Construction and classification of symmetry protected topological states in interacting fermion systems

Zhengcheng Gu

Q-R Wang, Z C Gu, Phys. Rev. X 8, 011055 (2018)
Q-R Wang, Z C Gu, arXiv:1811.00536 (to appear on PRX)
Yunqing Ouyang, Q-R Wang, Z C Gu, Yang Qi, arXiv:2005.06572

The Chinese University of Hong Kong

大學教育資助委員會
University Grants Committee

Vienna. Aug. 2020

AGENCE NATIONALE DE LA RECHERCHE
ANR
Research Grants Council
of Hong Kong
香港 研究資助局

Topological phases of quantum matter: beyond Landau's paradigm

Definition:

Gapped quantum phases without symmetry breaking and long range correlation, but can not be adiabatically connected to a trivial disorder phase without phase transition.

Two basic classes of topological phases:

Intrinsic topological phases (long-range-entanglement)

- adiabatical paths with no symmetry

Symmetry protected topological (SPT) phases

- adiabatical paths with symmetry

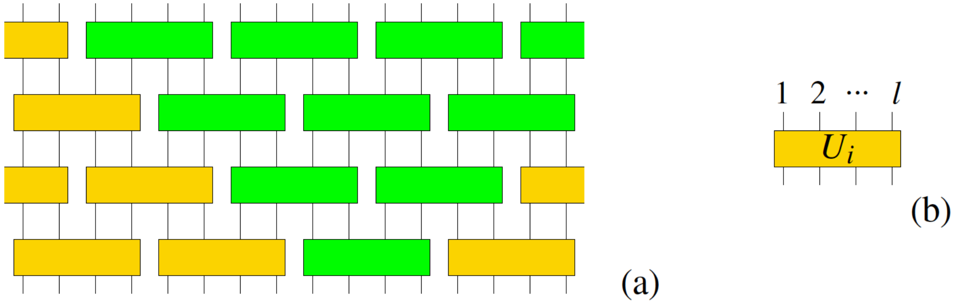
symmetry breaking Hamiltonians

SPT phases ⟺ The trivial disorder phase

(Z C Gu, X G Wen 2009)

Topological phases as equivalence class of local unitary transformation

- Two states describe the same intrinsic topological phase iff they are connected by finite depth local unitary(LU) transformation (acting on support space).



$$|\Phi(1)\rangle \sim |\Phi(0)\rangle \text{ iff } |\Phi(1)\rangle = U^M_{circ}|\Phi(0)\rangle$$

$$U^M_{circ} = U^{(1)}_{pwl} U^{(2)}_{pwl} \cdots U^{(M)}_{pwl} \qquad U_{pwl} = \prod_i U_i$$

- U_i is l-local for systems with local interactions
- U_i is fermion parity even for fermion systems

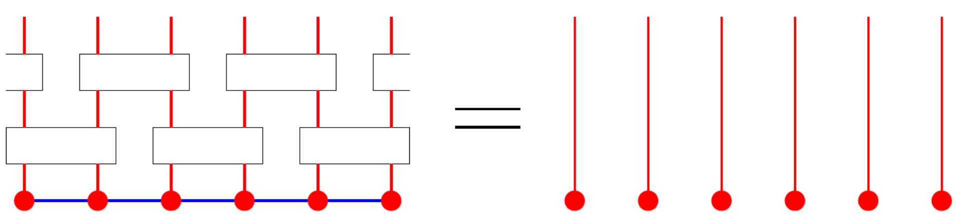
SPT phases as equivalence class of symmetric local unitary transformation

- Two states describe the same SPT/SET phase iff they are connected by finite depth *symmetric* local unitary(SLU) transformation (acting on support space).

- U_i is l-local and symmetric, e.g., by using the group element basis

$$U(gg_{i0}, gg_{i2}, gg_{i3}, \cdots) = U(g_{i0}, g_{i1}, g_{i2}, \cdots)$$

On the other hand, SPT state as short-range entangled state, and it can be connected to a product state without symmetry.



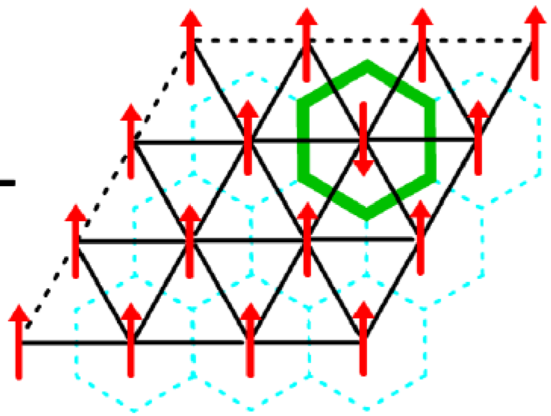
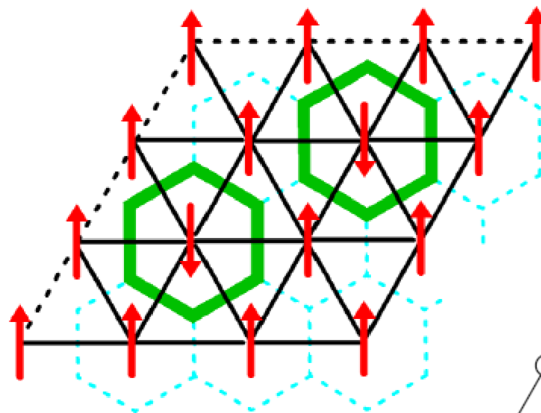
$$|\mathrm{SPT}\rangle = U_{circ}^{M}|\mathrm{Trivial}\rangle$$

