Applications of Toric Birational Geometry to String Compactifications

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Cornell University
The Unreasonable Effectiveness of Toric Geometry
July 8, 2026

Based on:

Calabi–Yau Threefolds from Vex Triangulations with Nate MacFadden (2512.14817)
Global Sections, the Minimal Model Program and Birational Tomography with Andrei Constantin and Andre Lukas (26XX.XXXXX)
Ongoing conversations with Liam McAllister, Jakob Moritz, & Mike Stillman

Some Goals

Research Objectives

See talks by Vishnu from Monday, Alessandro from yesterday!

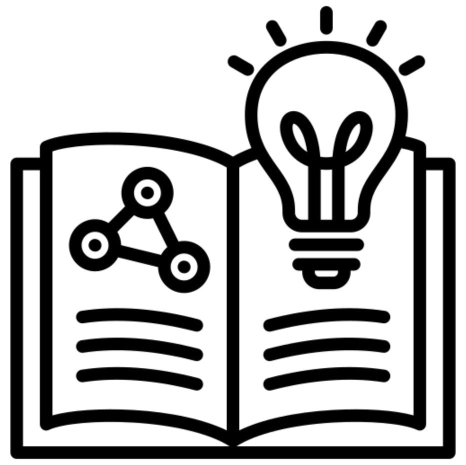
- Do **"string phenomenology"** — use string theory (specifically, compactifications of string theory on some Calabi–Yau variety X) to describe reality
- Compute aspects of the **topology and geometry** of X, as these contribute to the physical properties of string theory compactified on X
- Today: **birational geometry** of Calabi–Yau varieties, enabled in an important way by toric geometry

Sociological Objectives

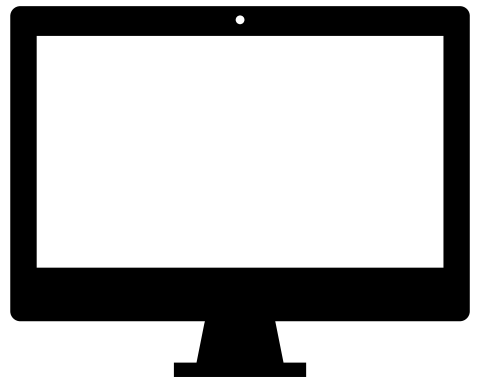
- **Learn from + find overlapping interests with attendees** ∴ I will mention an array of ideas, at varying levels of progress

Theory vs Computation & Role of Toric Geometry

String phenomenology depends crucially on
both **theory** and **computation**



General results on CYs determine **what physics is possible** in string compactifications



Explicit computations with CYs determine the **physics of a given compactification**

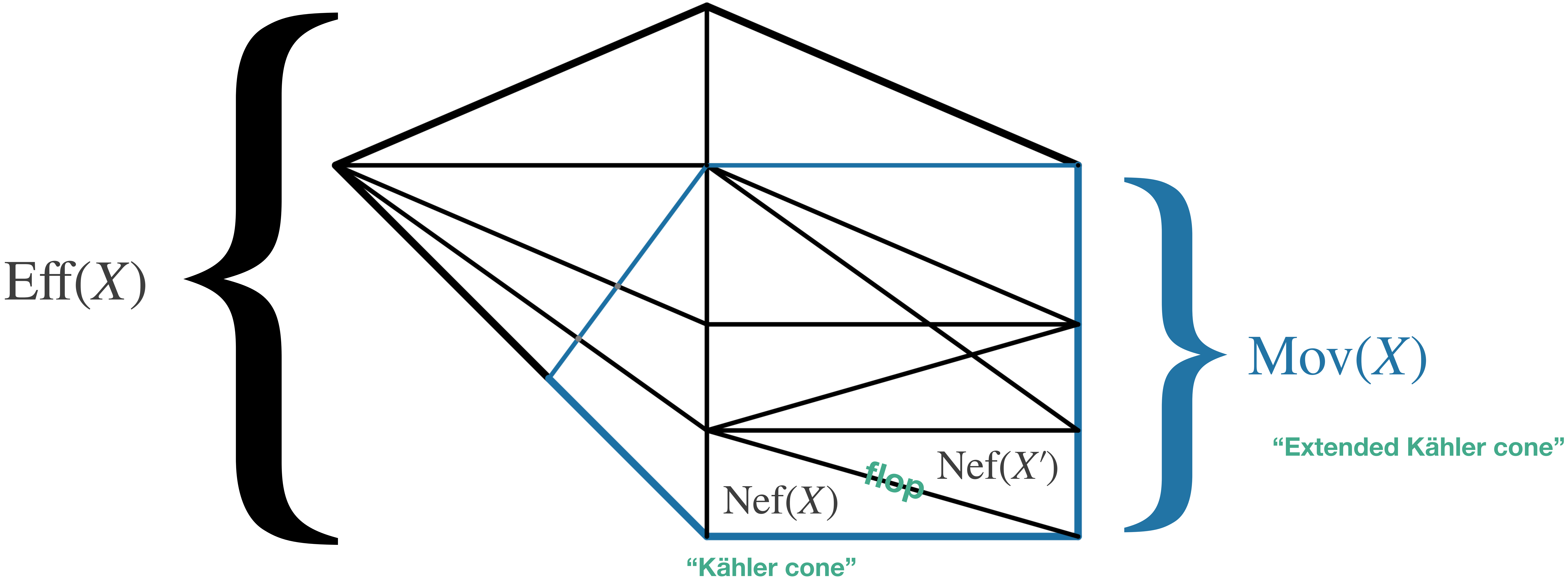
Toric geometry famously crucial for computations, aren't necessarily obviously related to general theory of CYs

Indeed, in string theory one fears a "lamppost effect" — being misled about quantum gravity by only studying toric subvariety CYs

In this talk, will emphasize (toric-enabled) computation, before culminating in the possible *theoretical* role of toric geometry for theory of CYs

Today's Question

How can I understand / compute the birational geometry (extended Kähler cone) of Calabi–Yau threefolds?



Talk Overview

I. Motivation

II. Toric hypersurface Calabi–Yau threefolds
   A. Vex triangulations
      (beyond fine, regular, star triangulations)
   B. “Non-inherited phases”
      (and if they can be made toric)

III. General Calabi–Yau threefolds
   (Cox rings, Mori dream spaces, and toric varieties)

Motivation

Why the extended Kähler cone (EKC)?

