Higher S-matrices & Poincaré duality for TQFTs

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based on joint work in progress w. Theo Johnson-Freyd

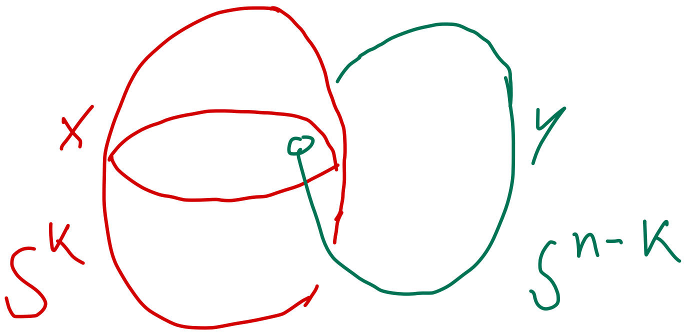
Slides available at www.davidreutter.com

[arXiv: 1812.11933 for background on fusion 2-categories. (w. C. Douglas)
arXiv: 2105.15167 for a first application of 2-cat. S-matrices (w. T. Johnson-Freyd)]

The story on one slide

In a semisimple anomalous $(n+1)$-dim. TQFT* the pairing

$$\underbrace{\left\{\begin{array}{c}\text{operators of}\\ \text{dim } k\end{array}\right\}}_{\text{defects btw. them}} \times \underbrace{\left\{\begin{array}{c}\text{operators of}\\ \text{dim } n-k\end{array}\right\}}_{\text{defects btw. them}} \longrightarrow \mathbb{C}$$



$\subseteq \mathbb{R}^{n+1}$

is non-degenerate.

Moreover, this completely characterizes anomalous TQFTs among relative $(n+2)/(n+1)$-d TQFTs.

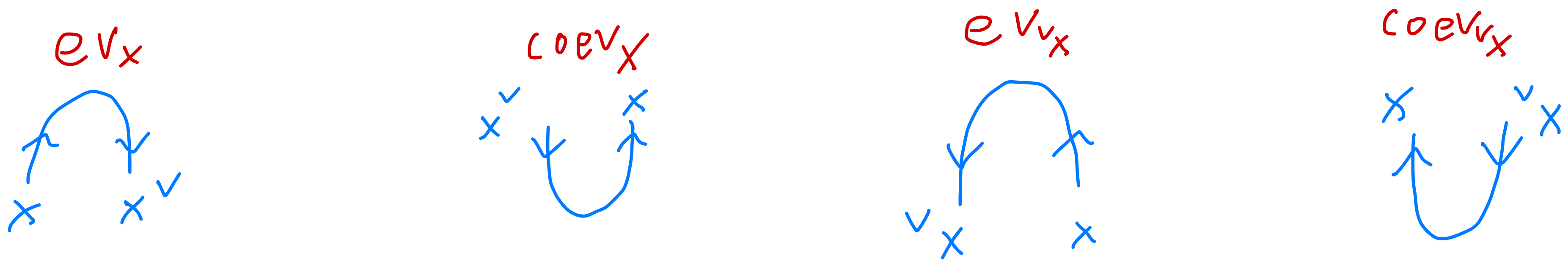
* with trivial local/point operators

I. 1-categorical warm-up

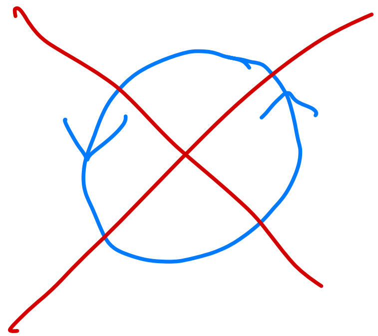
1-categorical warm-up: string diagrams

all objects have left & right duals.

Let $\mathcal{C}$ be a rigid monoidal 1-category, and $x \in \mathcal{C}$.



Notation somewhat unfortunate because I cannot form



since a priori $x^\vee \ncong {}^\vee x$

$\Rightarrow$ Need to record how often string has been twisted around.

