Equidistribution modulo finite groups in shifts

V. Berthé, H. Goulet-Ouellet, C.-F. Nyberg-Brodda, D. Perrin and K. Petersen

Uniform Distribution of Sequences

INSTITUT
DE RECHERCHE
EN INFORMATIQUE
FONDAMENTALE

Université Paris Cité

Equidistribution

Let α be irrational

For every x, for every interval I

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What about equidistribution modulo m?

$$\frac{1}{N}\,\mathrm{Card}\{0 \le n < N \mid \mathrm{Card}\{i \mid 0 \le i < n, i\alpha \in I\} \equiv k \mod m\} \to_{?} \frac{1}{m}$$

[Veech, Stewart, Merrill, Guesnais-Parreau, etc.]

Overview

- From rotations to symbolic dynamical systems
- The Fibonacci shift
- Skew products
- Density for group languages
- Arithmetic applications

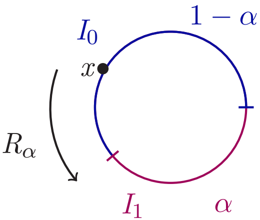
Sturmian words

Consider the rotation $(\mathbb{T}, R_\alpha)$ where $R_\alpha \colon x \mapsto x + \alpha \bmod 1$ and the coding map

$$\nu \colon \mathbb{T} \to \{0,1\}, \quad \nu(x) = a \quad \text{if } x \in I_a$$

The trajectory of x for R_α is coded by $u \in \{0,1\}^{\mathbb{Z}}$ with

$$u_n = \nu(R_\alpha^n(x)) \text{ for all } n$$



The Sturmian shift (X_α, S) is measurably isomorphic to $(\mathbb{T}, R_\alpha)$

Frequencies and symbolic discrepancy

Take a sequence $u = (u_n)_n \in \{1, \cdots, d\}^{\mathbb{N}}$

The frequency α_a of the letter a in u is defined as the following limit, if it exists

$$\alpha_a = \lim_{n \to \infty} \frac{1}{n} \mathrm{Card}\{k \in \{0, \ldots, n-1\} : u_k = a\} = \lim_n |u_0 \cdots u_{n-1}|_a$$

The discrepancy of $u = (u_n)_n$ is defined as

$$\Delta_{\boldsymbol{\alpha}}(u) = \max_a \sup_{n \in \mathbb{N}} \left| |u_0 \cdots u_{n-1}|_a - n\alpha_a \right|$$

One can also count occurrences of factors

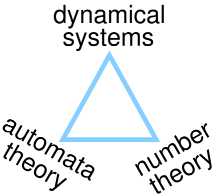
Equidistribution and beyond

- Bounded discrepancy/bounded remainder sets

$$|\operatorname{Card}\{i \mid 0 \leq i < n, i\alpha \in I\} - n|I|| \quad \text{is bounded iff} \quad |I| \in \mathbb{Z} + \alpha\mathbb{Z} \text{ [Kesten]}$$

- Equidistribution w.r.t. automata/rational languages

$$\frac{1}{N}\operatorname{Card}\{0 \leq n < N \mid \operatorname{Card}\{i \mid 0 \leq i < n, i\alpha \in I\} \equiv k \mod m\} \to_{?} \frac{1}{m}$$



Equidistribution and logic

- The first-order logical theory of Sturmian words is decidable [Hieronymi et al.]

- The MSO theory of the structure $\langle \mathbb{N}; <, P_s \rangle$ is decidable when s is an effective Sturmian word [B.-Karimov-Nieuwveld-Ouaknine-Vahanwala-Worrell]

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- The MSO theory of the following structures is decidable
  - $\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Pow}_3 \rangle$
  - $\langle \mathbb{N}; <, \mathrm{Pow}_2, \mathrm{Pow}_3, \mathrm{Pow}_5 \rangle$ assuming Schanuel's Conjecture

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Example

Are there infinitely many m and n such that

$$3^n < 2^m < 2^{m+1} < 3^{n+1} < 2^{m+2} < 2^{m+3} < 3^{n+2}$$

and

$$2^m \equiv 3^n \equiv 1 \pmod{13} \text{ ?}$$

Skew products

Skew product

- Let (X, S) be a minimal shift
- Let G be a finite group
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Skew product $G \times_\varphi X = (G \times X, T_\varphi)$

$$T_\varphi\colon G \times X \to G \times X, \quad T_\varphi(g, x) = (g\varphi(x_0), Sx)$$

Example Let X be the Fibonacci shift, $G = \mathbb{Z}/2\mathbb{Z}$ and

$$\varphi\colon \{a, b\}^* \to \mathbb{Z}/2\mathbb{Z}, a \mapsto 1, \; b \mapsto 0$$

