacobian ideal $J(S_0) = \langle \partial_1 S_0, \ldots, \partial_{n^2} S_0 \rangle$ over the ring $\mathcal{O}_{X_0}$

By introducing new variables of alternating parity: real/ Grass.

decreasing degree

construct a free resolution of $\mathcal{O}_{X_0}/J(S_0)$ that is a diff. $\mathcal{O}_{X_0}$-algebra $(A, d)$
The BV extension, that is, the Koszul-Tate resolution of the Jacobian ideal $J(S_0)$

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complex of fin. gen. $\mathcal{O}_{X_0}$-modules with a coboundary op. $d$

$(\{A_i\}, d_i)_{i \in \mathbb{Z}_{\leq 0}}$
The BV extension, that is, the Koszul-Tate resolution of the Jacobian ideal $J(S_0)$

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By introducing new variables of alternating parity: real/ Grass. decreasing degree

construct a free resolution of $\mathcal{O}_{X_0}/J(S_0)$ that is a diff. $\mathcal{O}_{X_0}$-algebra $(A, d)$

the sequence is exact:
$\begin{align*}
&\Rightarrow H^0(A) = \mathcal{O}_{X_0}/J(S_0) \\
&\Rightarrow H^k(A_k) = 0, \quad k < 0
\end{align*}$

complex of fin. gen. $\mathcal{O}_{X_0}$-modules with a coboundary op. $d$

$$(\{A_i\}, d_i)_{i \in \mathbb{Z} \leq 0}$$
The BV extension, that is, the Koszul-Tate resolution of the Jacobian ideal \( J(S_0) \)

\[(X_0, S_0) \xrightarrow{\text{BV extension}} \tilde{X} = \mathcal{F} \oplus \mathcal{F}^*[1]\]

To determine the (anti)-ghost fields content:
compute the **Koszul-Tate resolution** of the Jacobian ideal \( J(S_0) = \langle \partial_1 S_0, \ldots, \partial_{n^2} S_0 \rangle \) over the ring \( O_{X_0} \)

By introducing new variables of alternating parity: real/ Grass.

construct a **free resolution** of \( O_{X_0}/J(S_0) \) that is a diff. \( O_{X_0} \)-algebra \((A, d)\)

\[
\cdots A_{-n} \xrightarrow{d_{-n}} \cdots A_{-1} = O_{X_0} \langle x_1^*, \ldots, x_{n^2}^* \rangle \xrightarrow{d_{-1}} A_0 \xrightarrow{\pi} O_{X_0}/J(S_0) \rightarrow 0
\]

anti-gh. deg \(-n\) anti-gh. deg \(-1\)
The BV extension, that is, the Koszul-Tate resolution of the Jacobian ideal $J(S_0)$

$$(X_0, S_0) \xrightarrow{\text{BV extension}} \tilde{X} = F \oplus F^*[1]$$

To determine the (anti)-ghost fields content:
compute the Koszul-Tate resolution of the Jacobian ideal $J(S_0) = \langle \partial_1 S_0, \ldots, \partial_n S_0 \rangle$ over the ring $O_{X_0}$

By introducing new variables of alternating parity: real/ Grass.

construct a free resolution of $O_{X_0}/J(S_0)$ that is a diff. $O_{X_0}$-algebra $(A, d)$

the sequence is exact:

$\begin{align*}
H^0(A) &= O_{X_0}/J(S_0) \\
H^k(A_k) &= 0, \quad k < 0
\end{align*}$

complex of fin. gen. $O_{X_0}$-modules with a coboundary op. $d$

$$(\{A_i\}, d_i)_{i \in \mathbb{Z} \leq 0}$$

$\cdots A_{-n} \xrightarrow{d_{-n}} \cdots A_{-1} = O_{X_0} \langle x_1^*, \ldots, x_n^* \rangle \xrightarrow{d_{-1}} A_0 \cong O_{X_0} \xrightarrow{\pi} O_{X_0}/J(S_0) \to 0$

anti-gh. deg -n anti-gh. deg -1

Deg $-1$: we have to introduce the antifields corresp. to the initial fields $x_i$
The BV extension, that is, the Koszul-Tate resolution of the Jacobian ideal $J(S_0)$

$$(X_0, S_0) \xrightarrow{\text{BV extension}} \tilde{X} = \mathcal{F} \oplus \mathcal{F}^*[1]$$

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construct a free resolution of $\mathcal{O}_{X_0}/J(S_0)$ that is a diff. $\mathcal{O}_{X_0}$-algebra $(A, d)$

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complex of fin. gen. $\mathcal{O}_{X_0}$-modules with a coboundary op. $d$

$$(\{A_i\}, d_i)_{i \in \mathbb{Z} \leq 0}$$

$\begin{align*}
\cdots A_{-n} \xrightarrow{d_{-n}} \cdots A_{-1} &= \mathcal{O}_{X_0}(x_1^*, \ldots, x_{n^2}^*) \\
\text{anti-gh. deg } -n
\end{align*}$

$\begin{align*}
\cdots A_0 \xrightarrow{d_{-1}} \mathcal{O}_{X_0} &= \pi \mathcal{O}_{X_0}/J(S_0) \rightarrow 0 \\
\text{anti-gh. deg } -1
\end{align*}$

Deg $-1$: we have to introduce the antifields corresp. to the initial fields $x_i$

The number of lower degree new variables that have to be introduced at any step is determined by the higher order symmetries in $S_0$
Step 1 for the model: the extended configuration sp. & action (2)

**The extended configuration space**

We found a finite family of (anti)-ghost generators to extend $X_0$, which reflects the type of invariance of the action:

$$\tilde{X} = W^*[1] \oplus X_0^* \oplus X_0 \oplus W$$

with

$$W = \langle C_{ij} \rangle_{i<j} \oplus \langle E_{ijk} \rangle_{i<j<k} \oplus \cdots \oplus \langle E_1 \ldots (n^2 - 1) \rangle$$