1-categorical warm-up: 1d-tangle hypothesis

Let $\mathcal{C}$ be a rigid monoidal 1-category and $x \in \mathcal{C}$.
Then, x can be evaluated at an embedded 1-mfld $S \hookrightarrow \mathbb{R}^2$ with the following additional framing data:

- tangential framing $\tau : \underset{\text{tang. bundle}}{T_S} \simeq \underset{\text{triv. bundle}}{\underline{\mathbb{R}}}$
- normal framing $\nu : \underset{\text{normal bundle}}{N_S} \simeq \underline{\mathbb{R}}$
- a homotopy filling the triangle

$$\begin{tikzcd}
T_S \oplus N_S \arrow[rr, "\tau \oplus \nu"] \arrow[d, phantom, "\wr"] \arrow[drr, phantom, "\eta \ \wr"] & & \underline{\mathbb{R}}^2 \\
T_{\mathbb{R}^2}|_S \arrow[urr, "\text{blackboard framing}"'] & &
\end{tikzcd}$$

such an η does not exist for $S = S^1 \hookrightarrow \mathbb{R}^2$

Same data lets you draw defects on any framed 2-manifold M

Notation: $\displaystyle\int_{S,\tau,\nu,\eta} x = \int_S x \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$

[Baez–Dolan, Lurie, ...]

1-categorical warm-up: extra fractional dualizability

Now assume: $\mathcal{C}$ is fusion

semisimple, $|\pi_0 \mathcal{C}| < \infty$, I simple

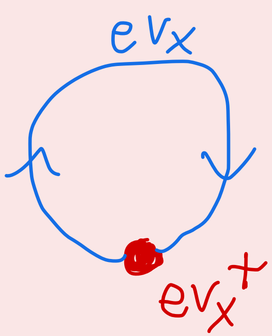
useful notation

$\pi_0 \mathcal{C} := \text{simples}/\text{iso}$

Observe: x simple $\Leftrightarrow$ $\underbrace{\int_{S^0} x}$ $= x \otimes x^\vee \cong I \oplus \ldots$ appears w. multiplicity 1.

$\Rightarrow$ Projection $\int_{D^1} x = ev_x : x x^\vee \xrightarrow{\curvearrowright} I$ splits uniquely: $x x^\vee \underset{ev_x}{\overset{ev_x^\dagger}{\rightleftarrows}} I$, $= id$

Think: value of x on $S^1_b := \partial D^2$ with "bounding 2-framing"



$\cong 1$

Think: value of x on D^2

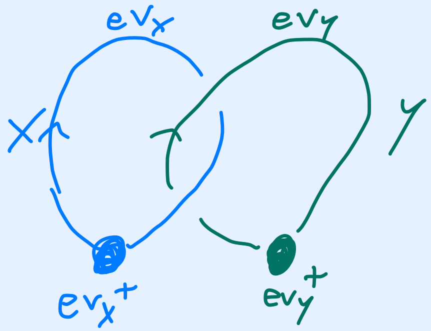
Leads to a canonical iso ${}^\vee x \cong x^\vee$. But: usually not monoidal!

A monoidal such iso is a pivotal structure.

1-categorical warm-up: framed S-matrices

Let $\mathcal{C}$ be a braided fusion category.

<u>Def</u> [Etingof?] The framed S-matrix of $\mathcal{C}$ is the $\pi_0\mathcal{C} \times \pi_0\mathcal{C}$ matrix

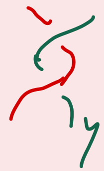
$\tilde{S}_{X,Y} :=$ 

(N.B.: If $\mathcal{C}$ is ribbon/oriented: $S_{X,Y} = \dim(X)\,\dim(Y)\,\tilde{S}_{X,Y}$
depend on ribbon structure)

<u>Theorem</u> [Etingof-Drinfeld-Gelaki-Nikshych-Ostrik, Freed-Teleman, Shimizu, Brochier-Jordan-Safronov-Snyder, ...].

$\tilde{S}$ invertible $\iff$ $Z_2(\mathcal{C})$ is trivial $\iff$ $\mathcal{C}$ is $\boxtimes$-invertible up to braided Morita equivalence

Müger center: subcategory of transparent objects

$Z_2(\mathcal{C}) := \{ X \in \mathcal{C} \mid$  $=$ $||_Y$ $\forall Y \}$

i.e. invertible in a certain Morita 4-cat.

