Double Copy of Yang-Mills & Double Field Theory

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Plan of the Talk

- quick reminder of double copy

- Lagrangian double copy of Yang-Mills
  $\rightarrow$ double field theory (DFT) to cubic order
  (modulo integrating out dilaton & imposing Siegel gauge)

- Algebraic double copy (gauge invariant and off-shell) including all dilaton couplings (to cubic order) via
  $\rightarrow$ Yang-Mills theory as L_∞-algebra $\quad \mathcal{K} \otimes \mathfrak{g}$
  $\rightarrow$ DFT as (cubic truncation of) L_∞-algebra on $\mathcal{K} \otimes \bar{\mathcal{K}}$

- One main goal: convince you that DFT is the definite framework to understand double copy.

- Outlook

Quick review of Double Copy

4-point amplitude of Yang-Mills theory

$$\mathcal{A}_{\text{YM}} = g^2\,\delta\Big(\sum_{i=1}^{4} p_i\Big)\ \Big\{\frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}\Big\}$$

with Mandelstam variables $s = (p_1 + p_2)^2$, etc.,

color factors $c_s = f_{a_1 a_2}{}^{e} f_{e a_3 a_4}$, and *kinematic numerators*

$$\begin{aligned} n_s(\epsilon, p, s) = &(\epsilon_1 \cdot \epsilon_2)\,(\epsilon_3 \cdot \epsilon_4)\,(p_1 \cdot p_3) + \cdots \\ &+ s\,(\epsilon_1 \cdot \epsilon_3)\,(\epsilon_2 \cdot \epsilon_4) - s\,(\epsilon_1 \cdot \epsilon_4)\,(\epsilon_2 \cdot \epsilon_3) \quad \text{etc.} \end{aligned}$$

Double Copy: Replace color by kinematic factors $c_i \to \bar{n}_i$

[Bern, Carrasco & Johansson (2008), Kawai-Lewellen-Tye (KLT) relations (1986)]

$$\mathcal{A}_{\text{grav}} \ \propto\ \sum_i \frac{n_i \bar{n}_i}{s_i}$$

$\rightarrow$ scattering amplitude of gravity (plus B-field and dilaton).

$\rightarrow$ “Gravity = (Yang-Mills)^2”

Part I: Lagrangian Double Copy

Lagrangian Derivation of Double Copy?

- Scattering amplitudes are on-shell and gauge fixed objects $\rightarrow$ derivation of double copy at level of Lagrangian that is gauge redundant and off-shell ??

- Hermann Nicolai: [From Grassmann to maximal (N=8) supergravity, 2010]

  "no amount of fiddling with the Einstein-Hilbert action will reduce it to a square of a Yang-Mills action."

- We'll show that replacing the color indices a by a *second* set of spacetime indices, $a \rightarrow \bar{\mu}$, corresponding to a *second* set of spacetime momenta $\bar{k}$:

$$A_\mu{}^a(k) \;\rightarrow\; e_{\mu\bar{\mu}}(k, \bar{k})$$

  yields double field theory (to cubic order).

Yang-Mills at Quadratic Order

Yang-Mills action in D dimensions,

$$S_{\mathsf{YM}} = -\frac{1}{4}\int d^D x\,\kappa_{ab}\,F^{\mu\nu\,a}F_{\mu\nu}{}^{b}$$

expanded to quadratic order in momentum space:

$$S^{(2)}_{\mathsf{YM}} = -\frac{1}{2}\int_k \kappa_{ab}\,k^2\,\Pi^{\mu\nu}(k)A_\mu{}^a(-k)A_\nu{}^b(k)\ ,$$

with $\int_k := \int d^D k$ and the projector $\quad [\,\Pi^{\mu\rho}\Pi_{\rho\nu} = \Pi^\mu{}_\nu\,]$

$$\Pi^{\mu\nu}(k) \equiv \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}$$

Gauge invariance

$$\delta A_\mu{}^a(k) = k_\mu \lambda^a(k)\,, \qquad \Pi^{\mu\nu}(k)k_\nu \equiv 0\ ,$$

where the gauge parameter $\lambda^a(k)$ is an arbitrary function.