No new hodge numbers — if Calabi–Yau threefolds related by small birational maps (flops) are basically “the same” to you, maybe not that interesting

vs. Marco’s talk from Monday

However

Many **new examples** of CY varieties
(mathematically interesting & new places to do physics)

Physical parameters (e.g., masses & couplings) depend on Kähler moduli:
the more we understand the EKC,
the **better equipped we are to match observed physics**

Vex Triangulations

Background (I)

Notation: $\Delta^\circ \subset N_{\mathbb{R}}$, $\Delta \subset M_{\mathbb{R}}$ V from fan Σ

Kreuzer & Skarke '00

Recall **fine, regular, star triangulations (FRSTs)** of Δ° induce toric varieties with smooth CY hypersurfaces (abuse notation by identifying polytope triangulation ~ fan)

Batyrev '93

Central fan $:=$ normal fan of Δ, fan from height vector $(1,\ldots,1)$, face fan of Δ°, …

Background (II)

Notation: $\Delta^\circ \subset N_{\mathbb{R}},\ \Delta \subset M_{\mathbb{R}}$ V from fan Σ

The following are equivalent characterization of (fans arising from) FRSTs

- FRSTs refine the central fan
- Secondary cones of FRSTs contain $(1,\ldots,1)$ / $-K_V$ See lectures by Hal from last week!
- Toric varieties from FRSTs have nef $-K_V$, in particular basepoint free $-K_V$
- $\forall \sigma \in \Sigma$, minimal generators $u_\rho \in N$ for $u_\rho \in \sigma$ lie in common facet (three-face) of Δ°

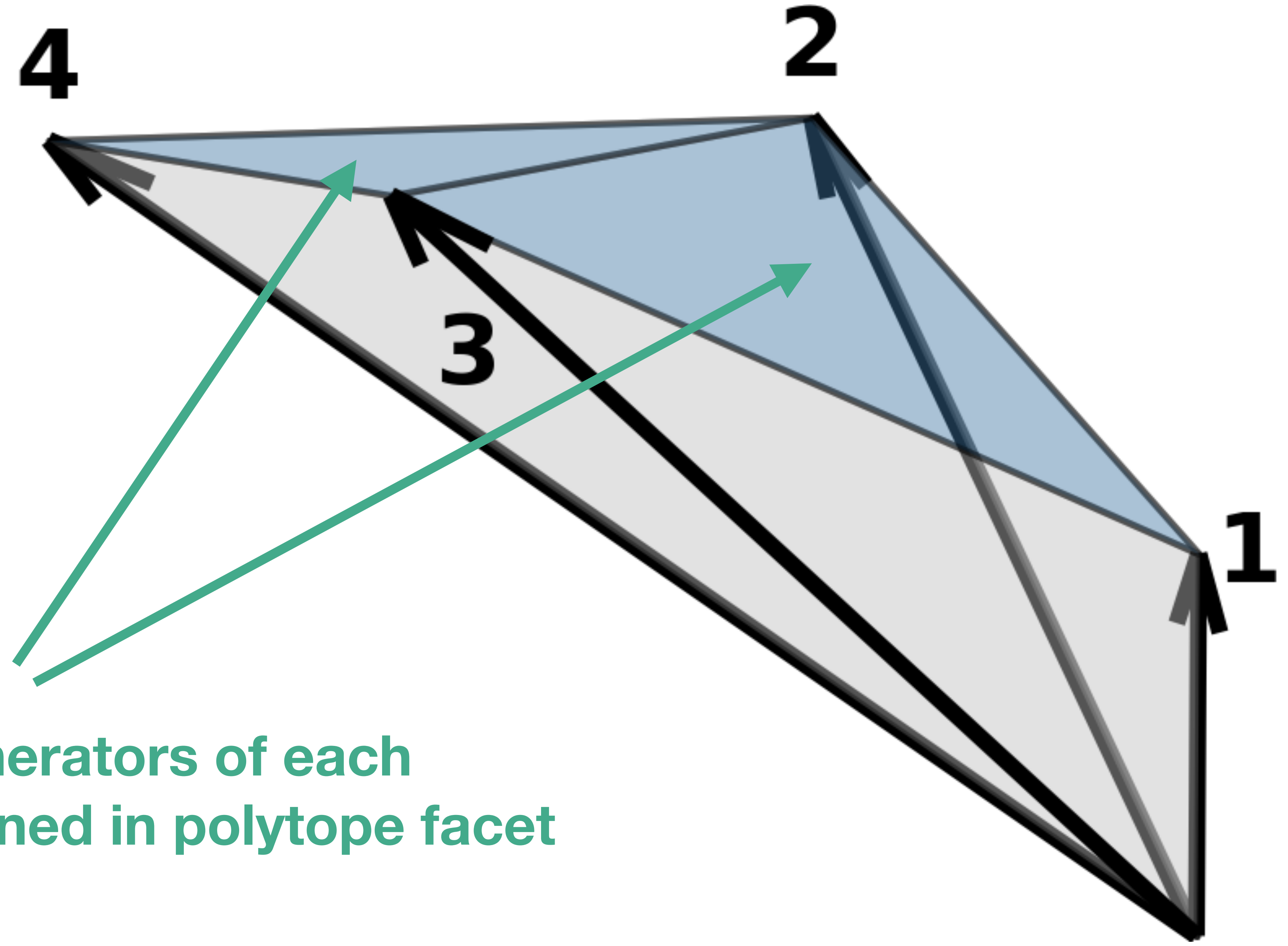
Vex triangulations are the fans which are not FRSTs
(i.e., do not have above properties), following references:
(still standard toric geometry, still using all rays)

Berglund & Hubsch '16 (x2)
Huang & Taylor '20
Berglund & Hubsch '22
Berglund & Lathwood '22
Berglund & Hubsch '24
Berglund, Huang, Taylor, & Wang (unpublished)

Visualizing Vexity

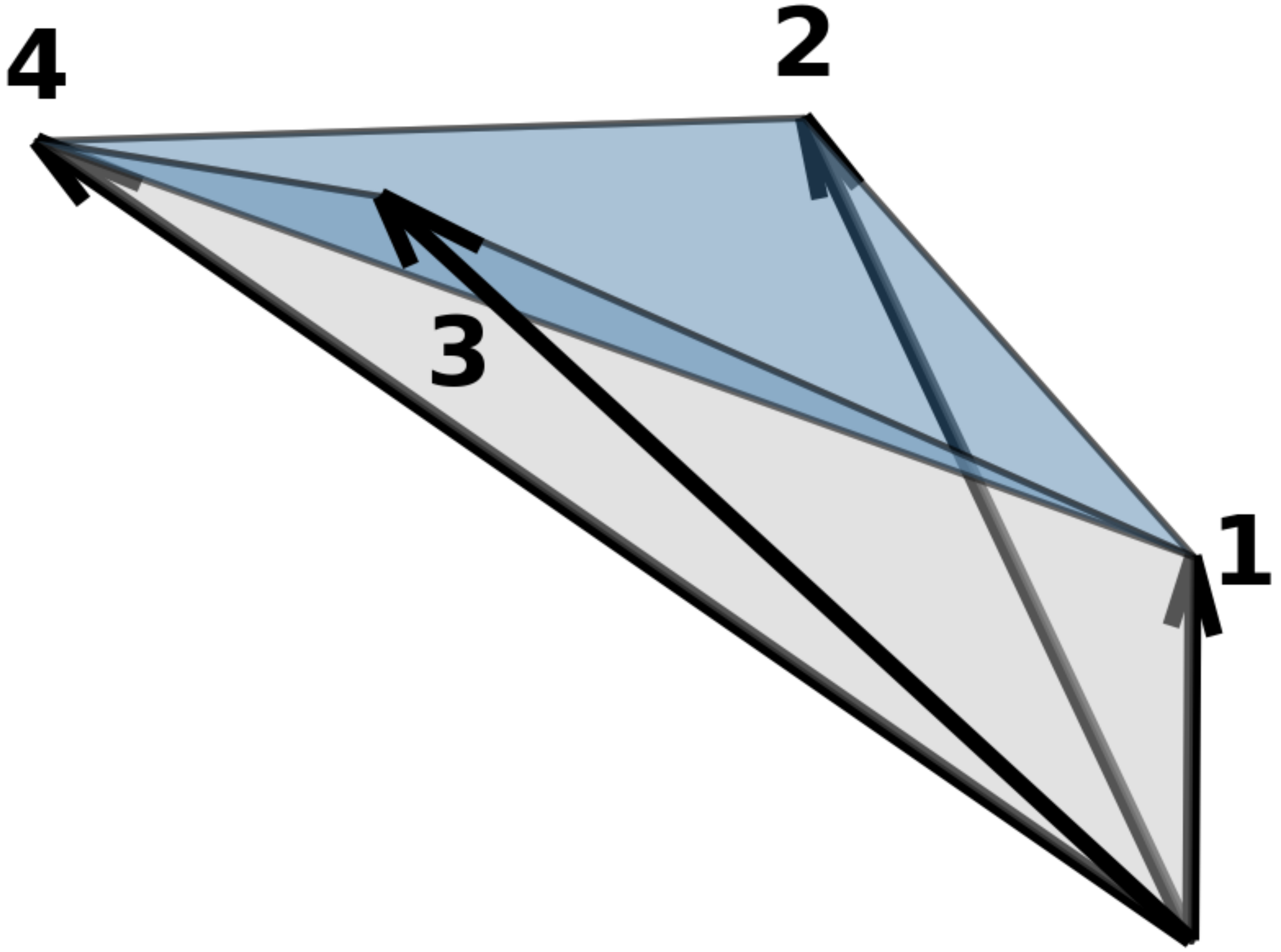
(Imagine this being just one cone in a complete fan / one part of a polytope)