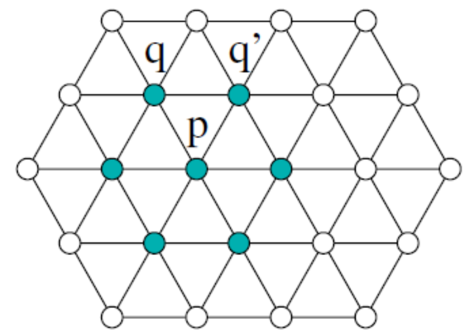
SPT phases are classified by equivalence class of symmetric LU transformation with one dimensional support space!

SPT states as fluctuating domain walls

An examples of bosonic SPT states with Ising symmetry

(M. Levin and Z.-C. Gu, Phys. Rev. B 86, 115109 (2012))

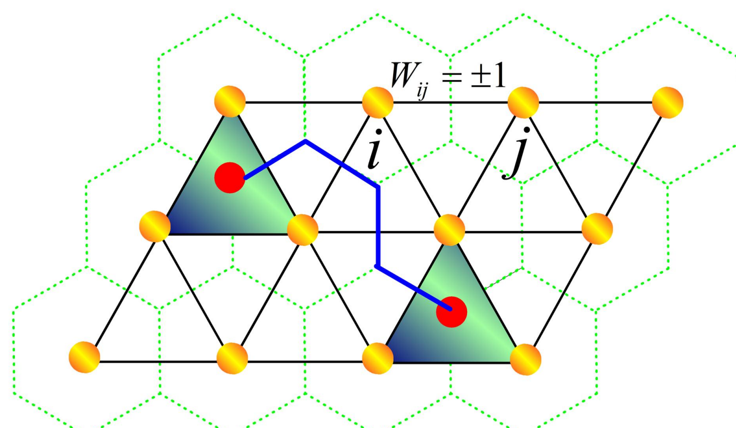
$| \, |\Psi_1\rangle = -$  $+$  $+ \ldots$



$$H_1 = -\sum_p B_p \ , \ B_p = -\sigma_p^x \prod_{\langle pqq' \rangle} i^{\frac{1-\sigma_q^z \sigma_{q'}^z}{2}} \qquad [B_p, B_{p'}] = 0$$

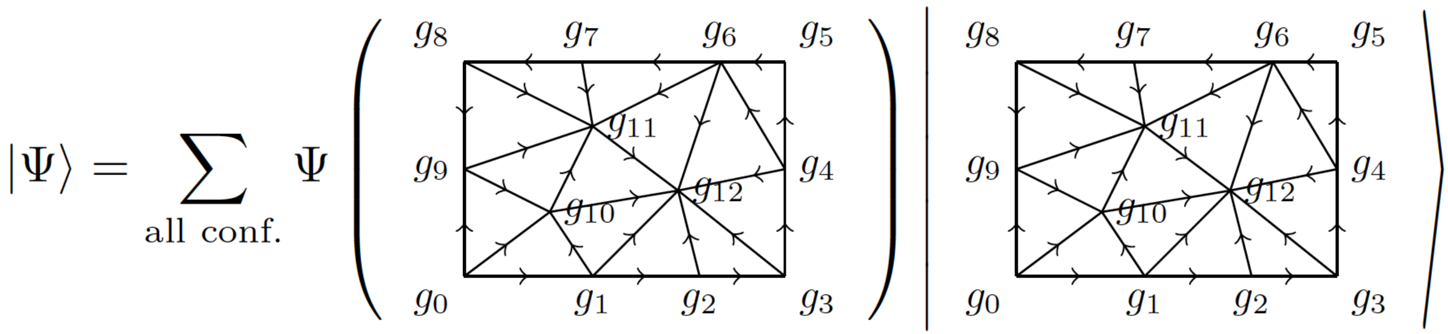
Ising spins carry Z_2 gauge charge and can couple to background Z_2 gauge field. Z_2 gauge flux carries semion statistics, classified by

$H^3(Z_2, U(1))$ (Dijkgraaf Witten, 1990)