Lagrangian Double Copy at Quadratic Order

Substitution ${A_\mu}^a(k) \;\to\; e_{\mu\bar{\mu}}(k,\bar{k})$ together with

$$\kappa_{ab} \;\to\; \bar{\Pi}^{\bar{\mu}\bar{\nu}}(\bar{k})$$

yields

$$S^{(2)}_{\text{DC}} = -\frac{1}{4}\int_{k,\bar{k}} k^2\,\Pi^{\mu\nu}(k)\bar{\Pi}^{\bar{\mu}\bar{\nu}}(\bar{k})e_{\mu\bar{\mu}}(-k,-\bar{k})e_{\nu\bar{\nu}}(k,\bar{k})$$

Democratic between k and $\bar{k}$ due to *level-matching constraint*

$$k^2 = \bar{k}^2$$

Gauge invariant under

$$\delta e_{\mu\bar{\mu}} \;=\; k_\mu\,\bar{\lambda}_{\bar{\mu}} + \bar{k}_{\bar{\mu}}\,\lambda_\mu\ ,$$

with *two* independent gauge parameters λ_μ and $\bar{\lambda}_{\bar{\mu}}$.

Double Field Theory at Quadratic Order

Claim: This *is* double field theory to quadratic order.

In order to eliminate non-local terms with $\frac{1}{k^2}$, we introduce 'auxiliary' scalar ϕ (the DFT dilaton):

$$S^{(2)}_{\text{DC}} \propto \int_{k,\bar{k}} \Big(k^2 e^{\mu\bar{\nu}} e_{\mu\bar{\nu}} - (k^\mu e_{\mu\bar{\nu}})^2 - (\bar{k}^{\bar{\nu}} e_{\mu\bar{\nu}})^2 - k^2\phi^2 + 2\phi\, k^\mu \bar{k}^{\bar{\nu}} e_{\mu\bar{\nu}} \Big)$$

Then integrating out ϕ,

$$\phi = \frac{1}{k^2} k^\mu \bar{k}^{\bar{\nu}} e_{\mu\bar{\nu}} \qquad (\Rightarrow\ \delta\phi = k\cdot\lambda + \bar{k}\cdot\bar{\lambda}\ )$$

we recover the double copy action above.

In (doubled) position space this is (free) DFT with derivatives $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\bar{\partial}_{\bar{\mu}} = \frac{\partial}{\partial \bar{x}^{\bar{\mu}}}$ subject to "weak constraint"

$$\Box \equiv \partial^\mu \partial_\mu = \bar{\partial}^{\bar{\mu}} \bar{\partial}_{\bar{\mu}}\,.$$

For $\partial_\mu = \bar{\partial}_{\bar{\mu}}$: (linearized) gravity + B-field + dilaton.

[Siegel (1993), Hull & Zwiebach (2009)]

Lagrangian Double Copy at Cubic Order

Cubic part of Yang-Mills action

$$S^{(3)}_{\mathsf{YM}} = -\mathsf{g}_{\mathsf{YM}} \int d^D x \, f_{abc} \, \partial^\mu A^{\nu a} \, A_\mu{}^b \, A_\nu{}^c$$

reads in momentum space

$$S^{(3)}_{\mathsf{YM}} \propto \mathsf{g}_{\mathsf{YM}} \int_{k_1,k_2,k_3} \delta(k_1+k_2+k_3) \, f_{abc} \, \Pi^{\mu\nu\rho}(k_1,k_2,k_3) \, A_{1\mu}{}^a A_{2\nu}{}^b A_{3\rho}{}^c$$

with short-hand notation $A_i \equiv A(k_i)$ and

$$\Pi^{\mu\nu\rho}(k_1,k_2,k_3) \equiv \eta^{\mu\nu} k_{12}^\rho + \eta^{\nu\rho} k_{23}^\mu + \eta^{\rho\mu} k_{31}^\nu \quad [\, k_{ij} \equiv k_i - k_j \,]$$