Is the face fan



Minimal generators of each
cone contained in polytope facet

"FRST"



"vex"

Calabi–Yau hypersurfaces from vex triangulations

Because $-K_V$ is no longer basepoint free,
you should be worried about **singularities**

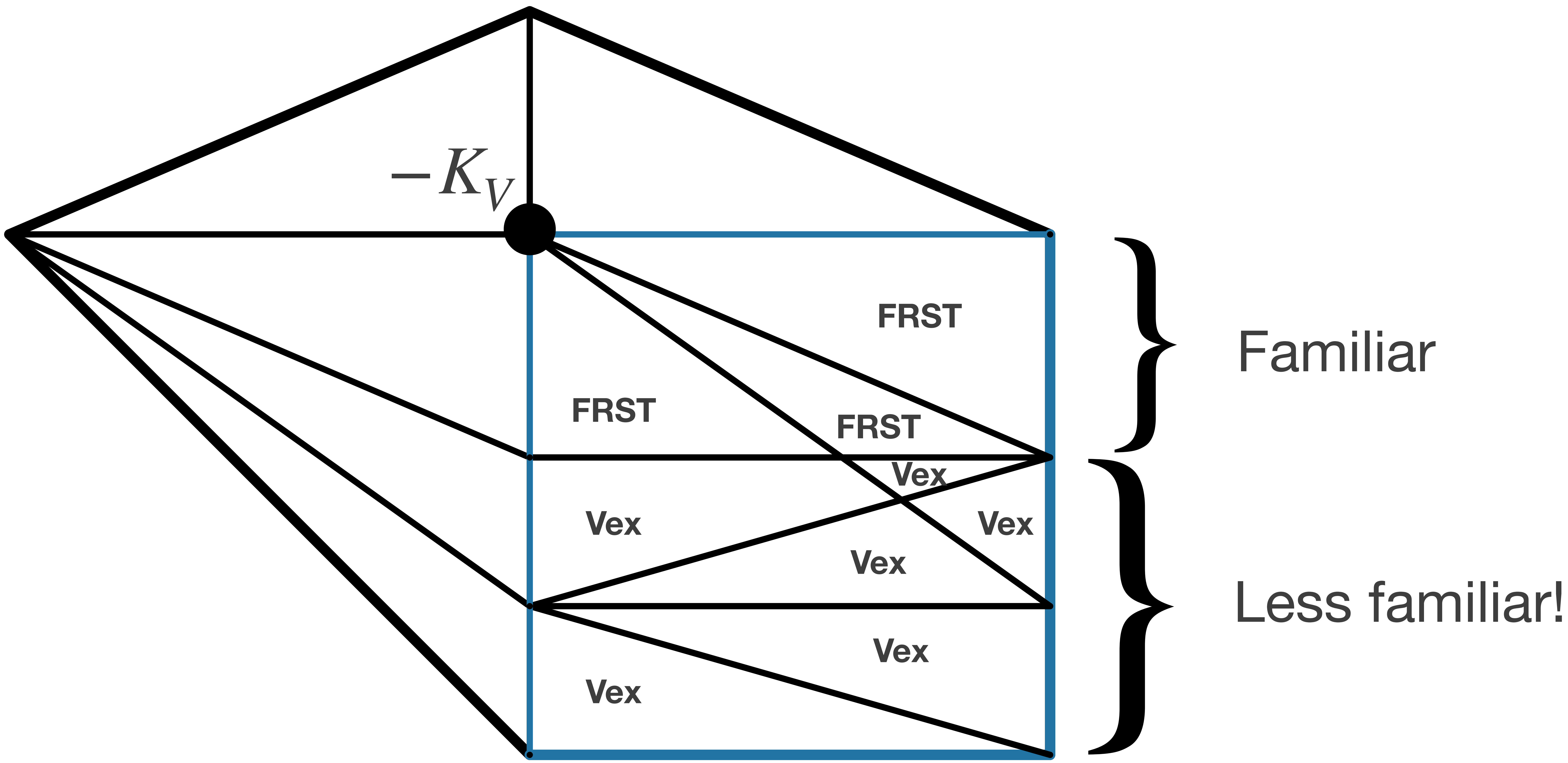
In particular, on the **base locus** you could have:

1) Intersections with **toric singularities**

2) **"Intrinsic singularities"** (failures of Jacobian criterion, solutions to $f = df = 0$)

However, non-trivially, it turns out that
for 4D reflexive polytopes, this never happens!

A Cartoon of the Secondary Fan



(Thanks to Jakob Moritz for the suggestion!)

Sketch of the Proof of CY Smoothness

Flops of smooth CY3s are smooth + CY3s from FRSTs are smooth + Vex triangulations connected to FRSTs by flips + Flips descend (at worst) to CY3 flops (small birational maps)

Kollar '89 | Batyrev '93 | (standard) | New!