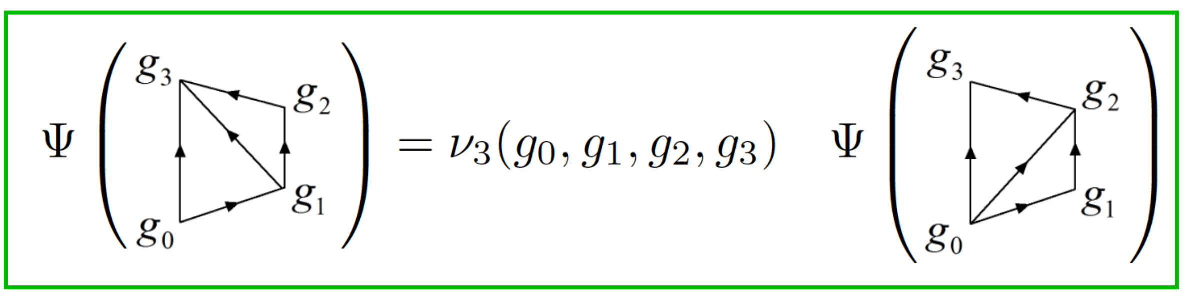


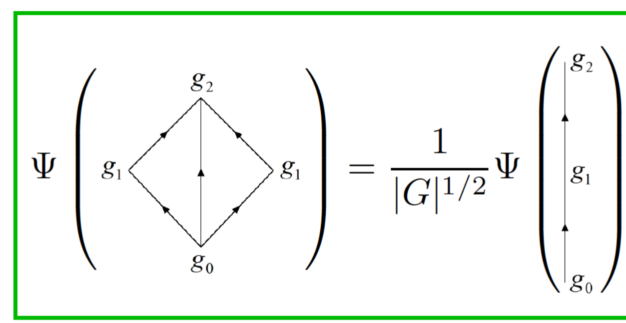
A general fixed point wavefunction for 2D SPT state

Fix point wavefunction on arbitrary branched triangulation



- Two types of fundamental SLU transformation which generates all the renormalization(retriangulation) moves for fixed point wavefunction

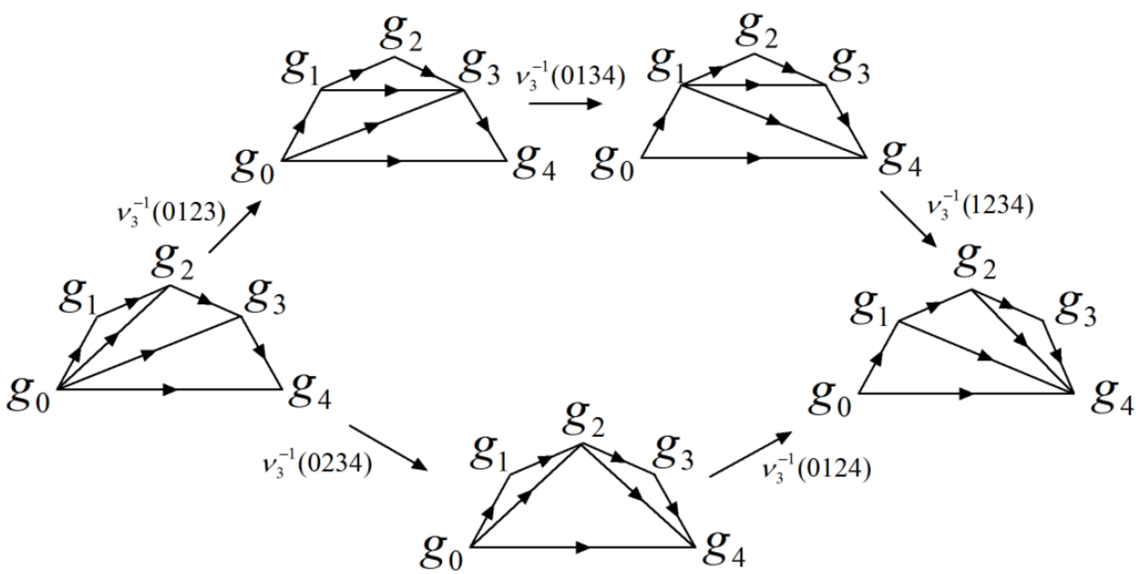




$\nu_3(gg_0, gg_1, gg_2, gg_3) = \nu_3(g_0, g_1, g_2, g_3)$: symmetric U(1) phase factor

Coherent condition and equivalent class

As a fixed point wavefucntion, different symmetric LU transformation must give rise to the same amplitude



Cocycle equation!

$$(\mathrm{d}\nu_3)(g_0,g_1,g_2,g_3,g_4) \equiv \frac{\nu_3(g_1,g_2,g_3,g_4)\nu_3(g_0,g_1,g_3,g_4)\nu_3(g_0,g_1,g_2,g_3)}{\nu_3(g_0,g_2,g_3,g_4)\nu_3(g_0,g_1,g_2,g_4)} = 1$$

Local basis change leads to equivalent solutions

$$|\{g_l\}\rangle' = U_{\mu_2}|\{g_l\}\rangle = \prod \mu_2(g_i,g_j,g_k)^{s_{\langle ijk\rangle}}|\{g_l\}\rangle$$

$$\nu_3'(g_0,g_1,g_2,g_3) = \nu_3(g_0,g_1,g_2,g_3)\frac{\mu_2(g_1,g_2,g_3)\mu_2(g_0,g_1,g_3)}{\mu_2(g_0,g_2,g_3)\mu_2(g_0,g_1,g_2)}$$

Coboundary equation!

- Bosonic SPT phases in 2+1D is classified by $H^3(G,U(1)_T)$

How about interacting fermions?

