The antifield-BRST approach to (gauge) field theories: an overview

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Program "Higher Structures and Field Theory"

Introduction

This lecture is a review of the BRST formulation of gauge theories with an emphasis on its algebraic properties that have a direct connection to the subject of this meeting.

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Economical descriptions not involving this excess luggage usually go beyond the Lie algebra framework.

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- They are present for any action and imply no restriction on S_0 (no ‘Noether identities”).
- They vanish on-shell and so have a trivial action on the space of solutions.

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for some λ^α and $M^{ij} = -M^{ji}$, which might depend on the fields.

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where $C^\gamma{}_{\alpha\beta}$ and $M^{jk}_{\alpha\beta}$ can depend on the fields ϕ^i.

Furthermore, when there is redundancy, there is some ambiguity in the "structure functions" $C^\gamma{}_{\alpha\beta}$, namely

$$C^\gamma{}_{\alpha\beta} \rightarrow C^\gamma{}_{\alpha\beta} + D^A_{\alpha\beta} Z^\gamma_A$$

(with appropriate redefinition of $M^{jk}_{\alpha\beta}$).

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since $C^{\gamma}{}_{\alpha\beta}$ may depend on the fields ϕ^i and $M^{jk}_{\alpha\beta}$ may not vanish.

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introducing new, higher order, structure functions $T^A_{\alpha\beta\gamma}$ and $T^{\delta j}_{\alpha\beta\gamma}$.

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All these structure functions, as well as the identities that they fulfill, can be economically encoded in the BRST formulation.

Furthermore, the ambiguity in the structure functions (starting from the R^i_α's themselves) can be very easily accounted for.

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The recursive construction leading to the higher order structure functions could then be related, in turn, to “homological perturbation theory” (and “strong homotopy Lie algebras L_∞”)

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for some $F^A_{\alpha\beta\delta}$.

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This different point of view turns out to be crucial for understanding the BRST construction and computing the BRST cohomology.

Covariant phase space

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[For local objects, jet space formalism can be used to put these considerations on a firmer footing.]

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Nilpotency of s implies $\delta^2 = 0$, $\delta\gamma + \gamma\delta = 0$, $\gamma^2 = 0$.

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We shall for this reason only focus here on the Koszul-Tate differential δ.

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$$D_\mu A_a^{*\mu} = \delta C_a^*.$$

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(If one includes the ghosts, one gets $C^\infty(\Sigma) \otimes \Lambda(C^a, \partial_\mu C^a, \cdots)$.)

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This is the Batalin-Vilkovisky construction, which works because the Koszul-Tate complex is a resolution.

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Procedure works and corresponding term in the solution of the master equation is $\sim \int d^n x C^{*\mu} \partial_\mu C$.

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Acyclicity of δ guarantees the existence of S_2 and of the successive terms.

Master equation

In terms of the solution S of the master equation $(S,S)=0$,

$$S = S_0 + S_1 + S_2 + \cdots$$

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Etc

Canonical transformations

Canonical transformations

Theorem : different choices of structure functions lead to solutions S of the master equation that are related by a canonical transformation (in the antibracket).

Conclusions and comments

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Finally, locality can be controlled, which leads to interesting developments (local BRST cohomology).

Conclusions and comments

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THANK YOU !

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