$\rightsquigarrow$ counting modulo 2 in the Fibonacci shift

Skew products and cocycles

One has

$$T_\varphi\colon\, G \times X \to G \times X, \quad T_\varphi(g,x) = (g\varphi(x_0), Sx)$$

This gives for $n \geq 0$

$$T_\varphi^n(g,x) = (g\varphi(x_0 \cdots x_{n-1}), S^n x)$$

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Toward modular equidistribution Let $\varphi : X \to \mathbb{Z}/2\mathbb{Z}$, $a \mapsto 1$, $b \mapsto 0$

One has

$$T_\varphi^n(0, x) = (|x_0 \cdots x_{n-1}|_a \mod 2, S^n x)$$

$$\frac{1}{N} \operatorname{Card}\{0 \leq n < N \mid \operatorname{Card}\{i \mid 0 \leq i < n, i\alpha \in I\} \equiv k \mod m\} \to_? \frac{1}{m}$$

An ergodic viewpoint

We are given a minimal shift (X, S, μ), a group G and a surjective morphism $\varphi : X \to G$

$$T_\varphi\colon G \times X \to G \times X, \quad T_\varphi(g, x) = (g\varphi(x_0), Sx)$$

An example of an invariant measure

- Let μ be a shift- invariant probability measure on X
- Let ν be the uniform probability distribution on G
- The measure $\nu \times \mu$ is an invariant probability measure on $(G \times X, T_\varphi)$

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- Let ν be an invariant measure on the skew product $(G \times X, T_\varphi)$ that projects to μ (the measure on the shift X)
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The pointwise ergodic theorem applied to the skew product T_φ gives

$$\frac{1}{n}\sum_{i=0}^{n-1} 1_g(T_\varphi^i(h, x)) = \frac{1}{n}\sum_{i=0}^{n-1} 1_g(h\varphi(x_0 \cdots x_{i-1})) \to \text{ some } \bar{f}_g(h, x) \ \nu \text{ a.e.}$$

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Take $h = 1$. One checks that

$$\frac{1}{n}\sum_{i=0}^{n-1} 1_g(\varphi(x_0 \cdots x_{i-1})) \to \text{ some } \bar{\psi}_g(x) \ \mu \text{ a.e.}$$

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$$\int_X \bar{\psi}_g(x)\, d\mu(x) = \int_X \lim \frac{1}{n}\sum_{i=0}^{n-1} 1_g(\varphi(x_0 \cdots x_{i-1}))\, d\mu(x)$$

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Consider the rational language $L_g = \varphi^{-1}(g)$

$$\begin{aligned}
\int_X \bar{\psi}_g(x)\, d\mu(x) &= \int_X \lim \frac{1}{n}\sum_{i=0}^{n-1} 1_g(\varphi(x_0\cdots x_{i-1}))\, d\mu(x) \\
&= \lim \frac{1}{n}\sum_{i=0}^{n-1} \int_X 1_g(\varphi(x_0\cdots x_{i-1}))\, d\mu(x) \\
&= \lim \frac{1}{n}\sum_{i=0}^{n-1} \sum_{w\in\mathcal{A}^i} \int_{[w]} 1_g(\varphi(x_0\cdots x_{i-1}))\, d\mu(x) \\
&= \lim \frac{1}{n}\sum_{i=0}^{n-1} \sum_{w\in L_g\cap\mathcal{A}^i} \mu[w] = \lim \frac{1}{n}\sum_{i=0}^{n-1} \mu(L_g\cap\mathcal{A}^i)
\end{aligned}$$

Skewing the Fibonacci shift

Consider the Fibonacci shift (X_σ, S)

$$\sigma\colon a \mapsto ab,\ \ b \mapsto a$$

$$\sigma^\infty(a) = abaababaabaababaababaabaababaabaababaababaa\cdots$$

- The shift (X_σ, S, μ) is a minimal and uniquely ergodic shift
- Let $\varphi\colon \{a, b\}^* \to G$ be a morphism onto the finite group G

We consider the skew product

$$T_\varphi\colon G \times X \to G \times X, \quad T_\varphi(g, x) = (g\varphi(x_0), Sx)$$

Theorem The Fibonacci shift has minimal and uniquely ergodic skew products $G \times X_\sigma$ with all finite groups

Density of regular languages

Density of regular patterns

Let (X, S) be a shift

Can we define a notion of frequency/density for a regular pattern in (X, S)?

Example Words with an even number of a given letter

How often do they occur in a given shift?