- Some special cases have been studied by reducing the free fermion classifications. (Lukasz Fidkowski and Alexei Kitaev,2010, Z.-C. Gu, M. Levin, 2013, Kitaev 2013, Chong Wang and T. Senthil 2014, M. A. Metlitski, L. Fidkowski, X. Chen, and A. Vishwanath, 2014)

- Stacking bosonic SPT states on top of free fermion SPT states, which is quite successful for the classification of interacting topological insulators.(C. Wang, A. C. Potter, and T. Senthil 2013, Chong Wang and T. Senthil 2014)

- Spin cobordism and invertible TQFT.(A. Kapustin, R. Thorngren, A. Turzillo, and Z. Wang, 2014, D S Freed 2014, A. Kapustin, R. Thorngren 2018)

- A general classification is still hard due to the Fock-space structure of interacting fermion systems.

Basic concepts of classifying SPT phases in interacting fermion systems

- 1D fermionic systems can be mapped to bosonic systems with an additional unbroken fermion parity symmetry. (Xie Chen, Z C Gu, X G Wen, Phys. Rev. B 84, 235128 (2011))

- The statistics of the gauge flux is still a good way to understand the 2D classification. (Z.-C. Gu, M. Levin, Phys. Rev. B 89, 201113(R) (2014) M. Cheng, Z. Bi. Y. You, Z. C. Gu. arXiv:1501.01313(2015))

- A special group super-cohomology theory is developed to classify a minimal subset(decoration of complex fermion on intersection points of symmetry domain walls). (Z.-C. Gu, X.-G. Wen, Phys. Rev. B 90, 115141 (2014))

- A further decoration of Kitaev's Majorana chains on symmetry domains leads to a complete classification for fermionic SPT states(for unitary symmetry)!

Symmetry group G^f for fermion systems

- A generic G^f is defined by the short exact sequence:

$$1 \to \mathbb{Z}_2^f \to G_f \to G_b \to 1$$

- It is called central extension and specified by a Z_2 coefficient cocycle:

$$\boxed{\omega_2 \in H^2(G_b, \mathbb{Z}_2 = \{0,1\})}$$

- For a given group element in the total group G_f:

$$g_f = (P_f^{n(g)}, g_b) \in \mathbb{Z}_2^f \times G_b,$$

$$g_f \cdot h_f = \left(P_f^{n(g)}, g_b\right) \cdot \left(P_f^{n(h)}, h_b\right) := \left(P_f^{n(g)+n(h)+\omega_2(g_b,h_b)}, g_b h_b\right)$$

$$P_f^{n(g)+n(h)+\omega_2(g_b,h_b)} \in \mathbb{Z}_2^f \text{ and } g_b h_b \in G_b.$$

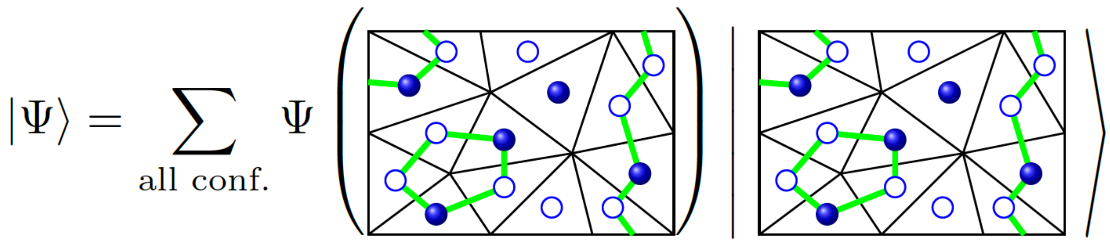
- The associativity of group multiplication holds naturally due to the fact that:

$$\omega_2(h,k) + \omega_2(gh,k) + \omega_2(g,hk) + \omega_2(g,h) = 0$$

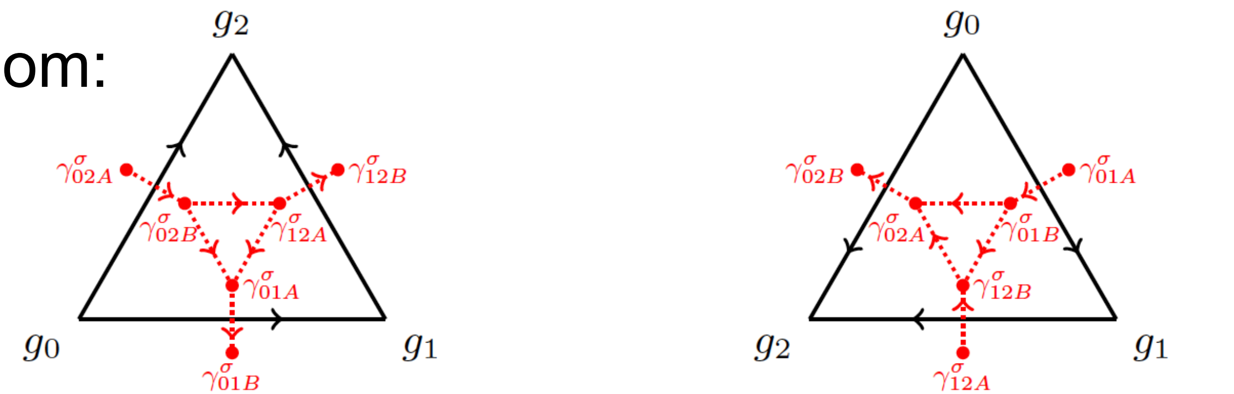
- It is easy to check: $\omega_2(e,g) = \omega_2(g,e) = 0$

Fixed point wavefunction of 2D SPT states with generic total symmetry G_f

- Fixed point wavefunction on arbitrary branched triangulation with discrete spin structure(Kasteleyn orientation).