\#(n^2 - 1)^i \#(n^2 - 1)^{i+1} \cdots \#(n^2 - 1)^{n^2 - 2} \text{ deg } 1 \text{ deg } 2 \text{ deg } n^2 - 2
Step 1 for the model: the extended configuration sp. & action (2)

The extended configuration space

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$$W = \langle C_{ij} \rangle_{i<j} \oplus \langle E_{ijk} \rangle_{i<j<k} \oplus \cdots \oplus \langle E_{1...(n^2-1)} \rangle$$

$$S_0 = \sum_i g_i (x_{n^2}^2)(x_1^2 + x_2^2 + x_3^2 + \cdots + x_{n^2-1})^i$$

$$\# \binom{n^2-1}{2} \quad \# \binom{n^2-1}{3} \quad \# \binom{n^2-1}{n^2-1}$$
Step 1 for the model: the extended configuration sp. & action (2)

The extended configuration space

We found a finite family of (anti)-ghost generators to extend \( X_0 \), which reflects the type of invariance of the action:

\[
\tilde{X} = W^*[1] \oplus X_0^* \oplus X_0 \oplus W
\]

with

\[
W = \langle C_{ij} \rangle_{i<j} \oplus \langle E_{ijk} \rangle_{i<j<k} \oplus \cdots \oplus \langle E_{1\ldots(n^2-1)} \rangle
\]

\[
S_0 = \sum_i g_i (x_{n^2}^2 + x_1^2 + x_2^2 + \cdots + x_{n^2-1})^i
\]

\[
\text{deg 1} \quad \sum \binom{n^2-1}{2} \\
\text{deg 2} \quad \sum \binom{n^2-1}{3} \\
\text{deg } n^2 - 2 \quad \sum \binom{n^2-1}{n^2-1}
\]
Step 1 for the model: the extended configuration sp. & action (2)

**The extended configuration space**

We found a *finite* family of (anti)-ghost generators to extend $X_0$, which reflects the type of invariance of the action:

$$\tilde{X} = W^*[1] \oplus X_0^* \oplus X_0 \oplus W$$

with

$$W = \langle C_{ij} \rangle_{i<j} \oplus \langle E_{ijk} \rangle_{i<j<k} \oplus \cdots \oplus \langle E_{1\cdots(n^2-1)} \rangle$$

$$\deg 1 \quad \deg 2 \quad \deg n^2 - 2$$

$$\#(n^2-1)_2 \quad \#(n^2-1)_3 \quad \#(n^2-1)$$

The extended action

$$\tilde{S} = S_0 + \sum_{p=1}^{n^2-1} S_{-p}^{lin} + \sum_{q=1}^{\left\lfloor \frac{p}{2} \right\rfloor} S_{-p}^{cor, \ q}$$

with

$$S_{-p}^{lin} = \sum_{i_1<\cdots<i_{p+1}} \left( \sum_{r=1}^{p+1} (-1)^{i_r} x_{i_r} E_{i_1\cdots\hat{i_r}\cdots i_{p+1}}^* \right) E_{i_1\cdots i_{p+1}}$$

$$\in \mathcal{F}_{\partial X_0} (-p, p) \rightsquigarrow \text{determined by the Koszul-Tate resolution}$$

$$S_{-p}^{cor, \ q} = \sum_{l_{p+1}:=(l_q, J), k} \epsilon(\sigma) E_{l_{p+1}}^* F_{l_q, k} G_{J, k}$$

$$\in \mathcal{F}(-p-2, q, p+2-q)$$
Step 1 for the model: the extended configuration sp. & action (2)

The extended configuration space

We found a finite family of (anti)-ghost generators to extend $X_0$, which reflects the type of invariance of the action:

\[ \tilde{X} = W^*[1] \oplus X_0^* \oplus X_0 \oplus W \]

with

\[ W = \langle C_{ij} \rangle_{i<j} \oplus \langle E_{ijk} \rangle_{i<j<k} \oplus \cdots \oplus \langle E_{1\cdots n^2-1} \rangle \]

\[ \begin{align*}
\text{deg 1} & \quad \#(n^2-1) \\
\text{deg 2} & \quad \#(n^2-1) \\
\text{deg } n^2-2 & \quad \#(n^2-1)
\end{align*} \]

The extended action

\[ \tilde{S} = S_0 + \sum_{p=1}^{n^2-1} S_{p}^{\text{lin}} + \sum_{q=1}^{\left\lfloor \frac{p}{2} \right\rfloor} S_{-p, q}^{\text{cor}} \]

with

\[ S_{-p}^{\text{lin}} = \sum_{i_1<\cdots<i_{p+1}} \left( \sum_{r=1}^{p+1} (-1)^{i_r} x_{i_1} E_{i_1 \cdots i_r \cdots i_{p+1}} \right) E_{i_1 \cdots i_{p+1}} \]

\[ \in \mathcal{F}_{\partial X_0}(-p, p) \leadsto \text{determined by the Koszul-Tate resolution} \]

\[ S_{-p, q}^{\text{cor}} = \sum_{l_{p+1}:=\{l_q, J\}, k \notin l_{p+1}} (-1)^{e(\sigma)} E_{l_{p+1}}^{\sigma} F_{l_q, k} G_{J, k} \]

\[ \in \mathcal{F}(-p-2, q, p+2-q) \]

\[ \begin{align*}
\text{linear in the antifields/antighost fields} \\
\text{quadratic in the ghost fields} \\
\text{exact and finite solution of the classical master equation}
\end{align*} \]
Step 2 & 3: the BV cohomology and the auxiliary fields