$\rightarrow$ obvious double copy rule:

$$f_{abc} \;\rightarrow\; \bar{\Pi}^{\bar{\mu}\bar{\nu}\bar{\rho}}(\bar{k}_1, \bar{k}_2, \bar{k}_3)$$

Double Field Theory at Cubic Order

Double copied cubic action (with $K = (k, \bar{k})$ and $e_i = e(K_i)$)

$$S_{\text{DC}}^{(3)} \propto \int dK_{123}\, \delta(K_1 + K_2 + K_3) \times \Pi^{\mu\nu\rho}(k_1, k_2, k_3)\, \bar{\Pi}^{\bar{\mu}\bar{\nu}\bar{\rho}}(\bar{k}_1, \bar{k}_2, \bar{k}_3)\, e_{1\,\mu\bar{\mu}}\, e_{2\,\nu\bar{\nu}}\, e_{3\,\rho\bar{\rho}}$$

yields in (doubled) position space

$$S_{\text{DC}}^{(3)} \propto \int dx d\bar{x}\; e_{\mu\bar{\mu}} \Big[2\partial^{\mu} e_{\rho\bar{\rho}}\, \bar{\partial}^{\bar{\mu}} e^{\rho\bar{\rho}} - 2\partial^{\mu} e_{\nu\bar{\rho}}\, \bar{\partial}^{\bar{\rho}} e^{\nu\bar{\mu}} - 2\partial^{\rho} e^{\mu\bar{\rho}}\, \bar{\partial}^{\bar{\mu}} e_{\rho\bar{\rho}} + \partial^{\rho} e_{\rho\bar{\rho}}\, \bar{\partial}^{\bar{\rho}} e^{\mu\bar{\mu}} + \bar{\partial}_{\bar{\rho}} e^{\mu\bar{\rho}}\, \partial_{\rho} e^{\rho\bar{\mu}} \Big]$$

Claim: This is cubic double field theory in Siegel gauge!

Cubic Double Field Theory in Siegel Gauge

In closed string theory string field contains

$$|\Psi\rangle \sim \Big(e_{\mu\bar{\mu}}\,,\; f_\mu\,,\; \bar{f}_{\bar{\mu}}\,,\; e\,,\; \bar{e} \Big)$$

with free Lagrangian

$$\begin{aligned}\mathcal{L}^{(2)} = &\; \tfrac{1}{4} e_{\mu\bar{\mu}}\,\Box\, e^{\mu\bar{\mu}} + 2\bar{e}\,\Box\, e - f_\mu f^\mu - \bar{f}_{\bar{\mu}} \bar{f}^{\,\bar{\mu}} \\ &- f^\mu \Big(\bar{\partial}^{\bar{\nu}} e_{\mu\bar{\nu}} - 2\partial_\mu \bar{e}\Big) + \bar{f}^{\,\bar{\nu}} \Big(\partial^\mu e_{\mu\bar{\nu}} + 2\bar{\partial}_{\bar{\nu}} e\Big)\end{aligned}$$

Integrating out f, $\bar{f}$ yields quadratic DFT with $\phi = e - \bar{e}$.

Integrating out e and $\bar{e}$ and picking Siegel gauge

$$f_\mu = \bar{f}_{\bar{\mu}} = 0$$

from cubic DFT yields precisely double copied action

$$S^{(3)}_{\text{DFT}} \to S^{(3)}_{\text{DC}} \;!!$$

Part II: Algebraic Double Copy

L_∞-algebra and Perturbative Field Theory

Algebraic interpretation of any (perturbative) field theory

[Zwiebach (1993), O.H. & Zwiebach (2017)]

$$S = \frac{1}{2}\Big\langle A\,, b_1(A)\Big\rangle + \frac{1}{3!}\Big\langle A\,, b_2(A,A)\Big\rangle + \frac{1}{4!}\Big\langle A\,, b_3(A,A,A)\Big\rangle + \cdots$$

on integer graded vector space

$$\mathcal{X} = \bigoplus_i X_i$$

structure: graded symmetric maps $b_1, b_2, b_3, \ldots$, $|b_n| = -1$, obeying generalized Jacobi identities:

- b_1 is nil-potent differential: $b_1^2 \;=\; 0$
- $b_1(b_2(x,y)) + b_2(b_1(x),y) + (-1)^x b_2(x,b_1(y)) \;=\; 0$
- Jacobi up to homotopy

$$b_2(b_2(x,y),z) + (-1)^{yz} b_2(b_2(x,z),y) + (-1)^x b_2(x,b_2(y,z))$$
$$+\, b_1(b_3(x,y,z)) + b_3(b_1(x),y,z) + \text{two terms} \;=\; 0$$

L_∞-algebra of Yang-Mills Theory I

Formulation of Yang-Mills following from string field theory:

$$S = \int d^D x \,\mathrm{Tr}\left\{\tfrac{1}{2}\, A^\mu \Box A_\mu - \tfrac{1}{2}\,\varphi^2 + \varphi\, \partial_\mu A^\mu - \partial_\mu A_\nu\, [A^\mu, A^\nu] - \tfrac{1}{4}\, [A^\mu, A^\nu]\, [A_\mu, A_\nu]\right\}$$

with Lie algebra valued fields, $A_\mu = A_\mu^a\, t_a$, etc.

Algebra: gauge parameters, fields, EoM & Noether identities:

$$\begin{array}{ccccccc} X_1 & \xrightarrow{b_1} & X_0 & \xrightarrow{b_1} & X_{-1} & \xrightarrow{b_1} & X_{-2} \\ \lambda & & \mathcal{A} & & \mathcal{E} & & \mathcal{N} \end{array}$$

with doublets $\mathcal{A} = (A^\mu, \varphi)$ and $\mathcal{E} = (E^\mu, E)$.

Linearized gauge invariance/EoM: $\delta\mathcal{A} = b_1(\lambda)$ and $b_1(\mathcal{A}) = 0$:

$$b_1(\lambda) = \begin{pmatrix} \partial^\mu \lambda \\ \Box \lambda \end{pmatrix} \in X_0\,, \quad b_1(\mathcal{A}) = \begin{pmatrix} \Box A^\mu - \partial^\mu \varphi \\ \partial \cdot A - \varphi \end{pmatrix} \in X_{-1}$$

L_∞-algebra of Yang-Mills Theory II

Non-linear Yang-Mills equations

$$b_1(\mathcal{A}) + \tfrac{1}{2}\, b_2(\mathcal{A},\mathcal{A}) + \tfrac{1}{6}\, b_3(\mathcal{A},\mathcal{A},\mathcal{A}) = 0$$

with, e.g.,

$$b_2^\mu(A_1, A_2) = \partial_\nu[A_1^\nu, A_2^\mu] + [\partial^\mu A_1^\nu - \partial^\nu A_1^\mu, A_{2\nu}] + (1 \leftrightarrow 2)$$

$$b_3^\mu(A_1, A_2, A_3) = [A_{\nu 1}, [A_2^\nu, A_3^\mu]] + 5 \text{ more terms}$$

Non-linear gauge transformations & algebra

$$\delta\mathcal{A} = b_1(\lambda) + b_2(\lambda, \mathcal{A})$$

where

$$b_2(\lambda, \mathcal{A}) = \begin{pmatrix} [A^\mu, \lambda] \\ \partial_\nu [A^\nu, \lambda] \end{pmatrix}, \qquad b_2(\lambda_1, \lambda_2) = -[\lambda_1, \lambda_2]$$

Classical consistency $\Leftrightarrow$ generalized Jacobi identities

The Kinematic Algebra: Stripping Off Color

Realize L_∞-algebra on $\mathcal{X}$ as tensor product:

[A. M. Zeitlin (2010), L. Borsten et. al. (2021)]

$$\mathcal{X} = \mathcal{K} \otimes \mathfrak{g}$$

$\mathfrak{g}$-valued field lives in tensor product, $x = x^a t_a \to x^a \otimes t_a$

L_∞ structure comes from algebraic structure on $\mathcal{K}$:

$$b_1(x) = m_1(x^a) \otimes t_a$$

$$b_2(x_1, x_2) = \pm f^a{}_{bc}\, m_2(x_1^b, x_2^c) \otimes t_a$$

$$b_3(A_1, A_2, A_3) = 2\, f^a{}_{be} f^e{}_{cd}\, m_3(A^b_{(1}, A^c_2, A^d_{3)}) \otimes t_a$$

Specifically, m_1, m_2, m_3 form <u>C_∞-algebra</u> (homotopy version of graded commutative and associative differential graded algebra)

$$m_2(m_2(u_1, u_2), u_3) - m_2(u_1, m_2(u_2, u_3)) = m_1(m_3(u_1, u_2, u_3)) + m_3(m_1(u_1), u_2, u_3) + \text{two terms}$$

$\mathbb{Z}_2$ Grading of Kinematic Algebra

Kinematic vector space $\mathcal{K} = \bigoplus_{i=-3}^{0} K_i$ admits $\mathbb{Z}_2$ grading into $\mathcal{K}^{(0)} \oplus \mathcal{K}^{(1)}$:

$$\begin{array}{ccccccc} K_0 & \xrightarrow{m_1} & K_{-1} & \xrightarrow{m_1} & K_{-2} & \xrightarrow{m_1} & K_{-3} \\ \lambda & & A_\mu & & E & & \\ & & \varphi & & E^\mu & & \mathcal{N} \end{array}$$

Write each line in basis (with $M = (+, \mu, -)$):

$$|\theta_M\rangle = \Big\{ |\theta_+\rangle \ , \ |\theta_\mu\rangle \ , \ |\theta_-\rangle \Big\}$$

$$|c\,\theta_M\rangle = c\,|\theta_M\rangle$$

where c is odd nilpotent operator, $c^2 = 0, |c| = -1$.

Also: odd nilpotent element $b \sim \frac{\partial}{\partial c}$, $|b| = +1$, conjugate to c

$$b^2 = 0 \,, \quad bc + cb = 1 \,, \quad m_1 b + b m_1 = \Box\,\mathbf{1}$$

'bc-ghost system' (string field theory or worldline quantization)

L_∞-algebra of Double Field Theory via Double Copy

L_∞-algebra of DFT defined on subspace

$$\mathcal{V}_{\mathsf{DFT}} \;\subset\; \mathcal{K}\otimes\bar{\mathcal{K}}$$

as follows. Setting

$$c^{\pm} := c\otimes\mathbf{1}\pm\mathbf{1}\otimes\bar{c} \qquad\qquad b^{\pm} := \tfrac{1}{2}(b\otimes\mathbf{1}\pm\mathbf{1}\otimes\bar{b})$$

we have for $\Delta := \frac{1}{2}(\Box-\bar{\Box})$

$$\mathcal{V}_{\mathsf{DFT}} = \Big\{\Psi\in\mathcal{K}\otimes\overline{\mathcal{K}}\mid b^{-}\Psi=0\,,\;\Delta\Psi=0\Big\}$$

DFT differential and 2-bracket:

$$B_1 := m_1\otimes\mathbf{1}+\mathbf{1}\otimes\bar{m}_1$$

$$B_2 := -\frac{1}{2}\mathcal{P}_{\Delta}b^{-}m_2\otimes\bar{m}_2$$

well-defined and defines (cubic truncation of) L_∞-algebra.