- Degrees of freedom:



(a) Positive oriented triangle (b) Negative oriented triangle

- $|G_b|$ level bosonic (spin) state $|g_i\rangle$ ($g_i \in G_b$) on each vertex i.
- $|G_b|$ species of complex fermions c^σ_{ijk} ($\sigma \in G_b$) at the center of each triangle $\langle ijk\rangle$.
- $|G_b|$ species of complex fermions (split to Majorana fermions) $a^\sigma_{ij} = (\gamma^\sigma_{ij,A} + i\gamma^\sigma_{ij,B})/2$ ($\sigma \in G_b$) of each link $\langle ij\rangle$.

Kitaev's Majorana chain and Kasteleyn orientation

- Hamiltonian of Majorana chain:

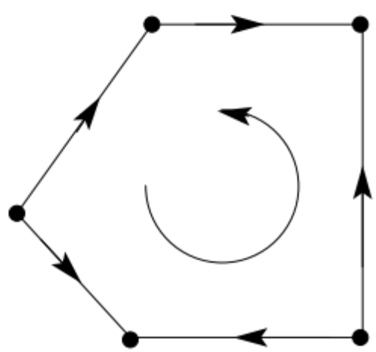
$$H = -\sum_j (c_j^\dagger c_{j+1} + \mathrm{h.c.}) - \sum_j (c_j c_{j+1} + \mathrm{h.c.})$$
$$= i\sum_j \gamma_{2j}\gamma_{2j+1}$$

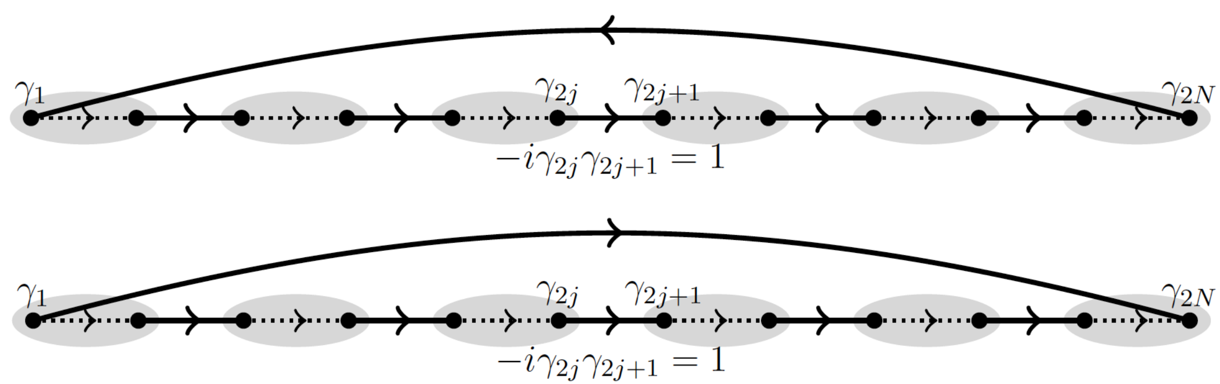
$$\begin{cases} \gamma_{2j-1} = c_j + c_j^\dagger \\ \gamma_{2j} = \frac{1}{i}(c_j - c_j^\dagger) \end{cases}$$

Kasteleyn orientation:

•For a graph with edge orientation, the number of clockwise-oriented edges at every face boundary is odd

•Two Majorana dimer states have the same fermion parity, if and only if the transition graph is Kasteleyn oriented





Symmetry transformation and equivalent ways of decorations

The rule of symmetry transformation:

$$
\begin{aligned}
U(g)|g_i\rangle &= |gg_i\rangle, \\
U(g)c^{\sigma}_{ijk}U(g)^{\dagger} &= (-1)^{\omega_2(g,\sigma)}c^{g\sigma}_{ijk}, \\
U(g)\gamma^{\sigma}_{ij,A}U(g)^{\dagger} &= (-1)^{\omega_2(g,\sigma)}\gamma^{g\sigma}_{ij,A}, \\
U(g)\gamma^{\sigma}_{ij,B}U(g)^{\dagger} &= (-1)^{\omega_2(g,\sigma)+s_1(g)}\gamma^{g\sigma}_{ij,B}.
\end{aligned}
$$

$$s_1 \in H^1(G_b, \mathbb{Z}_2)$$

$$
s_1(g) = \begin{cases} 0, & g \text{ is unitary} \\ 1, & g \text{ is antiunitary} \end{cases}
$$

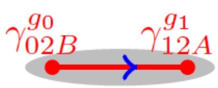
$$g \in G_b$$

- Majorana chain decoration:

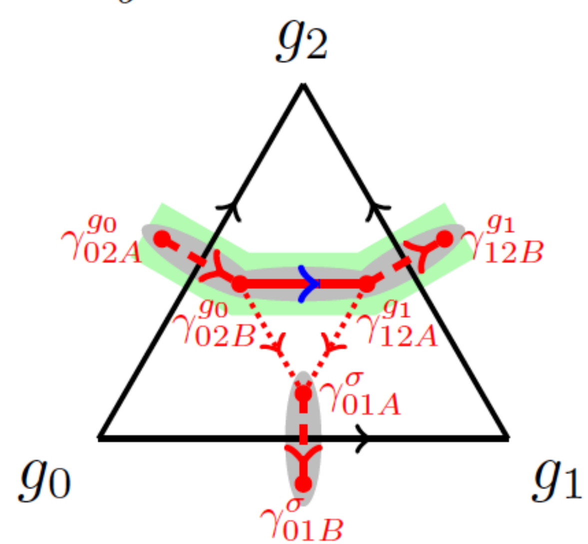
$$n_1 \in H^1(G_b, \mathbb{Z}_2) \qquad n_1(g_i, g_j) \in \mathbb{Z}_2$$

- Example of Fermion parity violation:

$$\Longleftrightarrow \qquad -i\gamma^{\sigma}_{ij,A}\gamma^{\sigma}_{ij,B} = 1$$

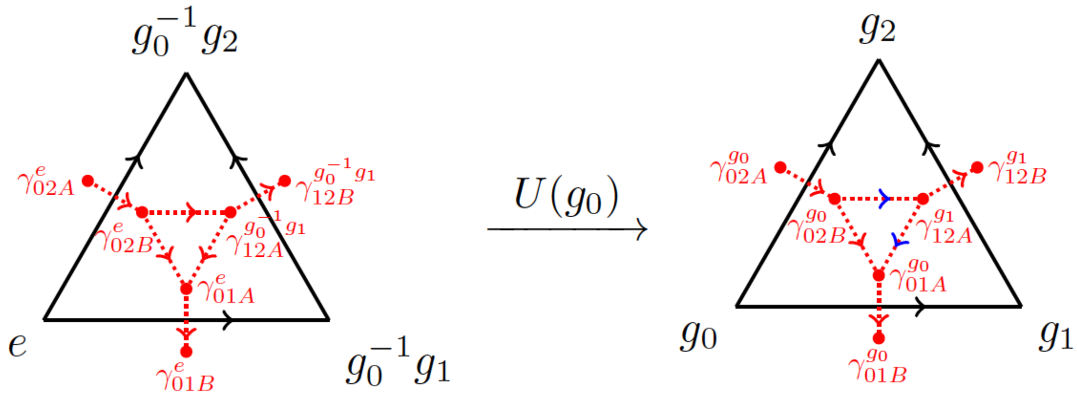


$$\Longleftrightarrow \qquad -i\gamma^{g_0}_{02B}\gamma^{g_1}_{12A} = (-1)^{\omega_2(g_0, g_0^{-1}g_1)}.$$



we only put $\gamma^{g_i}_{ij,A}$ and $\gamma^{g_i}_{ij,B}$ to be in the nontrivial pairing

- Even fermion parity for reference pair: $-i\gamma^{e}_{i,C}\gamma^{g^{-1}h}_{j,D}=1$



$$P^{g_i,g_i}_{ijA,ijB}=U(g_0)P^{g_0^{-1}g_i,g_0^{-1}g_i}_{ijA,ijB}U(g_0)^{-1}=\frac{1}{2}\left(1-i\gamma^{g_i}_{ij,A}\gamma^{g_i}_{ij,B}\right)$$

$$P^{g_0,g_0}_{02B,01A}=U(g_0)P^{e,e}_{02B,01A}U(g_0)^{\dagger}=\frac{1}{2}\left(1-i\gamma^{g_0}_{02B}\gamma^{g_0}_{01A}\right),$$

$$P^{g_0,g_1}_{02B,12A}=U(g_0)P^{e,g_0^{-1}g_1}_{02B,12A}U(g_0)^{\dagger}=\frac{1}{2}\left[1-(-1)^{\omega_2(g_0,g_0^{-1}g_1)}i\gamma^{g_0}_{02B}\gamma^{g_1}_{12A}\right],$$

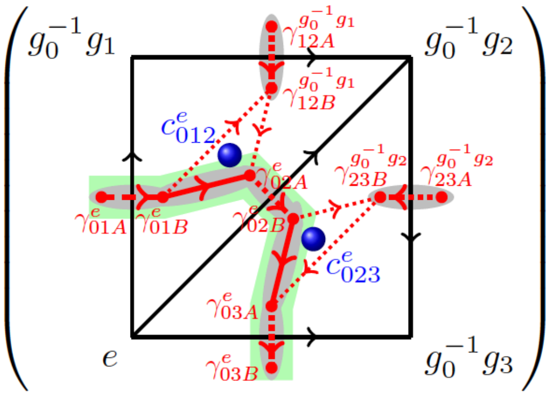
$$P^{g_1,g_0}_{12A,01A}=U(g_0)P^{g_0^{-1}g_1,e}_{12A,01A}U(g_0)^{\dagger}=\frac{1}{2}\left[1-(-1)^{\omega_2(g_0,g_0^{-1}g_1)+s_1(g_0)}i\gamma^{g_1}_{12A}\gamma^{g_0}_{01A}\right]$$