Main lemma:

Cox, Little, Schenck '11

Lemma 8.3.6. *Let m, m' be distinct lattice points on the boundary of a reflexive polytope P. Then exactly one of the following holds:*

(a) *m and m' lie in a common edge of P,*

(b) *$m + m' = 0$, or*

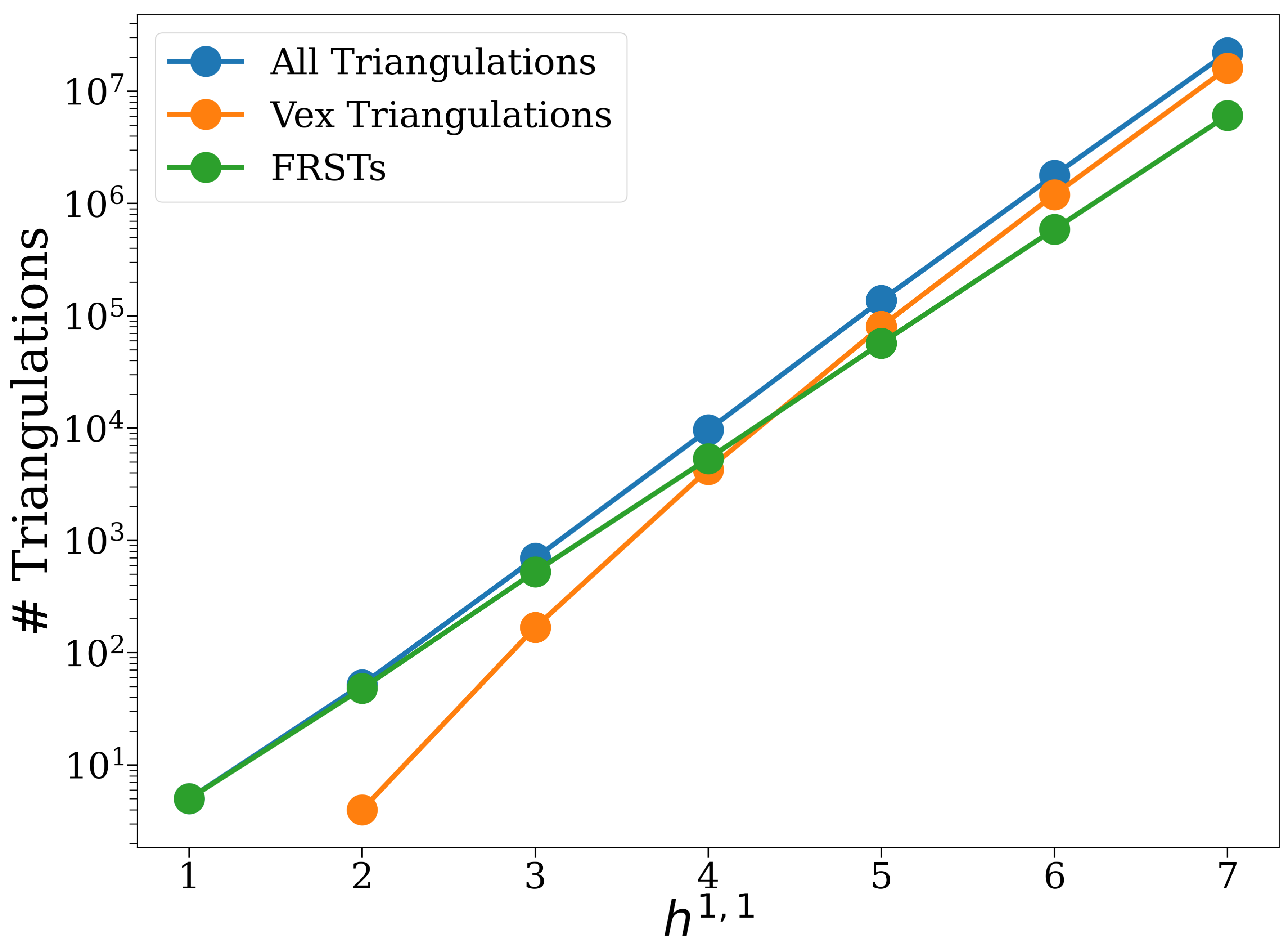
(c) *$m + m'$ is also on the boundary of P.*

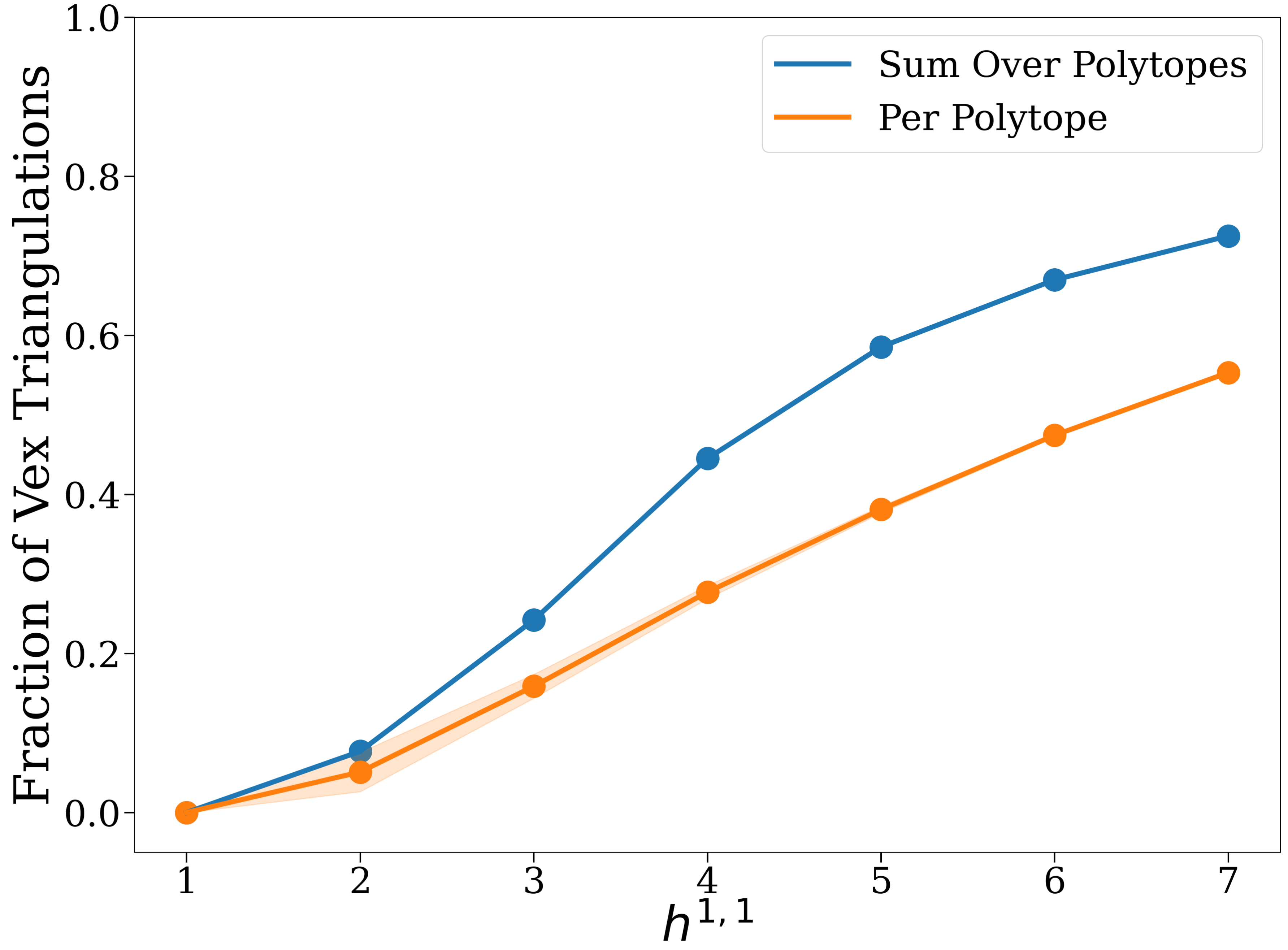
$\Longrightarrow$ Base locus of $-K_V$ has codimension > 2

A *small* birational modification descends to a *non-small* birational modification on a hypersurface only if the exceptional locus is codim 2 & contained in base locus

Contradiction!