Modular categories := ribbon/oriented + one of these equiv. conditions.

1- categorical warm-up: TQFT interpretation

A braided fusion cat. $\mathcal{B}$ leads to a framed, fully local/extended

4d TQFT + 3d boundary theory

Crane-Yetter/Walker-Wang $\qquad$ Reshetikhin-Turaev

Moreover, $\mathcal{B}$ = category of line & point defects/operators in the ∂-theory.

<u>Def:</u> An $(n+1)$d TQFT is invertible if its values on all manifolds is invertible.

<u>Def:</u> An anomalous nd TQFT is a n d boundary theory of an invertible $(n+1)$d theory.

TQFT interpretation of S-matrix theorem

$\mathcal{B}$ is $\underbrace{\text{non-degenerate}}_{:= \text{ S-matrix of } \mathcal{B} \text{ is non-degenerate}}$ $\Longleftrightarrow$ bulk 4d theory is invertible, i.e. $\mathcal{B}$ defines an anomalous 3d theory.

[perspective due to Walker, Freed, Freed-Teleman, ...]

II. n-categorical version

n-categorical version: higher fusion categories

Slogan: $\dfrac{\text{multifusion } (n+1)\text{-cat.}}{\text{multifusion } n\text{-cat.}} = \dfrac{\text{multifusion } 1\text{-cat.}}{\text{f.d. semisimple algebra}}$

Def [Douglas-R. for n=2]

A multifusion n-category is a $\mathbb{C}$-linear monoidal n-cat. $\mathcal{C}$, s.t.

- it has direct sums and all n-idempotents split
  (n-cat. version of idempotent [n>2: Gaiotto-Johnson-Freyd]   physical interpretation: all operators/defects which can be condensed from operators in $\mathcal{C}$ are already in $\mathcal{C}$)
- Locally multifusion: $\forall X \in \mathcal{C}$, $End(X)$ is a multifusion $(n-1)$-category
- rigid: all objects have duals.
- finite: $|\pi_0 \mathcal{C}| < \infty$ (defined on next slide!)

Prop: X is simple $\Leftrightarrow$ $\oplus$-indecomposable $\Leftrightarrow$ $End(id_X^{n-1}) \cong \mathbb{C}$

fusion := multifusion + I simple

n-categorical version: higher fusion categories

Def: Simples X, Y are Schur connected if $\underbrace{\exists\, X \xrightarrow{\neq 0} Y}$

$n=1$: connected $\Leftrightarrow$ iso $n \geq 2$ connected $\overset{\Leftarrow}{\not\Rightarrow}$ iso

Higher-cat. Schur's lemma

This is an equivalence relation.

Schur components

$\pi_0 \mathcal{C} := \text{simples} / \text{Schur connectivity} \quad \left(= \text{simples} / \text{condensation} \right)$

---

fusion cats generalize finite groups $(\mathrm{Vec}[G])$ ← cat. of f.d. G-graded vectorspaces

fusion n-cats ⟶ finite n-groups $(n\mathrm{Vec}[G])$

$(:=$ loop spaces w. $|\pi_k| < \infty \ \forall k$, $\pi_{\geq n} = 0)$

Slogan: Higher fusion cat. theory is a common generalization of
a) f.d. linear algebra b) finite homotopy theory

Def: $\pi_k \mathcal{C} := \pi_0 \Omega^k \mathcal{C}$ ← $\Omega := \mathrm{End}(I)$

generalizes ⟶ homotopy groups $\left(\pi_k\, n\mathrm{Vec}[G] = \pi_k G \right)$

Generally no groups but basis of a (certain kind of) fusion ring.

n-categorical version: tangle hypothesis

Let $\mathcal{C}$ be a monoidal n-category. A fully dualizable $(n-k)$-morphism $X \in \Omega^{n-k}\mathcal{C}$ can be evaluated at an embedded k-dimensional manifold $M^k \hookrightarrow \mathbb{R}^{n+1}$ with framing data:

- tangential framing $\tau : T_M \simeq \underline{\mathbb{R}^k}$

  tang. bundle   triv. bundle

- normal framing $\nu : N_M \simeq \underline{\mathbb{R}^{n+1-k}}$

  normal bundle

- a homotopy filling the triangle

$$\begin{tikzcd}
T_M \oplus N_M \arrow[d, "\cong"'] \arrow[drr, "\tau\oplus\nu"] \arrow[dr, phantom, "\eta\ \cong"] & & \\
T_{\mathbb{R}^{n+1}}|_M \arrow[rr, "\text{blackboard framing}"'] & & \underline{\mathbb{R}^{n+1}}
\end{tikzcd}$$

Same data lets you draw defects on any framed $(n+1)$-manifold W

Notation: $\int_{M,\tau,\nu,\eta} X = \int_M X$

[Baez-Dolan, Lurie, ...]

n-categorical version: extra fractional dualizability

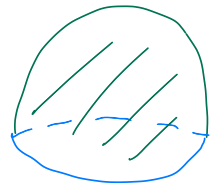
Let $\mathcal{C}$ be a fusion n-category and $x \in \Omega^{n-k}\mathcal{C}$

$$x \in \Omega^{n-k}\mathcal{C} \text{ simple} \iff \underbrace{\Omega^{n-1}\mathcal{C}}_{\text{fusion 1-category}} \ni \int_{S_b^{k-1}} x \cong \mathbb{1} \oplus \dots$$

appears with multiplicity 1

sphere w. k-framing induced by $S_b^{k-1} = \partial D^k$

$\Rightarrow$ evaluation morphism $\int_{D^k} x : \int_{S_b^{k-1}} x \longrightarrow I$



admits a unique splitting  $: I \longrightarrow \int_{S_b^{k-1}} x$

<u>Conclude</u>: can place x on S_b^k: 

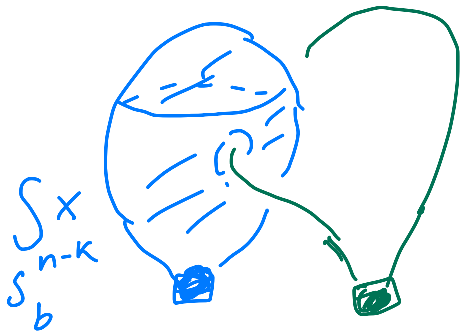
$(k+1)$-framing: one more than dualizability allows

This needs semisimplicity!

n-categorical version: framed S-matrix

$\tilde{S}$ generalizes Whitehead bracket $\pi_k \Omega X \times \pi_{n-k} \Omega X \to \pi_n \Omega X$

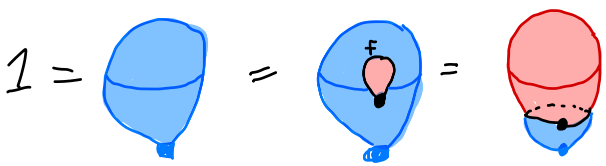
For $x \in \Omega^k e$, $y \in \Omega^{n-k} e$ simple, $0 \le k \le n$ define:

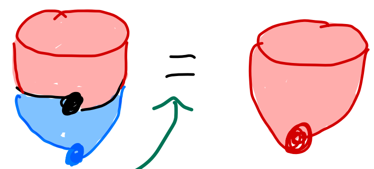
$\tilde{S}^k_{x,y} :=$ 

$\in \Omega^n e \cong \mathbb{C}$

Thm: $\tilde{S}^k$ only depends on components $\tilde{S}^k : \pi_k e \times \pi_{n-k} e \to \mathbb{C}$

Pf: Supp. x, x' in same comp. Pick simple $f: x \to x'$ non-zero. Then:



(2) Uniqueness of splittings $\Rightarrow$ 

)

f and y don't link since $\dim f + \dim y = (n-k-1) + k < n$

S-matrices and invertibility

A (separable) fusion n-cat. $\mathcal{C}$ leads to a framed, fully local/extended

[Conjecture: automatic over char. zero field]

$(n+2)d$ TQFT + $(n+1)$-d boundary theory

Morita class of $\mathcal{C}$ $\quad$ $\mathcal{C}$ itself

Moreover, $\mathcal{C}$ = n-cat of codimension ≥ 1 defects/operators in ∂-theory.