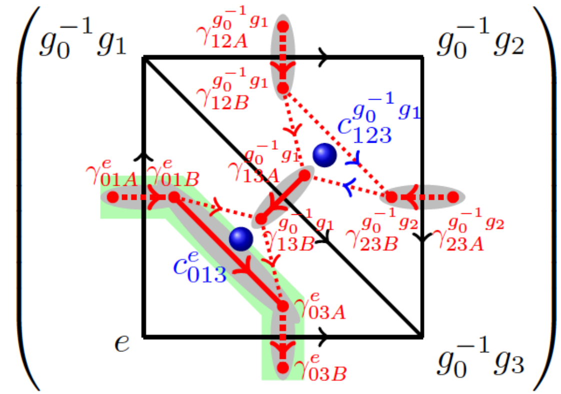
- Fermion parity change for a single triangle:

$$\begin{aligned}\Delta P_f^{\gamma}(012)&=(-1)^{\omega_2(g_0,g_0^{-1}g_1)[n_1(g_0,g_2)n_1(g_1,g_2)+n_1(g_0,g_1)n_1(g_1,g_2)]+s_1(g_0)n_1(g_0,g_1)n_1(g_1,g_2)}\\&=(-1)^{\omega_2(g_0,g_0^{-1}g_1)n_1(g_1,g_2)+s_1(g_0)n_1(g_0,g_1)n_1(g_1,g_2)}\\&=(-1)^{(\omega_2\smile n_1+s_1\smile n_1\smile n_1)(g_0,g_0^{-1}g_1,g_1^{-1}g_2)},\end{aligned}$$

Wavefunction renormalization as fermionic symmetric local unitary transformation

A particular re-triangulation move:

Ψ 

$= F(e, g_0^{-1}g_1, g_0^{-1}g_2, g_0^{-1}g_3)$ Ψ 

$$F(e,\bar{0}1,\bar{0}2,\bar{0}3) = \nu_3(\bar{0}1,\bar{1}2,\bar{2}3)\left(c_{012}^{e\dagger}\right)^{n_2(012)}\left(c_{023}^{e\dagger}\right)^{n_2(023)}\left(c_{013}^{e}\right)^{n_2(013)}\left(c_{123}^{g_0^{-1}g_1}\right)^{n_2(123)} X_{0123}[n_1].$$

$$X_{0123}[n_1] = P_{0123}[n_1]\cdot\left(\gamma_{23B}^{g_0^{-1}g_2}\right)^{\mathrm{d}n_2(0123)},$$

$$P_{0123}[n_1] = \left(\prod_{\text{loop } i} 2^{(L_i-1)/2}\right)\left(\prod_{\text{Majorana pairs }\langle a,b\rangle\text{ in }\mathcal{T}} P_{a,b}^{g_a,g_b}\right)\left(\prod_{\text{link }\langle ij\rangle\notin\mathcal{T}}\ \prod_{\sigma\in G_b} P_{ijA,ijB}^{\sigma,\sigma}\right)$$

We used the abbreviation $\bar{i}j$ for $g_i^{-1}g_j$ in the arguments of F.

$$\nu_3(g_0^{-1}g_1, g_1^{-1}g_2, g_2^{-1}g_3) = \nu_3(e, g_0^{-1}g_1, g_0^{-1}g_2, g_0^{-1}g_3)$$

Generic F-move and total fermion parity change

- Define the following generic F-move

$$F(g_0,g_1,g_2,g_3) = {}^{g_0}F(e,g_0^{-1}g_1,g_0^{-1}g_2,g_0^{-1}g_3) := U(g_0)F(e,g_0^{-1}g_1,g_0^{-1}g_2,g_0^{-1}g_3)U(g_0)^\dagger$$

$$F(g_0,g_1,g_2,g_3) = \nu_3(g_0,g_1,g_2,g_3)(c_{012}^{g_0\dagger})^{n_2(012)}(c_{023}^{g_0\dagger})^{n_2(023)}(c_{013}^{g_0})^{n_2(013)}(c_{123}^{g_1})^{n_2(123)}X_{0123}[n_1]$$

$$X_{0123}[n_1] = P_{0123}[n_1]\cdot(\gamma_{23B}^{g_2})^{\mathrm{d}n_2(0123)}$$

It is a symmetric local unitary transformation if we require

$$n_1(g_0,g_1) = n_1(e,g_0^{-1}g_1) = n_1(g_0^{-1}g_1),$$

$$n_2(g_0,g_1,g_2) = n_2(e,g_0^{-1}g_1,g_0^{-1}g_2) = n_2(g_0^{-1}g_1,g_1^{-1}g_2),$$

$$\nu_3(g_0,g_1,g_2,g_3) = {}^{g_0}\nu_3(g_0^{-1}g_1,g_1^{-1}g_2,g_2^{-1}g_3) = \nu_3(g_0^{-1}g_1,g_1^{-1}g_2,g_2^{-1}g_3)^{1-2s_1(g_0)}\cdot\mathcal{O}_4^{\mathrm{symm}}(g_0,g_1,g_2,g_3)$$