Statistics of Vex Triangulations in KS





Computed using excellent Python library **regfans** developed by Nate MacFadden

$h^{1,1}$	# Polytopes	# Fine Regular Triangulations	# Vex Triangulations
1	5	5	0
2	36	52	4
3	244	694	168
4	1197	9639	4291
5	4990	137620	80570
6	17101	1786783	1196698
7	50376	22089147	16014955
Total	73949	24023940	17296686

Upshot of vex triangulations

Many new examples of smooth CY3s from Kreuzer–Skarke database
(seem to be **exponentially more vex triangulations than FRSTs**)

$$X \subset V \implies \mathrm{Mov}(V) \subset \mathrm{Mov}(X)$$

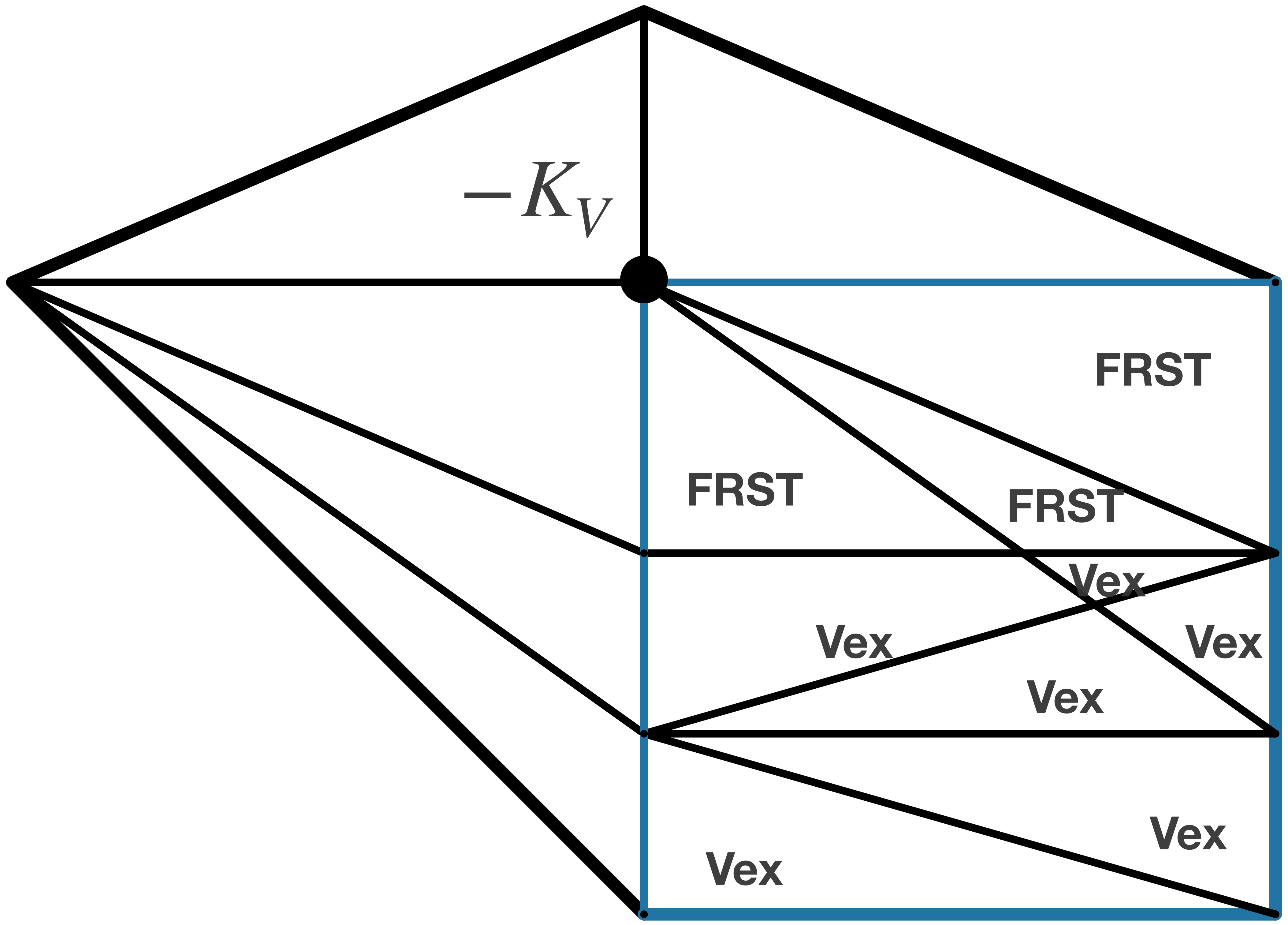
***Entire* movable cone of toric variety** contained in
movable cone (EKC) of CY hypersurface, for 4D reflexive polytopes

Non-inherited phases

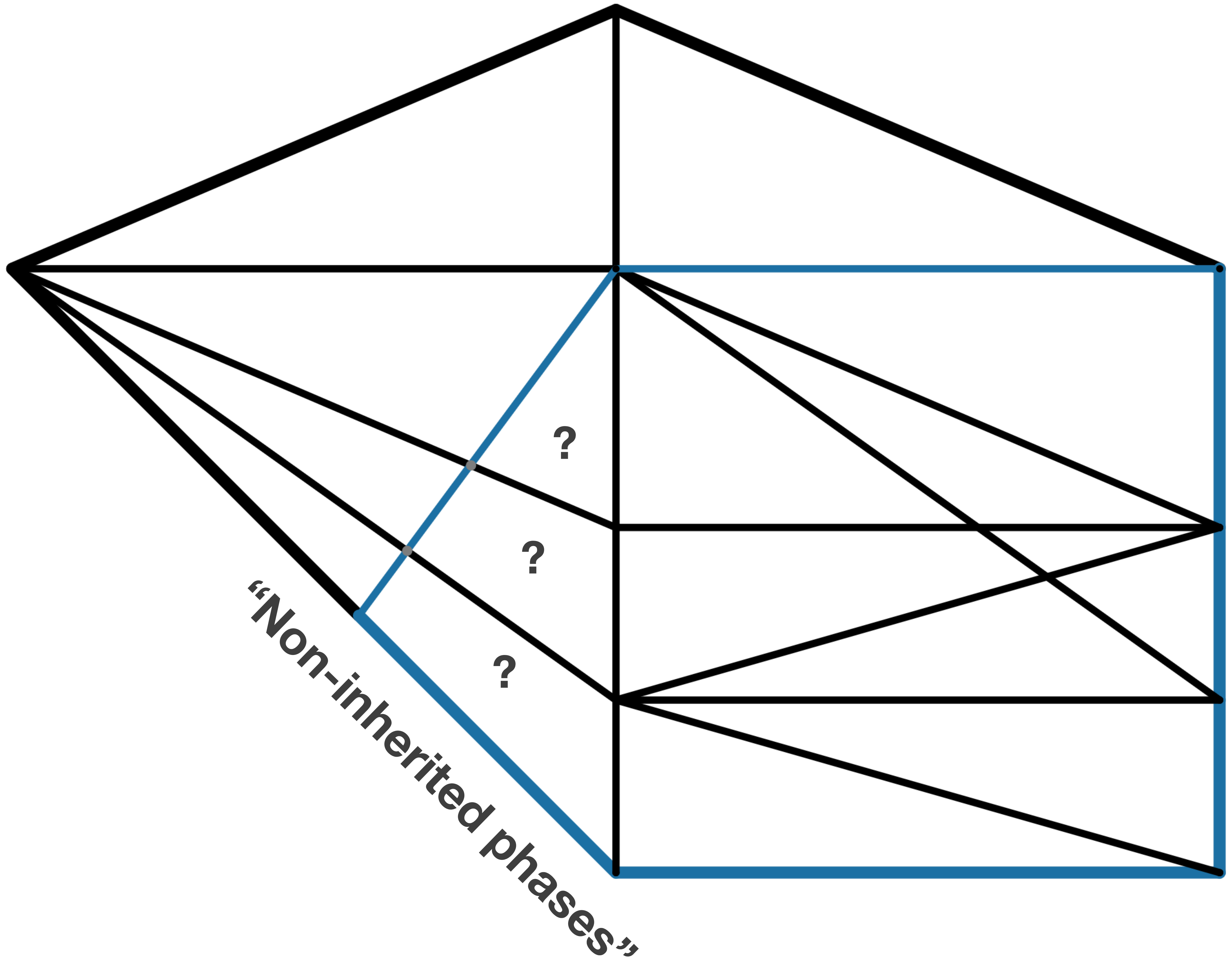
Vex isn't enough — "non-inherited phases"