**Step 2:** an extended theory $(\tilde{X}, \tilde{S})$, with $\{\tilde{S}, \tilde{S}\} = 0$ naturally induces a **BV complex** with

- **Cochain spaces:** $C^i(\tilde{X}, d\tilde{S}) = [\mathcal{O}_{\tilde{X}}]^i \cong \text{Sym}^i_{\mathcal{O}_{X_0}} (W^*[1] \oplus X_0^*[1] \oplus W)$

- **Coboundary op.:** $d\tilde{S} := \{\tilde{S}, -\} : C^\bullet(\tilde{X}, d\tilde{S}) \to C^{\bullet+1}(\tilde{X}, d\tilde{S}), \quad d^2\tilde{S} = 0$
Step 2 & 3: the BV cohomology and the auxiliary fields

**Step 2:** an extended theory \((\tilde{X}, \tilde{S})\), with \(\{\tilde{S}, \tilde{S}\} = 0\) naturally induces a BV complex with

- **Cochain spaces:** \(C^i(\tilde{X}, d_{\tilde{S}}) = [O_{\tilde{X}}]^i \cong \text{Sym}^i O_{X_0^*}[1] \oplus X_0^*[1] \oplus W\)

- **Coboundary op.:** \(d_{\tilde{S}} := \{\tilde{S}, -\} : C^\bullet(\tilde{X}, d_{\tilde{S}}) \rightarrow C^{\bullet+1}(\tilde{X}, d_{\tilde{S}}), \quad d_{\tilde{S}}^2 = 0\)

**Step 3:** the functional \(\tilde{S}\) is in a form which is not suitable for an analysis via perturbation theory: the anti-fields/anti-ghost fields have to be removed both from \(\tilde{X}\) and \(\tilde{S}\). ⇛ **gauge-fixing procedure**
Step 2 & 3: the BV cohomology and the auxiliary fields

Step 2: an extended theory \((\tilde{X}, \tilde{S})\), with \(\{\tilde{S}, \tilde{S}\} = 0\) naturally induces a BV complex with

- **Cochain spaces**: \(C^i(\tilde{X}, d_{\tilde{S}}) = [O_{\tilde{X}}]^i \cong \text{Sym}^i_{O_{X_0}} (W^* [1] \oplus X^*_0 [1] \oplus W)\)
- **Coboundary op.**: \(d_{\tilde{S}} := \{\tilde{S}, -\} : C^* (\tilde{X}, d_{\tilde{S}}) \to C^{*+1} (\tilde{X}, d_{\tilde{S}}), \quad d_{\tilde{S}}^2 = 0\)

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Gauge-fixing fermion \(\Psi \in [O_{X_0} \oplus W]^{-1}\)

We have to introduce **auxiliary fields** (i.e. ghost fields with negative ghost degree) to define \(\Psi\).
Step 2 & 3: the BV cohomology and the auxiliary fields

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\]

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restrict to the Lagrangian submfd. defined by the gauge-fixing conditions \( \varphi^*_i = \frac{\partial \Psi}{\partial \phi_i} \).

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Cochain sp: $C^i(X_t|\Psi, d_{S_t}|\Psi) = \text{Sym}(X_0 \oplus W_t)^i$

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The whole construction

$(X_0, S_0)$ initial gauge theory $\xrightarrow{\text{gh./anti-gh.}} (\widetilde{X}, \widetilde{S})$ + auxiliary flds $\xrightarrow{\text{BV-extended th.}} (X_t, S_t)$ total th $\xrightarrow{\text{gauge-fixing}} (X_t | \Psi, S_t | \Psi)$ gauge-fixed th.

$C_{\text{BV}}^\bullet(X_t, d_{S_t}) \cong C_{\text{BV}}^\bullet(\widetilde{X}, d_{\widetilde{S}})$ BV complex $C_{\text{BV}}^\bullet(X_t, d_{S_t})$ total complex $C_{\text{BRST}}^\bullet(X_t, d_{S_t})$ BRST complex
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Physically relevant:

$\downarrow H^0(\tilde{X}, d_{\tilde{S}}) = \{\text{Cl. observable of } (X_0, S_0)\}$

$\downarrow H^i(\tilde{X}, d_{\tilde{S}}) = \{\text{obstruction to quantization}\}$

The whole construction

$(X_0, S_0)$ initial gauge theory

$\tilde{X}, \tilde{S}$ auxiliary flds

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$(X_t|\Psi, S_t|\Psi)$ gauge-fixed th.

$C_{BV}^\bullet(\tilde{X}, d_{\tilde{S}})$ BV complex

$C_{BV}^\bullet(X_t, d_{S_t})$ total complex

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The goals

The discovery of the BRST complex made it evident that the ghost fields are not just a tool to solve the problem of computing path integrals but they encode the structure of the gauge symmetries of the theory.
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Questions and goals:

► Considering the deep connection existing between NCG and gauge theories, can the BV construction be described in terms of spectral triples?
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\[(M_n(\mathbb{C}), \mathbb{C}^n, D, f)\]

initial spectral triple

\[\tilde{\mathcal{X}}, \tilde{\mathcal{S}}\]

extended theory

\[(X_0, S_0)\]

initial gauge theory

\[C_{BV}^\bullet(\tilde{\mathcal{X}}, d_{\tilde{\mathcal{S}}}) \cong \ ???\]

BV complex \cong ??

\[C_{BV}^\bullet(X_t, dS_t) \cong \ ???\]

BV tot complex \cong ??

\[C_{BRST}^\bullet(X_t, dS_t)|_\Psi \cong \ ???\]

BRST complex \cong ??

\[(X_t|_\Psi, S_t|_\Psi)\]

gauge-fixed theory

noncommutative geometry

BV construction
Step 1: the BV construction in NCG (1)