Density

A shift $\rightsquigarrow$ A measure

Let (X, S, μ) be a shift with a shift invariant probability measure μ

$$[w] = \{x \in X \mid x_0 \cdots x_{|w|-1} = w\}$$

$$\sum_{a \in \mathcal{A}} \mu[aw] = \sum_{a \in \mathcal{A}} \mu[wa] = \mu[w] \text{ for all } w \in \mathcal{A}^*$$

Density

Let (X, S, μ) be a shift with a shift invariant probability measure μ

Let L be a rational language on $\mathcal{A}$ (i.e., a language recognised by a finite automaton)

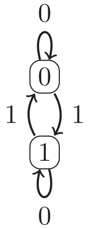
The density of L under the measure μ is defined as the following limit whenever it exists

$$\begin{aligned}\delta_\mu(L) &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(L \cap \mathcal{A}^i) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu\{x \in \mathcal{A}^{\mathbb{Z}} \mid x_0 \cdots x_{i-1} \in L\}\end{aligned}$$

- Since $\mu(w) = 0$ when $w \notin \mathcal{L}(X)$, $\delta_\mu(L) = \delta_\mu(L \cap \mathcal{L}(X))$

Counting modulo 2

Question What is the density of finite words having an even number of a's in the shift (X, S)? Does it exist?



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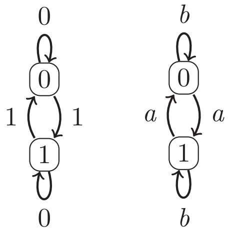

Algebraically Let $\varphi\colon \{a, b\}^* \to \mathbb{Z}/2\mathbb{Z}$, $\varphi(a) = 1$, $\varphi(b) = 0$

$$L = \varphi^{-1}(0) = \{w \mid |w|_a \equiv 0 \mod 2\}$$

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The sequence $(\mu(L \cap \mathcal{A}^n))_{n\in\mathbb{N}}$ does not have a limit, as

$$\lim_{n\to\infty} \mu(L \cap \mathcal{A}^{F_{4n}}) = 1, \quad \lim_{n\to\infty} \mu(L \cap \mathcal{A}^{F_{4n+2}}) = 0$$

but its Cèsaro mean does

$$\frac{1}{n} \sum_{i=0}^{n-1} \lim_{n\to\infty} \mu(L \cap \mathcal{A}^i) = 1/2$$

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Density of a rational language

The density of a language L with respect to a probability measure μ on $\mathcal{A}^{\mathbb{Z}}$ is

$$\delta_\mu(L) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(L \cap \mathcal{A}^i)$$

whenever the limit exists

Our aim is

- to show that the density of a rational language exists for every invariant measure μ
- to give a way to compute it (via ergodicity)

The result is known when μ is a Bernoulli measure [Berstel, 1972]

On the existence of the density

Theorem [Group language] [B., Goulet-Ouellet, Nyberg-Brodda, Petersen, Perrin, 2024] Let X be a shift space on a finite alphabet $\mathcal{A}$ with an ergodic measure μ and let $\varphi \colon \mathcal{A}^* \to G$ be a morphism onto a finite group G with uniform probability measure ν.
If the product measure $\nu \times \mu$ is ergodic on $G \times X$, then for every group language $L = \varphi^{-1}(K)$ with $K \subseteq G$

$$\delta_\mu(L) = |K|/|G|$$

Theorem [Rational language] [B.-Goulet-Ouellet-Perrin, 2025] Let μ be an invariant measure on $\mathcal{A}^{\mathbb{Z}}$. Then every rational language on the alphabet $\mathcal{A}$ has a density with respect to μ.

Ergodicity and density

Theorem

- Let (X, S, μ) be a shift space on $\mathcal{A}$
- Let $\varphi \colon \mathcal{A}^* \to G$ be a morphism onto a finite group G with uniform probability measure ν

If the skew product $G \times_\varphi X$ is ergodic, then for all $K \subseteq G$

$$\delta_\mu(\varphi^{-1}(K)) = |K|/|G|$$

We recall that

$$\delta_\mu(L) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(L \cap \mathcal{A}^i)$$

Proof The density $\delta_\mu(L)$ can be rewritten as an ergodic sum in $G \times_\varphi X$

Toolbox

- Group language $L = \varphi^{-1}(g), \ g \in G$
  - Minimality can be characterized in terms of return words
  - Ergodicity via Anzai's criterium (coboundaries) and essential values
- Monoid case $L = \varphi^{-1}(m), \ m \in M$
  - Green's relations and J-class

Back to the Fibonacci shift

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Back to the Fibonacci shift

- Let (X, S) be the Fibonacci shift over $\{a, b\}$
- Let $G = \mathbb{Z}/m\mathbb{Z}$
- Let $\varphi : \{a, b\}^* \to \mathbb{Z}/m\mathbb{Z},\ a \mapsto 1,\ b \mapsto 0$