Gendler, Heidenreich, McAllister, Moritz, & Rudelius '22
Gendler, MacFadden, McAllister, Moritz, Nally, Schachner, & Stillman '23

Toric Variety



Calabi–Yau hypersurface



∃ flops of the hypersurface not inherited from flips of toric variety

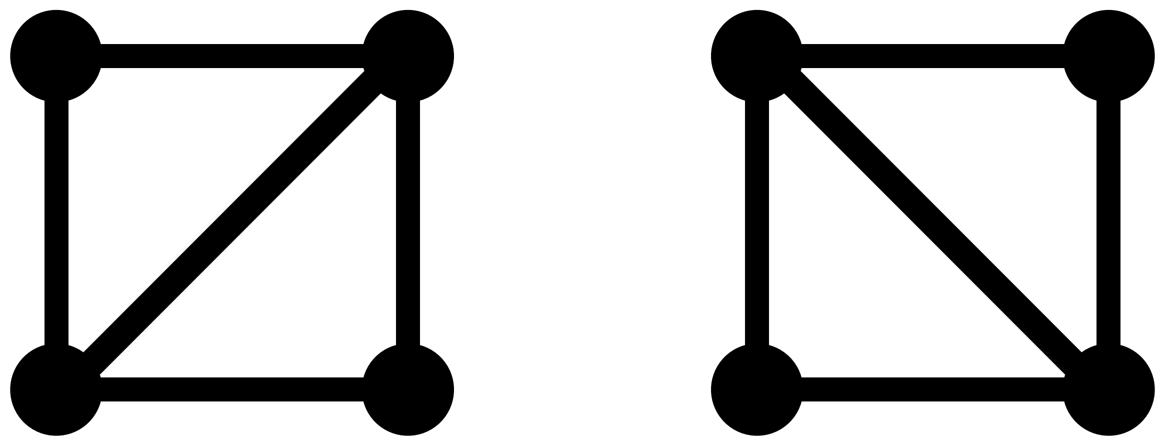
Understanding Non-Inherited Phases

Recall that flips of toric varieties are prescribed by **circuits**, or minimal linear relations among generators u_ρ of one-cones

See lectures by Hal from last week!

Circuits have **signatures** $\in \mathbb{Z}^2_\geq$ given by number of $+/-$ terms in relation

e.g.,



$$u_1 + u_2 - u_3 - u_4 = 0$$

$\therefore$ signature $(2,2)$

Understanding Non-Inherited Phases

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or minimal linear relations among generators u_ρ of one-cones

Circuits have **signatures** $\in \mathbb{Z}^2_{\geq}$ given by number of $+/-$ terms in relation

(m, n) — flip $(m,1)$ — contracted divisor $(m,0)$ — fibration

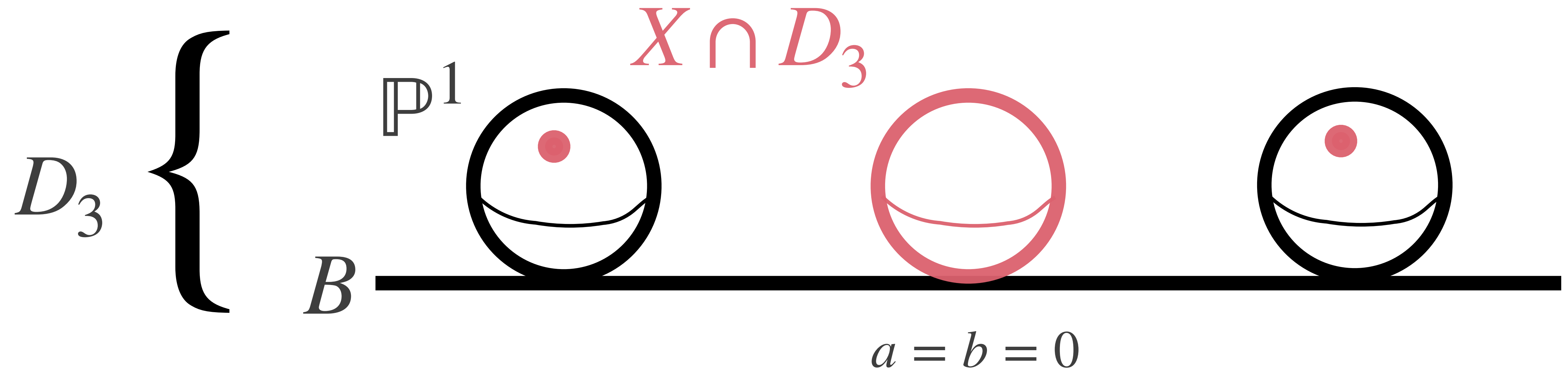
Vex triangulation story handles (m, n) circuits, but how do
divisorial contractions & fibrations (boundaries of toric movable cone)
affect the anticanonical hypersurface?

$(m,1)$ usually induces CY divisorial contraction

$(m,0)$ induces fibration of CY Kreuzer & Skarke '00

But then there are special cases yielding flops & "non-inherited phases"—
if CY birational geometry isn't inherited from toric birational geometry,
can we keep using toric methods somehow?

Example — (2,1)



Consider circuit with linear relation $u_1 + u_2 - u_3 = 0$

On ambient V, D_3 is $\mathbb{P}^1_{x_1,x_2}$ fibration over a surface / blowup of surface B — circuit corresponds map $V \to V'$ contracting that fiber

Can argue that CY equation factors as $f = x_1 a - x_2 b$

$D_3 \cap X$ is a non-flat fibration over B with generic fiber given by a point but on locus $a = b = 0$ contains entire $\mathbb{P}^1$ fiber

Toric contraction descends to contraction of these isolated $\mathbb{P}^1$s — a flop to some CY not captured by birational class of V (& GV from intersection theory!)