**Question 1**: Can the BV extended theory $(\tilde{X}, \tilde{S})$ be described as a new **BV-spectral triple**? Can we encode the BV-extension process in the language of NCG?

$$(A_0, \mathcal{H}_0, D_0) \& f \xrightarrow{?} (X_0, S_0) \xrightarrow{BV\ construction} (\tilde{X}, \tilde{S})$$
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- finite spectral triple are naturally defined over $\mathbb{C}$
- in $\tilde{S}$ there appear **Grassmannian** variables
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**Problem 1:** going from \(\mathbb{C}\) to \(\mathbb{R}\) \(\sim\) real structure

**Def.** For a spectral triple \((A, \mathcal{H}, D)\), a **real structure** \(J\) is an antilinear isometry \(J : \mathcal{H} \to \mathcal{H}\) such that:

\[
\begin{align*}
J^2 &= \pm \text{Id} \quad JD = \pm DJ \\
[a, Jb^* J^{-1}] &= 0, \quad [[D, a], Jb^* J^{-1}] = 0, \quad \forall a, b \in A
\end{align*}
\]

Then \((A, \mathcal{H}, D, J)\) is called a (odd) **real spectral triple**

<table>
<thead>
<tr>
<th>KO-dim.</th>
<th>1</th>
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<th>5</th>
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Step 1: the BV construction in NCG (2)

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**Spectral action:** $S[D + \varphi] = Tr(f(D + \varphi));$

- for $f$ a regular function (good decay, cut off...);
- for $\varphi$ a self-adjoint element, with $\varphi = \sum_j a_j[D, b_j], a_j, b_j \in A$

**Fermionic action:** $S[\psi] = \frac{1}{2} \langle (J)\psi, D\psi \rangle,$

- for $\langle , \rangle$ the inner product structure on $\mathcal{H};$
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Questions:

- **ghost fields:** where are they encoded in the initial spectral triple? Which role are they going to play in the BV-spectral triple?
- **extended action:** how can we determine $\tilde{S}$ starting from $(D_0, f)$?
Step 1: the BV construction in NCG (3)

For the model:

\[ \mathcal{H}_0 = \mathbb{C}^2 \oplus \text{ghost/anti-ghost fields} \rightarrow \mathcal{H}_{BV} = \mathcal{Q} \oplus \mathcal{Q}^*[1] \]
Step 1: the BV construction in NCG (3)

For the model:

$$\mathcal{H}_0 = \mathbb{C}^2 \xrightarrow{\text{+ ghost/anti-ghost fields}} \mathcal{H}_{BV} = Q \oplus Q^*[1]$$

Symmetries of $S_0$

- 3 indep. ones among pairs of coord. $\leadsto$ 3 ghost in deg 1
- 1 involving all three coordinates $\leadsto$ 1 ghost in deg 2

$$S_0 = \sum_i g_i(x_4)(x_1^2 + x_2^2 + x_3^2)^i$$
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Hence:

\[ \mathcal{H}_{BV,f} = Q_f^*[1] \oplus Q_f \quad \text{for} \quad Q_f = [i\text{su}(2)]_0 \oplus [i\text{su}(2)]_1 \oplus [u(1)]_2 \]

The BV-Hilbert space:

\[ \mathcal{H}_{BV} = Q^*[1] \oplus Q \quad \text{for} \quad Q = [M_2(\mathbb{C})]_0 \oplus [M_2(\mathbb{C})]_1 \oplus [M_2(\mathbb{C})]_2 \]
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The real structure:
\[ J_{BV}: \mathcal{H}_{BV} \rightarrow \mathcal{H}_{BV} \quad \text{with} \quad J_{BV}(\varphi) = \varphi^\dagger \]
Properties of the action $S_{BV} := \tilde{S} - S_0$

- it has total degree 0
- it is 0 restricted to $X_0$
- it is linear in the antifields/antighost fields
- it is quadratic in the ghost fields
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Given:

$Ad(x)/Ab(x) : M_2(\mathbb{C}) \rightarrow M_2(\mathbb{C})$;

$\phi \mapsto [\alpha(x_i), \phi]_-/+$,

The BV operator $D_{BV}$

$D_{BV} = \begin{pmatrix} 0 & R \end{pmatrix}
\begin{pmatrix} 0 & 0 & 0 \\ R^* & 0 & 0 \\ S \end{pmatrix}
$ for

$R := \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}Ad(C) & Ab(X) \\ \frac{1}{2}Ad(C) & -\frac{1}{2}Ad(X) & 0 \end{pmatrix}$,

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\]

**The algebra** $\mathcal{A}_{BV}$:

Given $(\mathcal{H}_{BV}, D_{BV}, J_{BV})$ as defined above, the maximal unital algebra completing them to a spectral triple is

\[
\mathcal{A}_{BV} = M_2(\mathbb{C}) = \mathcal{A}_0.
\]
Theorem [R.I.]