Theorem The skew product $\mathbb{Z}/m\mathbb{Z} \times X$ is uniquely ergodic

$\rightsquigarrow$ equidistribution results on the congruence of the number of visits of $R_\alpha : x \mapsto x + \alpha \mod 1$ to the interval $[0, \alpha)$, $\alpha = \frac{\sqrt{5}-1}{2}$

Corollary For every $x \in X$, $k \in \mathbb{Z}/m\mathbb{Z}$ and $a \in \{0, 1\}$, one has

$$\frac{1}{N}\mathrm{Card}\{0 \leq n \leq N-1 \mid |x_0 \cdots x_{n-1}|_a \equiv k \mod m\} \to \frac{1}{m}$$

$$\frac{1}{N}\mathrm{Card}\{0 \leq n \leq N-1 \mid \mathrm{Card}\{i \mid 0 \leq i < n, i\alpha \in [0, \alpha)\} \equiv k \mod m\} \to \frac{1}{m}$$

Continued fractions

- Let (X, S) be the Fibonacci shift X over the alphabet $\{1, 2\}$
- Let $G = \mathrm{GL}(2, \mathbb{Z}/2\mathbb{Z})$
- Let
$$\varphi\colon \{1,2\}^* \to \mathrm{GL}(2, \mathbb{Z}/2\mathbb{Z}), \quad k \mapsto \begin{pmatrix} 0 & 1 \\ 1 & \overline{k} \end{pmatrix},$$
where $\overline{k}$ stands for the congruence class of the integer k modulo 2
- The map $\varphi : \{1,2\}^* \to \mathrm{GL}(2, \mathbb{Z}/2\mathbb{Z})$ is onto

Continued fractions cocycles

For any $x \in X$, consider the real number in $[0,1]$ that admits $(x_n)_{n\geq 1}$ as its sequence of partial quotients

$$x = (x_n)_n \in X \rightsquigarrow \text{ sequence of partial quotients} \rightsquigarrow [0; x_1, x_2, \dots]$$

$$\varphi\colon \mathcal{A}^* \to \mathrm{GL}(2, \mathbb{Z}/2\mathbb{Z}), \quad k \mapsto \begin{pmatrix} 0 & 1 \\ 1 & \overline{k} \end{pmatrix}$$

Let $(p_n(x)/q_n(x))_n$ stand for the associated sequence of rational approximations in the corresponding continued fraction expansion. One has

$$q_{-1}(x) = 0, p_{-1}(x) = 1,\ q_0(x) = 1,\ p_0(x) = 0$$

$$q_{n+1}(x) = x_{n+1} q_n(x) + q_{n-1}(x),\ p_{n+1}(x) = x_{n+1} p_n(x) + p_{n-1}(x) \text{ for all } n$$

For $n \geq 0$, one has

$$\varphi^{(n)}(x) = \begin{pmatrix} p_{n-1}(x) & p_n(x) \\ q_{n-1}(x) & q_n(x) \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}/2\mathbb{Z})$$

- Let (X, S) be the Fibonacci shift over the alphabet $\{1, 2\}$
- The skew product $\mathrm{GL}(2, \mathbb{Z}/2\mathbb{Z}) \times X$ is uniquely ergodic
- By ergodicity

$$\begin{pmatrix} p_{n-1}(x) & p_n(x) \\ q_{n-1}(x) & q_n(x) \end{pmatrix}$$

  equidistributes in the group $\mathrm{GL}(2, \mathbb{Z}/2\mathbb{Z})$.
- For every $x \in X$

$$\lim_{N\to\infty} \frac{1}{N} \mathrm{Card}\{1 \leq n \leq N \mid q_n(x) \equiv 0 \mod m\} = \frac{1}{3},$$
$$\lim_{N\to\infty} \frac{1}{N} \mathrm{Card}\{1 \leq n \leq N \mid q_n(x) \equiv 1 \mod m\} = \frac{2}{3}.$$

- One recovers the behaviour of a random irrational number [Jager-Liardet]
  Certain residue classes are attained more frequently than others.

- Let (X, S) be the Fibonacci shift over the alphabet $\{1, 2\}$
- Real numbers having as a sequence of partial quotients elements of the Fibonacci shift behave like a.e. real number
- One has for a.e. real number in $[0, 1]$

$$\lim_n \frac{\log q_n}{n} = \frac{\pi^2}{12 \log 2} \text{ a.e. (Lévy's theorem)}$$

- We take now the cocycle with values in $\mathrm{GL}(2, \mathbb{Z})$
- For every $x \in (X, S)$

$$\lim_n \frac{\log q_n}{n} \text{exists}$$

[Walters, Fan-Wu]