<u>Theorem</u> (in progress w. Johnson-Freyd)

$\tilde{S}^k : \Omega_k \mathcal{C} \times \Omega_{n-k}\mathcal{C} \to \mathbb{C}$ is invertible $\forall\, 0 \leq k \leq n$ $\iff$ $Z(\mathcal{C})$ is trivial $\iff$ $\mathcal{C}$ Morita invertible

Drinfeld center

$\iff$ bulk $(n+2)$-d theory invertible

$\iff$ $\mathcal{C}$ defines an anomalous $(n+1)d$ theory

[proves a conjecture of Kong-Wen '14 in condensed matter theory]

In any semisimple anomalous TQFT, there is a perfect pairing between operators of complimentary dimension (modulo lower-dim. operators)

Thm. applies more generally to:

- non-extended semisimple open/closed TQFT
- expectation: (some) non-semisimple non-compact open/closed TQFT, if one uses the correct generalization of $\mathbb{C}[\Omega_k \mathcal{C}]$ [cf. Shimizu '16]

Some consequences

- **Higher Verlinde formula:** $\mathcal{C}$ Morita inv., $1 \le k \le n-1$, $x, y, z \in \pi_k \mathcal{C}$:

(normalized) fusion rules for $\pi_k \mathcal{C}$

$$\tilde{N}_{x,y}^{z} = \sum_{w \in \pi_{n-k}\mathcal{C}} \tilde{S}_{x,w} \tilde{S}_{y,w} \left(\tilde{S}^{-1}\right)_{w,z}$$

operators of complimentary dimension!

- **Structure theory of (not only higher) fusion categories:**

E.g.: $\mathcal{C}$ Morita invertible $\Rightarrow |\pi_k \mathcal{C}| = |\pi_{n-k} \mathcal{C}| \; \forall \, 0 \le k \le n$

- $k=0$: $\mathcal{C}$ inv. and fusion ($\pi_n \mathcal{C} = *$) $\Rightarrow \pi_0 \mathcal{C} = *$

Interpretation: anomalous s.s. TQFT without point operators ($\pi_n \mathcal{C} = *$) $\Rightarrow$ all codim. 1 op's have ∂ to vacuum & can be recovered from codim. ≥ 2 op's.

Consequence: $\mathcal{C}$ inv. fusion n-cat $\Rightarrow \mathcal{C} = \mathrm{Mod}$(braided fusion $(n-1)$-cat.)
$\mathcal{C}$ inv. fusion 1-cat $\Rightarrow \mathcal{C} = \mathrm{Vect}$.

- cf. Dmitri Nikshych's talk: Minimal non-deg. extensions of a braided fusion cat. $\mathcal{B}$ controlled by $\mathcal{Z}(\mathrm{Mod}\text{-}\mathcal{B})$.

in arXiv:2105.15167, w. Johnson-Freyd, we study $\mathcal{Z}(\mathrm{Mod}\text{-}\mathcal{B})$ via its S-matrix.

Outlook: Surgery for non-degenerate fusion n-cats

Invertibility of $\tilde{S}: \pi_k \mathcal{C} \times \pi_{n-k} \mathcal{C} \to \mathbb{C}$ feels like Poincaré duality.

Slogan: $\dfrac{\text{fusion } n\text{-cat.}}{\text{top. space}} \sim \dfrac{\text{Morita invertible fusion } n\text{-cat.}}{n\text{-manifold / } n\text{-Poincaré space}}$

$\Rightarrow$ Nondegeneracy of $\tilde{S}$ is a first step towards a classification of Morita invertible higher fusion categories via surgery:

$\rightsquigarrow$ Kill high π_k's via "ungauging"

$\rightsquigarrow$ Poincare duality: have also killed π_{n-k}

$\rightsquigarrow$ possible S-matrix obstruction in middle dimension

Work in progress w. Johnson-Freyd...

Thank you for your attention!