For a finite spectral triple \((A_0, \mathcal{H}_0, D_0) = (M_2(\mathbb{C}), \mathbb{C}^2, D_0)\) with induced gauge theory \((X_0, S_0)\), the associated BV-spectral triple is:

\[
(A_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) = (M_2(\mathbb{C}), \mathbb{Q} \oplus \mathbb{Q}[1], D_{BV}, J_{BV})
\]

In other words:

\[
\tilde{X} \cong \Omega^1(A_0) + [\Omega^1(A_0)]^* [1] + \mathcal{H}_{BV}, f
\]

\[
S_{BV} := \tilde{S} - S_0 = \frac{1}{2} S_{ferm}
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\[
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Step 1: the BV construction in NCG (5)

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\]

The structure of the BV-spectral triple gives a “geometric interpretation” of some physical aspects:

- the *parity* of fields, antifields, ghost fields and antighost fields is determined by the fact that:
  \[
  JD_1 = -D_1 J \quad JD_2 = +D_2 J;
  \]

Hence, the operator \(D_{BV}\) *partially commutes and partially anticommutes* with the real structure \(J_{BV} \rightsquigarrow \text{mixed KO-dim} \)
Step 1: the BV construction in NCG (6)

Hence:

\[(\mathcal{A}_0, \mathcal{H}_0, D_0) \& f \xrightarrow{\text{BV construction}} (\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})\]

The structure of the BV spectral triple is:

- \(\mathcal{A}_{BV}\) \(\rightsquigarrow\) it simply completes the triple \((\mathcal{H}_{BV}, D_{BV}, J_{BV})\) to a (mixed KO-dim.) real spectral triple;
- \(\mathcal{H}_{BV}\) \(\rightsquigarrow\) encodes the ghost/antighost sector & it keeps track of the ghost degree;
- \(D_{BV}\) \(\rightsquigarrow\) determines the action \(S_{BV}\);
- \(J_{BV}\) \(\rightsquigarrow\) the presence of a real structure is required to describe real fields.
Step 1: the BV construction in NCG (6)

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\[X_0 = \Omega^1(A_0) \quad S_0 = \text{Tr}(f(D_0 + \varphi))\]

\[\tilde{X} = X_0 + X_0^*[1] + \mathcal{H}_{BV,f} \quad S_{BV} = \frac{1}{2} S_{\text{ferm}}\]
Step 2: the BV cohomology in NCG

As it happens for the BV-extended theory \((\tilde{X}, \tilde{S})\) that naturally induces the BV cohomology, we expect also the BV spectral triple to determine a cohomology complex

\[
(\tilde{X}, \tilde{S}) \longrightarrow C^\bullet_{BV}(\tilde{X}, d\tilde{S}) \quad \rightarrow \quad (A_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \rightarrow \quad ?
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Idea: to look at cohomology theories naturally appearing in the context of NCG

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Def. For a graded coalgebra $(\mathcal{B}, \Delta)$ and a bi-comodule $(\mathcal{M}, \omega_L, \omega_R)$ over it, the Hochschild complex is given by:

$$C^q_H(\mathcal{M}, \mathcal{B}) := \text{Hom}_{K}(\mathcal{M}, \mathcal{B} \otimes^q) \quad \& \quad d_H : C^q_H(\mathcal{M}, \mathcal{B}) \rightarrow C^{q+1}_H(\mathcal{M}, \mathcal{B}) \quad \text{with}$$

$$d_H(\varphi)|_m := (\varphi \otimes Id) \circ \omega_R|_m + \sum_{i=1}^q (-1)^{|-|+1+\cdots+|i-1+1|} (Id \otimes \cdots \otimes \Delta \otimes \cdots \otimes Id) \circ \varphi|_m + (-1)^{q+1} (Id \otimes \varphi) \circ \omega_L|_m$$
Step 2: the BV cohomology in NCG

As it happens for the BV-extended theory \((\tilde{X}, \tilde{S})\) that naturally induces the BV cohomology, we expect also the BV spectral triple to determine a cohomology complex

\[
(\tilde{X}, \tilde{S}) \rightarrow C^\bullet_{BV}(\tilde{X}, d_{\tilde{S}}) \quad (A_{BV}, H_{BV}, D_{BV}, J_{BV}) \rightarrow ?
\]

Idea: to look at cohomology theories naturally appearing in the context of NCG

<table>
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Def. For a graded coalgebra \((\mathcal{B}, \Delta)\) and a bi-comodule \((\mathcal{M}, \omega_L, \omega_R)\) over it, the Hochschild complex is given by:

\[
C^q_H(\mathcal{M}, \mathcal{B}) := \text{Hom}_{\mathbb{K}}(\mathcal{M}, \mathcal{B}^{\otimes q}) \quad \& \quad d_H : C^q_H(\mathcal{M}, \mathcal{B}) \rightarrow C^{q+1}_H(\mathcal{M}, \mathcal{B}) \quad \text{with}
\]

\[
d_H(\varphi)|_m := (\varphi \otimes \text{Id}) \circ \omega_R|_m + \sum_{i=1}^{q} (-1)^{1+\cdots+1-i+1}(\text{Id} \otimes \cdots \otimes \Delta \otimes \cdots \otimes \text{Id}) \circ \varphi|_m + (-1)^{q+1}(\text{Id} \otimes \varphi) \circ \omega_L|_m
\]

Try to determine: \((\mathcal{B}, \Delta) \quad \& \quad (\mathcal{M}, \omega_L, \omega_R)\)
Step 2: the BV cohomology in NCG (2)

The bi-comodule $\mathcal{M}$

$$\mathcal{M} := \langle \Omega^1(A_0) \rangle \cong \mathcal{O}_{\mathcal{X}_0}, \quad \text{for} \quad \Omega^1(A_0) = \{ \varphi = \sum_i a_j [D_0, b_j] : \varphi^* = \varphi, a_j, b_j \in A_0 \}$$
Step 2: the BV cohomology in NCG (2)

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\[ \mathcal{M} := \langle \Omega^1(A_0) \rangle \cong O_{X_0}, \quad \text{for} \]
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The coalgebra $\mathcal{B}$

\[ \mathcal{B} := \Omega^1(A_0) \otimes \mathcal{H}_{BV,f} = \Omega^1(A_0) \otimes (Q_f \oplus Q^*_f[1]) \]
\[ \text{for} \quad Q_f = [isu(2)]_0 \oplus [isu(2)]_1 \oplus [u(1)]_2, \]
Step 2: the BV cohomology in NCG (2)