Cubic Double Field Theory from Double Copy

The cubic action for this L_∞-algebra

$$S = \frac{1}{2}\Big\langle \Psi, B_1(\Psi)\Big\rangle + \frac{1}{6}\Big\langle \Psi, B_2(\Psi,\Psi)\Big\rangle + \mathcal{O}(\Psi^4)\ ,$$

yields cubic DFT:

$$\begin{aligned}
\mathcal{L} = {}& \tfrac{1}{4}\, e^{\mu\bar\nu}\Box e_{\mu\bar\nu} + 2\,\bar e\,\Box e - f^\mu f_\mu - \bar f^{\bar\mu}\bar f_{\bar\mu} \\
& - f^\mu\Big(\bar\partial^{\bar\nu} e_{\mu\bar\nu} - 2\,\partial_\mu \bar e\Big) + \bar f^{\bar\nu}\Big(\partial^\mu e_{\mu\bar\nu} + 2\,\bar\partial_{\bar\nu} e\Big) \\
& + \tfrac{1}{8}\, e^{\mu\bar\nu}\Big(\bar\partial^{\bar\lambda} e_{\mu\bar\lambda}\,\partial^\rho e_{\rho\bar\nu} + \partial^\lambda e_{\lambda\bar\rho}\,\bar\partial^{\bar\rho} e_{\mu\bar\nu} + 2\,\partial_\mu e_{\lambda\bar\rho}\,\bar\partial_{\bar\nu} e^{\lambda\bar\rho} \\
& \qquad - 2\,\partial_\mu e^{\lambda\bar\rho}\,\bar\partial_{\bar\rho} e_{\lambda\bar\nu} - 2\,\bar\partial_{\bar\nu} e^{\lambda\bar\rho}\,\partial_\lambda e_{\mu\bar\rho}\Big) \\
& + \tfrac{1}{2}\, e^{\mu\bar\nu}\Big(f_\mu - \partial_\mu e\Big)\Big(\bar f_{\bar\nu} - \bar\partial_{\bar\nu}\bar e\Big)
\end{aligned}$$

Only last line missed in ‘naive’ attempt.

Field-redefinition equivalent (but significant simplification!) to [Hull-Zwiebach (2009)].

DFT Gauge Transformations from Yang-Mills

Consider cubic vertex of Yang-Mills theory encoded in 2-bracket:

$$b_2(A,A)^a_\mu = f^a{}_{bc}(A^b \bullet A^c)_\mu$$

unambiguously defines 'kinematic' product

$$(v \bullet w)_\mu = v^\nu \partial_\nu w_\mu + (\partial_\mu v^\nu - \partial^\nu v_\mu)w_\nu + (\partial_\nu v^\nu)w_\mu - (v \leftrightarrow w)$$

Structure of generalized Lie derivative of DFT. In fact,

$$\delta^{(1)}_\lambda e_{\mu\bar{\mu}} = (\lambda \bullet e_{\bar{\mu}})_\mu + \text{(auxiliary fields)} \,,$$

with similar term under $\bar{\lambda}_{\bar{\mu}}$.

Diffeomorphisms directly encoded in 3-vertex of Yang-Mills !!

Summary & Outlook

- simplest double copy implementation at Lagrangian level:

  $\rightarrow$ gauge invariant DFT to quadratic order

  $\rightarrow$ Siegel gauge fixed DFT to cubic order

  (modulo integrating out scalars)

- Off-shell/local/gauge invariant double copy to cubic order:

  $\rightarrow$ Yang-Mills as homotopy Lie algebra or

  L_∞-algebra $\mathcal{K} \otimes \mathfrak{g}$ in terms of 'kinematic' C_∞-algebra $\mathcal{K}$

  $\rightarrow$ L_∞-algebra of DFT on $\mathcal{V} = \left[\mathcal{K} \otimes \bar{\mathcal{K}}\right]_{\text{level-matched}}$

- to all orders? $\rightarrow$ diffeomorphism structure already there!

- classical solutions? $\rightarrow$ black holes!

- non-trivial backgrounds $\rightarrow$ cosmology!

  [O.H., Allison Pinto, 2207.14788]