Example — (2,1)

Anticanonical hypersurface $X' \subset V'$ in blowdown is conifold transition of X, but we know how to construct resolutions of X':

$$\det \begin{pmatrix} a & b \\ x_2 & x_1 \end{pmatrix} = 0 \qquad \to \qquad \begin{pmatrix} a & b \\ x_2 & x_1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = 0$$

Both small resolutions are codim-2 complete intersections in $\mathbb{P}^1_{s,t}$-bundle over V'

Example of Example — Nodal Quintic

Example of the (2,1) signature in Kreuzer–Skarke at $h^{1,1} = 2$

$V =$ blowup of surface on $\mathbb{P}^4$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$V' = \mathbb{P}^4$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$\mathbb{P}^1$ bundle over V':

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

7

I assume that you mean $x_1 = x_2 = 0$ in your description of the divisors. I believe that you only really get two different resolutions, but yes, they are both projective varieties. Hodge numbers of birational complex Calabi-Yau varieties are always the same.

You can think of this as a determinantal variety for the matrix A of the form

$$\begin{pmatrix} \deg 1 & \deg 1 \\ \deg 4 & \deg 4 \end{pmatrix}$$

in the variables x. The resolutions correspond to considering degenerate matrices together with either left or right kernel vectors.

This is also very much a toric example, in the sense that both resolutions are complete intersections of two hypersurfaces that correspond to "double-mirror" nef partitions, see for example the paper of my student Zhan Li "On the birationality of complete intersections associated to nef-partitions", arXiv:1310.2310.

Specifically, if I got my calculations right, one resolution is a complete intersection of degrees $(1,1)$ and $(1,4)$ in $\mathbb{P}^1 \times \mathbb{P}^4$, and the other is a complete intersection in the $\mathbb{P}^1$ bundle $\mathbb{P}(\mathcal{O}(-1) + \mathcal{O}(-4))$ over $\mathbb{P}^4$ of the half-anticanonical divisor that corresponds to the tautological line bundle $\mathcal{O}(1)$ of this fibration. The singular variety in question is the projection to the $\mathbb{P}^4$ in both cases.

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answered Oct 25, 2014 at 2:15

Lev Borisov

Add a comment

Upshot

Why so important to find toric descriptions for non-inherited phases?

They're there, we should study them!

But also, certain phenomena seem empirically to only arise in KS
but **only in non-inherited phases**: e.g.,

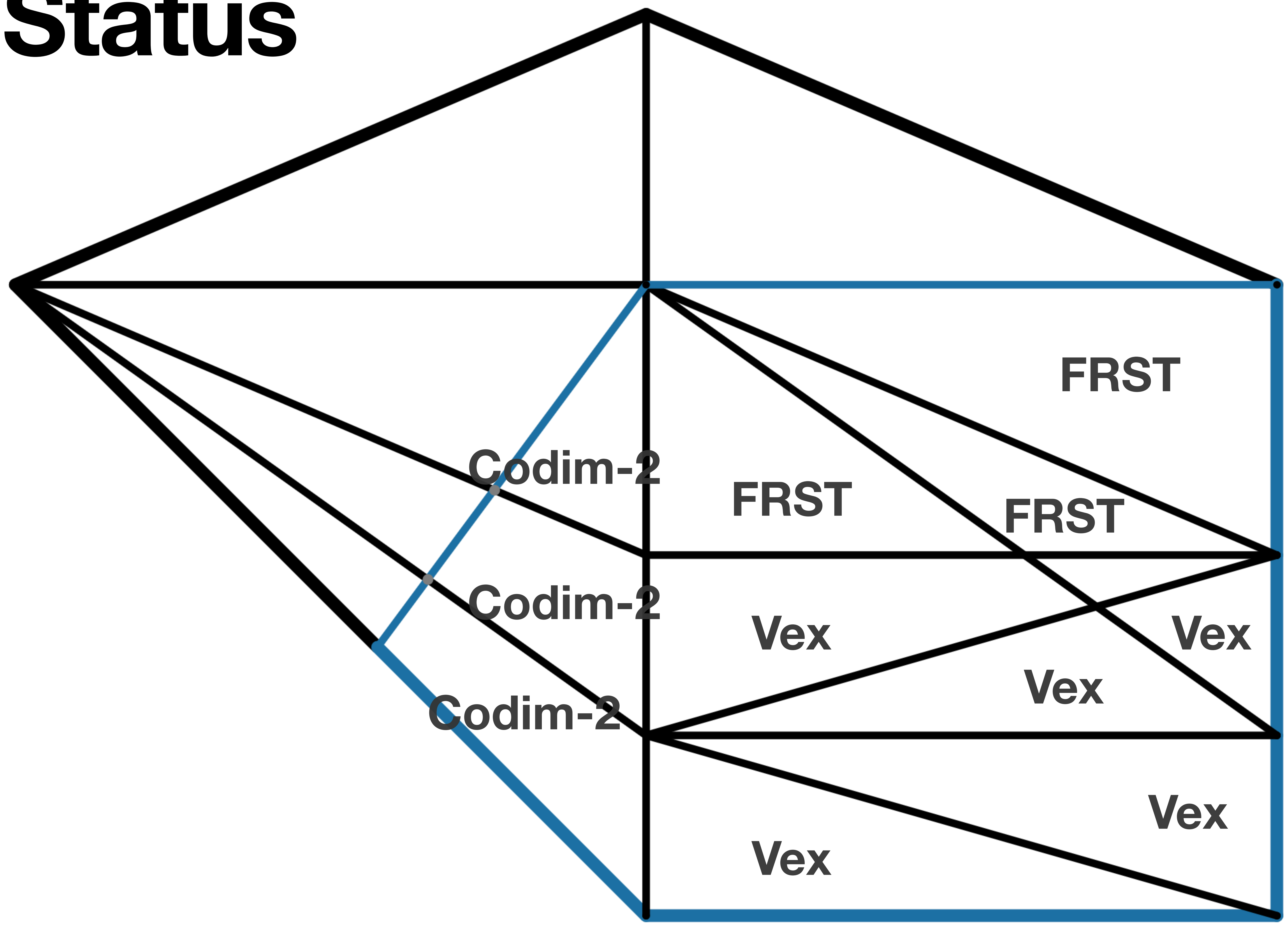
Flops of length > 1

Fibrations with non-toric bases

One imagines a program where one iteratively encounters non-toric phases
& renders them toric by "paying" with codimension —
but I don't know the procedure for codim >1!

General Calabi–Yau threefolds

Current Status



… but is there one toric variety to rule them all? Can we realize all of the phases in a single toric variety?

Recall Mori Dream Spaces

Mori dream spaces are roughly varieties whose nef and movable cones are finitely generated

Hu & Keel '00

Consider the Cox ring of all sections of all effective line bundles — Mori dream spaces are also varieties whose **Cox rings are finitely generated**

Example: **toric varieties**, with Cox rings given by the homogeneous coordinate ring

(in fact, toric variety $\Longleftrightarrow$ Cox ring is polynomial ring)

The "Good Toric Embedding"

Mori dream spaces inherit all of their birational geometry from a toric variety!