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\]

Left/Right module structure:

\[
\omega_L(g) := \frac{1}{8} \sum_k \frac{\partial g}{\partial x_k} \{\langle J_{BV}(-), D_{BV}(-) \rangle, x_k^0\}
\]

\[
= -\omega_R(g)
\]

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\[
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\]

for \( Q_f = [\mathfrak{su}(2)]_0 \oplus [\mathfrak{su}(2)]_1 \oplus [\mathfrak{u}(1)]_2 \),
**Step 2: the BV cohomology in NCG (2)**

**The bi-comodule $\mathcal{M}$**

$$\mathcal{M} := \langle \Omega^1(A_0) \rangle \cong \mathcal{O}x_0,$$

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**Coproduct:**

$$\Delta(\alpha) := \frac{1}{4} \sum_{r \neq 0, k} \{ \langle J_{BV}(-), D_{BV}(-) \rangle, a^r_k x^k_r \}$$

where $\{-,-\}$ is the antibracket structure induced by the pairing fields/anti-fields and $x^r_k$ is a basis of $\mathcal{H}_{BV,f}$. 
Step 2: the BV cohomology in NCG (2)

The bi-comodule $\mathcal{M}$

\[ \mathcal{M} := \langle \Omega^1(A_0) \rangle \cong \mathcal{O}_{X_0}, \quad \text{for} \]
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BV-spectral triple

\[ (A_{BV}, H_{BV}, D_{BV}, J_{BV}) \]

Hochschild complex

\[ \mathcal{M} \quad \mathcal{B} \]

\[ \text{deg } 0 : \omega_L, \omega_R \]
\[ \text{deg } \neq 0 : \Delta \]

\[ (A_0, H_0, D_0) \rightarrow (A_{BV}, H_{BV}, D_{BV}, J_{BV}) \rightarrow (C^*_H(\mathcal{M}, \mathcal{B}), d_H) \rightarrow \ldots \]
Step 2: the BV cohomology in NCG (2)

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$$\mathcal{M} := \langle \Omega^1(A_0) \rangle \cong \mathcal{O}x_0,$$

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$$\mathcal{B} := \Omega^1(A_0) \otimes \mathcal{H}_{\text{BV}, f} = \Omega^1(A_0) \otimes (Q_f \oplus Q^*_f[1])$$

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where $\{\cdot, \cdot\}$ is the antibracket structure induced by the pairing fields/anti-fields and $x^r_k$ is a basis of $\mathcal{H}_{\text{BV}, f}$.

Theorem: [R.I.]

Given a BV spectral triple $(A_{\text{BV}}, \mathcal{H}_{\text{BV}}, D_{\text{BV}}, J_{\text{BV}})$, let $(\mathcal{B}, \mathcal{M})$ be defined as above. Then it holds that:

$$(C^\bullet_H(\mathcal{M}, \mathcal{B}), d_H) \cong (C^\bullet_{\text{BV}, f}((\tilde{X}, d_5), d_{5,f}), f).$$
Step 3: auxiliary fields

\[
(\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \quad + \quad \text{auxiliary fields/anti-fields} \quad \Rightarrow \quad (\mathcal{A}_{t}, \mathcal{H}_{t}, D_{t}, J_{t})
\]
Step 3: auxiliary fields

\[(A_{BV}, H_{BV}, D_{BV}, J_{BV}) \xrightarrow{\text{+ auxiliary fields/anti-fields}} (A_t, H_t, D_t, J_t)\]

- \(A_t = A_{BV} = A_0 = M_2(\mathbb{C}) \rightarrow\) no changes for the algebra

- \(H_t := H_{BV} \oplus H_{aux}, \quad H_{BV} = [M_2(\mathbb{C})]^{\oplus 6}, \quad H_{aux} = [M_2(\mathbb{C})]^{\oplus 12}\)
  where \(H_{aux,f} = \mathcal{R}_f \oplus \mathcal{R}_f^*[1]\)
  for \(\mathcal{R}_f := [u(1)]_{-2} \oplus [u(1) \oplus su(2)]_{-1} \oplus [isu(2) \oplus u(1)]_0 \oplus [u(1)]_1\).

- \(D_t\)
  \[D_t = \begin{pmatrix} D_{BV} & 0 \\ 0 & D_{aux} \end{pmatrix}\]
  for \(D_{aux} = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}\)

- \(T := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i \cdot \text{Id} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{Id} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ i \cdot \text{Id} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\)

determined by the pairing between contractible pairs

Roberta A. Iseppi

The BV construction for finite spectral triples
Step 3: auxiliary fields

\[(A_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \xrightarrow{\text{+ auxiliary fields/anti-fields}} (A_t, \mathcal{H}_t, D_t, J_t)\]

\[A_t = A_{BV} = A_0 = M_2(\mathbb{C}) \quad \leadsto \text{no changes for the algebra}\]

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for \(\mathcal{R}_f := [u(1)]_{-2} \oplus [u(1) \oplus su(2)]_{-1} \oplus [isu(2) \oplus u(1)]_0 \oplus [u(1)]_1.\)

\[D_t = \begin{pmatrix} D_{BV} & 0 \\ 0 & D_{aux} \end{pmatrix} \quad \text{for} \quad D_{aux} = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix} \quad \text{where} \quad T := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & i \cdot \text{Id} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \text{Id} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i \cdot \text{Id} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\]

Properties:

\[X_t = X_0 + X_0^*[1] + \mathcal{H}_{t,f} \quad S_t = S_0 + \frac{1}{2} \langle J_t \varphi, D_t \varphi \rangle\]

\[\text{Completely determined by the level of reducibility of the theory, that is, by the maximal degree in } \mathcal{H}_{BV}\]

\[\text{The induced Hochschild/BV complex is s.t. } C^\bullet_{BV,t}(X_t, d_{S_t}) \cong C^\bullet_H(\mathcal{M}_t, \mathcal{B}_t) \text{ quasi isom. } C^\bullet_{BV}(\tilde{X}, d) \cong C^\bullet_H(\mathcal{M}, \mathcal{B})\]
Step 4 & 5: gauge-fixing and BRST cohomology

\[(A_0, H_0, D_0) \quad \xrightarrow{\text{+ ghost fields}} \quad (A_{BV}, H_{BV}, D_{BV}, J_{BV}) \quad \xrightarrow{\text{+ auxiliary fields}} \quad (A_t, H_t, D_t, J_t)\]

\[(X_0, S_0) \quad \xrightarrow{\text{+ ghost fields}} \quad (\tilde{X}, \tilde{S}) \quad \xrightarrow{\text{+ auxiliary fields}} \quad (X_t, S_t)\]

\[C^\bullet_t(\tilde{X}, d_{\tilde{S}}) \cong C^\bullet_H(\mathcal{M}, \mathcal{B})\]

\[C^\bullet_t(X_t, d_{S_t}) \cong C^\bullet_H(M_t, B_t)\]
Step 4 & 5: gauge-fixing and BRST cohomology

\[ (\mathcal{A}_0, \mathcal{H}_0, D_0) \xrightarrow{+ \text{ ghost fields}} (\mathcal{A}_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \xrightarrow{+ \text{ auxiliary fields}} (\mathcal{A}_t, \mathcal{H}_t, D_t, J_t) \]

(bosonic) induced th.

\[ (X_0, S_0) \xrightarrow{+ \text{ ghost fields}} (\widetilde{X}, \widetilde{S}) \xrightarrow{+ \text{ auxiliary fields}} (X_t, S_t) \]

(fermionic) induced th.

\[ C^\bullet(\widetilde{X}, d_{\widetilde{S}}) \cong C^\bullet(\mathcal{M}, \mathcal{B}) \]

\[ C^\bullet(X_t, d_{S_t}) \cong C^\bullet(\mathcal{M}_t, \mathcal{B}_t) \]

The construction is all coherent and consistent.
Step 4 & 5: gauge-fixing and BRST cohomology

$$\left( A_0, H_0, D_0 \right) \xrightarrow{+ \text{ ghost fields}} \left( A_{BV}, H_{BV}, D_{BV}, J_{BV} \right) \xrightarrow{+ \text{ auxiliary fields}} \left( A_t, H_t, D_t, J_t \right)$$

$$\left( X_0, S_0 \right) \xrightarrow{+ \text{ ghost fields}} \left( \tilde{X}, \tilde{S} \right) \xrightarrow{+ \text{ auxiliary fields}} \left( X_t, S_t \right)$$

The construction is all coherent and consistent

$$C^\bullet_t(\tilde{X}, d_{\tilde{S}}) \cong C^\bullet_H(M, B)$$

$$C^\bullet_t(X_t, d_{S_t}) \cong C^\bullet_H(M_t, B_t)$$

What happens if we apply the gauge-fixing procedure?

What can we say about the BRST cohomology complex?
Step 4 & 5: gauge-fixing and BRST cohomology

What it happens if we apply the gauge-fixing procedure?

What can we say about the BRST cohomology complex?

Theorem: [R.I.]

Given a total spectral triple \((A_t, H_t, D_t, J_t)\) and a gauge-fixing fermion \(\psi \in [O_{Q_f} \oplus R_f]^{-1}\), it holds that:

\[
(C_t^\bullet(X_t, d_{S_t})|_{\psi}, d_{S_t}|_{\psi}) = (C_H^\bullet(M_\psi, B_\psi), d_H),
\]

where

\[
B_\psi := \Omega^1(A_0) \otimes [Q_f \oplus \{ \varphi_i^* = \frac{\partial \psi}{\partial \varphi_j} \} \oplus R_f \oplus \{ \chi_i^* = \frac{\partial \psi}{\partial \chi_j} \}] \text{ and } M_\psi := \langle \Omega^1(A_0) \rangle
\]
Where we are:

Noncommutative geometry and QFT

The BV construction and the BV/BRST cohomology

BV construction in NCG

Where we are:

Noncommutative geometry

(Mn(C), Cn, D, f) → (ABV, HBV, DBV, JBV) ► (At, Ḥt,Dt, Jt) ► (Xt, St) ► (Xt, St) |Ψ

initial spectral triple

BV spectral triple

total spectral triple

total theory

gauge-fixed theory

Ψ ∈ F−1(OX₀) (HBV, f ⊕ Ḥaux,f)
gauge-fixing fermion

C(H,M, B) ≈ CBV(X, dS)
BV/Hochschild complex

C(H,M, Bt) ≈ CBV(Xt, dSt)
BV tot/Hochschild complex

C(H,M, BΨ) ≈ CBRST(Xt, dSt)|Ψ
BRST/Hochschild complex

BV construction & BRST cohomology

Roberta A. Iseppi

The BV construction for finite spectral triples
Noncommutative geometry and QFT

The BV construction and the BV/BRST cohomology

BV construction in NCG

Where we are:

Noncommutative geometry

\((M_n(\mathbb{C}), \mathbb{C}^n, D, f)\) \quad \text{initial spectral triple}

\((A_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})\) \quad \text{BV spectral triple}

\((A_t, \mathcal{H}_t, D_t, J_t)\) \quad \text{total spectral triple}

\(\Psi \in F_{\mathcal{O}_{X_0}}^{-1}(\mathcal{H}_{BV, f} \oplus \mathcal{H}_{aux,f})\) \quad \text{gauge-fixing fermion}

Results and reached goals:

\(C^*_H(\mathcal{M}, \mathcal{B}) \cong C^*_BV(\tilde{X}, d_{\tilde{S}})\) \quad \text{BV/ Hochschild complex}

\(C^*_H(\mathcal{M}, \mathcal{B}_t) \cong C^*_BV(X_t, d_{S_t})\) \quad \text{BV tot / Hochschild complex}

\(C^*_H(\mathcal{M}, \mathcal{B}_\Psi) \cong C^*_BRST(X_t, d_{S_t})|_\Psi\) \quad \text{BRST/ Hochschild complex}

\((X_0, S_0)\) \quad \text{initial gauge theory}

\((\tilde{X}, \tilde{S})\) \quad \text{extended theory}

\((X_t, S_t)\) \quad \text{total theory}

\((X_t, S_t)|_\Psi\) \quad \text{gauge-fixed theory}

BV construction & BRST cohomology
Where we are:

Noncommutative geometry

\[(M_n(\mathbb{C}), \mathbb{C}^n, D, f) \rightarrow (A_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})\]

initial spectral triple

\[(A_t, \mathcal{H}_t, D_t, J_t)\]

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BV/ Hochschild complex

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BRST/ Hochschild complex

Results and reached goals:

- Determined how the information about the BV extended theory can be extracted from the initial spectral triple

Roberta A. Iseppi

The BV construction for finite spectral triples
Noncommutative geometry and QFT

The BV construction and the BV/BRST cohomology

BV construction in NCG

Where we are:

Noncommutative geometry

\[ (M_n(\mathbb{C}), \mathbb{C}^n, D, f) \rightarrow (A_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV}) \]
initial spectral triple

\[ \rightarrow (A_t, \mathcal{H}_t, D_t, J_t) \]
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gauge-fixed theory

Results and reached goals:

- Determined how the information about the BV extended theory can be extracted from the initial spectral triple
- Established the (noncom.) geometrical role played by ghost/anti-ghost fields in the BV spectral triple

Roberta A. Iseppi
The BV construction for finite spectral triples
Noncommutative geometry and QFT

The BV construction and the BV/BRST cohomology

Where we are:

Noncommutative geometry

\[(M_n(\mathbb{C}), \mathbb{C}^n, D, f) \rightarrow (A_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})\]

BV spectral triple

\[C_H^*(\mathcal{M}, \mathcal{B}) \cong C_{BV}^*(\tilde{X}, d_{\tilde{S}})\]

BV/ Hochschild complex

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initial gauge theory

extended theory

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Results and reached goals:

- Determined how the information about the BV extended theory can be extracted from the initial spectral triple
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- Discovered the relation existing between BV/BRST cohomology and Hochschild cohomology
Where we are:

Noncommutative geometry

$(M_n(\mathbb{C}), \mathbb{C}^n, D, f)$ → $(A_{BV}, \mathcal{H}_{BV}, D_{BV}, J_{BV})$

initial spectral triple

$C^*_H(\mathcal{M}, \mathcal{B}) \cong C^*_{BV}(\tilde{X}, d_{\tilde{s}})$

BV/ Hochschild complex

$(X_0, S_0)$

initial gauge theory

$(\tilde{X}, \tilde{S})$

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BV tot / Hochschild complex

$(X_t, S_t)$

total theory

$\Psi \in \mathcal{F}^{-1}_{\mathcal{O}X_0} (\mathcal{H}_{BV, f} \oplus \mathcal{H}_{aux, f})$

gauge-fixing fermion

$C^*_H(\mathcal{M}, \mathcal{B}_\Psi) \cong C^*_{BRST}(X_t, d_{S_t})|_{\Psi}$

BRST/ Hochschild complex

$(X_t, S_t)|_{\Psi}$
gauge-fixed theory

BV construction & BRST cohomology

**Results and reached goals:**

- Determined how the information about the BV extended theory can be extracted from the initial spectral triple
- Established the (noncom.) geometrical role played by ghost/anti-ghost fields in the BV spectral triple
- Discovered the relation existing between BV/BRST cohomology and Hochschild cohomology
Where we want to go

What’s next?

► To go quantum, to impose the action to solve the *quantum master equation*. In this case the BV complex would encode the quantum observables
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⇝ It is expected to change the construction at the level of the operator $D_{BV}$ and leave the Hilbert space $\mathcal{H}_{BV}$ unchanged.
Where we want to go

What’s next?

- To go quantum, to impose the action to solve the quantum master equation. In this case the BV complex would encode the quantum observables.

  It is expected to change the construction at the level of the operator $D_{BV}$ and leave the Hilbert space $\mathcal{H}_{BV}$ unchanged.

- To extend the construction to the case of 4-dim. spacetime, that is, to consider a spectral triple over $C^\infty(M)$. 
Where we want to go

What’s next?

▶ To go quantum, to impose the action to solve the *quantum master equation*. In this case the BV complex would encode the quantum observables

\[ \rightsquigarrow \text{It is expected to change the construction at the level of the operator } D_{BV} \text{ and leave the Hilbert space } \mathcal{H}_{BV} \text{ unchanged} \]

▶ To extend the construction to the case of 4-dim. spacetime, that is, to consider a spectral triple over \( C^\infty(M) \)

\[
M = \text{compact Riem. spin manifold with canonical spectral triple } (C^\infty(M), L^2(M, S), D_M, J_M, \gamma_M) \\

F = \text{finite noncomm. space with finite real spectral triple } (\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \mathbb{C}^{96}, D_{SM}, J_{SM}, \gamma_{SM})
\]

\[
S_0 = \text{spectral action } \quad Tr(f(D_M/\Lambda)) + \langle J_{SM}(\psi), D_{SM}\psi \rangle = \text{fermionic action}
\]