Let Mori dream space X have Cox ring R

Then $R \cong \mathbb{C}[x_1, \ldots, x_n]/I$

Algebraic geometry principle: maps of rings induce maps of varieties (the other way)

$\mathbb{C}[x_1, \ldots, x_n] \to R$ induces $X \hookrightarrow V$

(V toric variety with above homogeneous coordinate ring)

Theorem: this embedding is **favorable**, and Mori chambers of V **refine** those of X

(i.e., all birational maps of X are **inherited** from V)

Hu & Keel '00

Implications for Calabi–Yau Varieties

Famously, Calabi–Yau varieties are in general **not Mori dream spaces** (infinitely generated nef cones & "infinite flop chains")

Brodie, Constantin, Lukas, & Ruehle '21
Gendler, Kim, McAllister, Moritz, & Stillman '22

But the **Kawamata–Morrison cone conjecture** asserts that the nef cone and movable cones of Calabi–Yau varieties admit finite rational polyhedral fundamental domains

Morrison '93
Morrison '96
Kawamata '97

[AHL10]. More generally, the cone conjecture asserts that Calabi-Yau varieties have similar structures as Mori dream spaces "up to the action of birational automorphism groups"

Okawa '15
Choi, Li, Li, & Zhou '25

There is Mori dream space technology for this — KM means that fundamental domain of effective cone for Calabi–Yau varieties is a **"Mori dream region"** **Hu & Keel '00**

∴ a fundamental domain of the effective cone of any CY should be inherited from a toric variety

Küronya & Urbinati '19

Why Should I Care?

Disclaimer — Cox rings are very hard to compute, so finding the "good embedding" is hard!

Batyrev '03
Laface & Velasco '07
Herrera, Laface, & Ugaglia '24
...

Maybe toric geometry isn't quite the lamppost we might've thought — maybe Calabi–Yau varieties were "meant to be toric"

& can this formalism let us continue to use toric geometry, away from KS & CICY lampposts?

The "good embedding" is a great place to do computations — so much is inherited directly from the toric variety

Even when not constructible, the mere existence of the "good embedding" could be useful

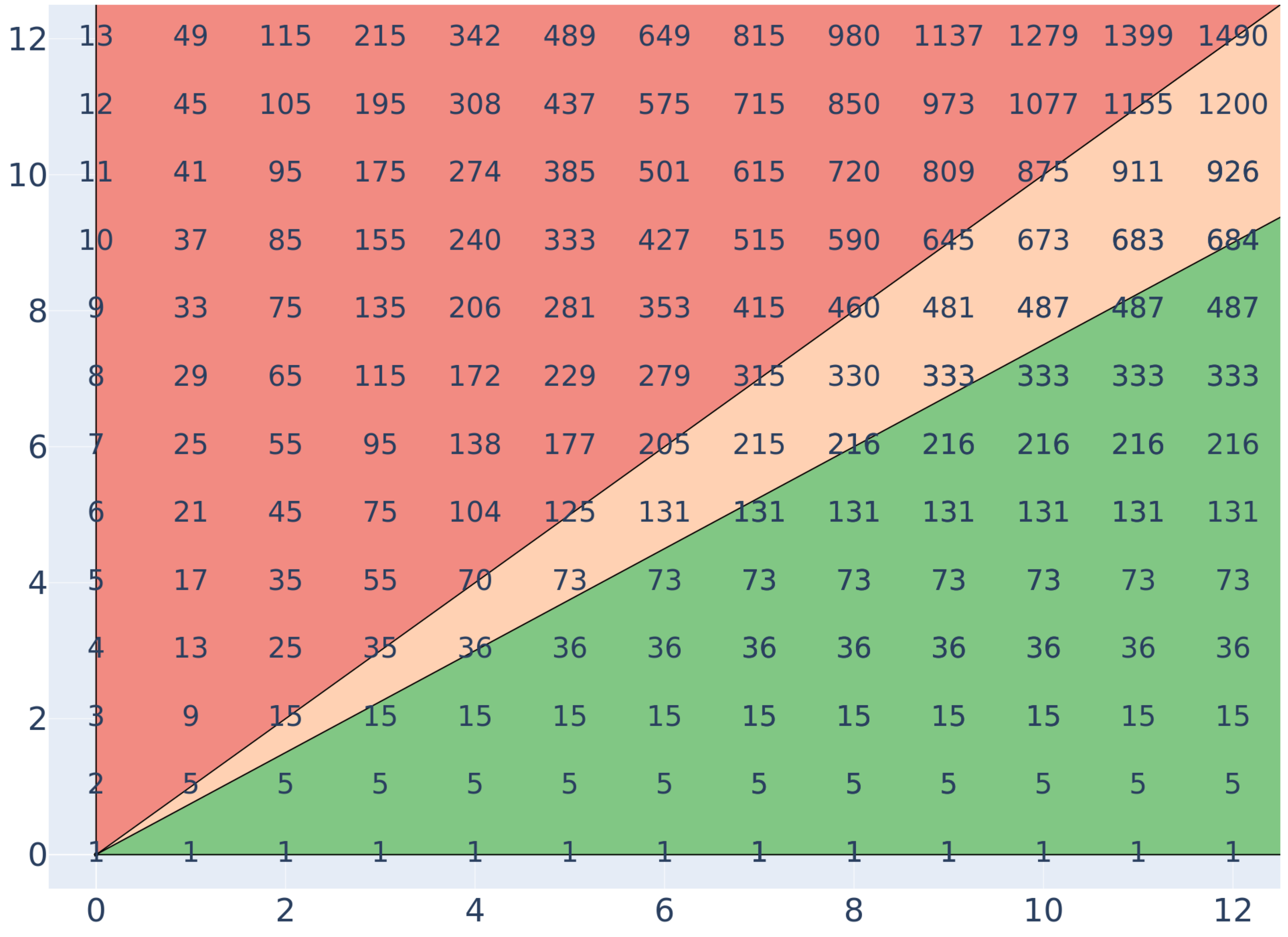
Like, e.g., CY metric, or log resolutions

Also, interesting invariants: if $R \cong C[x_1, \ldots, x_n]/I$, what is smallest n or simplest I?

Example — Nodal Quintic, Again

Using the methods from Andrei's talk, can easily compute the # of global sections of all divisors on the $h^{1,1} = 2$ example from earlier

This obscures the ring structure of the Cox ring, but assume a complete intersection and deduce the generators / relations the Cox ring must have by eye



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Use Wall's theorem to check your answer, & conclude that a "good embedding" is a complete intersection of two sections of (4,4) in

$$Q = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 0 & 0 & 1 & 4 \end{pmatrix}$$

(A nef partition!)

Conclusions

Thank you for your time!

- Constructing the **birational geometry / extended Kähler cone** of Calabi–Yau threefolds is an important problem motivated by string compactifications
- For 4D reflexive polytopes, **there is terrain beyond FRSTs**
  - Vex triangulations furnish many more computable smooth CY3s
  - **Non-inherited phases** also abound, but can maybe be systematically be rendered toric (at higher codimension)
- The **Mori dream spaces** & the **"good toric variety embedding"** together with the **Kawamata–Morrison cone conjecture** furnishes an interesting framework for understanding Calabi–Yau threefolds generally & computationally

Would be grateful to learn from you all about the theory & computation of this!