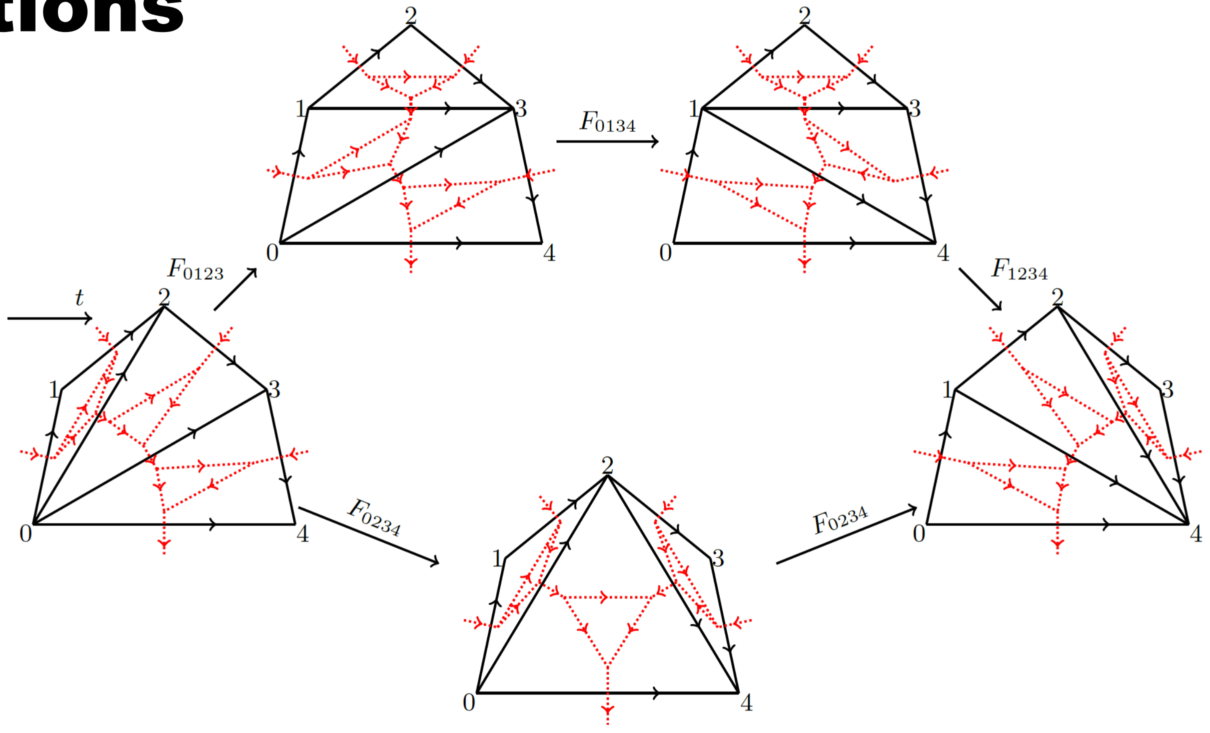
Total Fermion parity change:

$$\Delta P_f^\gamma(F) = (-1)^{(\omega_2\smile n_1+s_1\smile n_1\smile n_1)(g_0^{-1}g_1,g_1^{-1}g_2,g_2^{-1}g_3)}.$$

H^3 obstruction function:

$$\mathrm{d}n_2 = \omega_2\smile n_1 + s_1\smile n_1\smile n_1.$$

H^4 obstruction from RG Fixed point conditions



$$\mathrm{d}\nu_3 = \mathcal{O}_4[n_2], \quad \mathcal{O}_4[n_2] = \mathcal{O}_4^{\text{symm}}[n_2] \cdot \mathcal{O}_4^{c}[n_2] \cdot \mathcal{O}_4^{c\gamma}[\mathrm{d}n_2] \cdot \mathcal{O}_4^{\gamma}[\mathrm{d}n_2].$$

$$\mathcal{O}_4^{\text{symm}}[n_2](01234) = (-1)^{(\omega_2 \smile n_2 + s_1 \smile \mathrm{d}n_2)(01234) + \omega_2(013)\mathrm{d}n_2(1234)},$$

$$\mathcal{O}_4^{c}[n_2] = (-1)^{n_2 \smile n_2 + \mathrm{d}n_2 \smile_1 n_2},$$

$$\mathcal{O}_4^{c\gamma}[\mathrm{d}n_2] = (-1)^{\mathrm{d}n_2 \smile_2 \mathrm{d}n_2},$$

$$\mathcal{O}_4^{\gamma}[\mathrm{d}n_2](01234) = (-1)^{\mathrm{d}n_2(0124)\mathrm{d}n_2(0234)}(-i)^{\mathrm{d}n_2(0123)[1-\mathrm{d}n_2(0124)] \pmod 2}.$$

Classifying Fermionic SPT in 2D and 1D

$$n_1 \in H^1(G_b, \mathbb{Z}_2),$$
$$n_2 \in C^2(G_b, \mathbb{Z}_2)/B^2(G_b, \mathbb{Z}_2)/\Gamma^2,$$
$$\nu_3 \in C^3[G_b, U(1)_T]/B^3[G_b, U(1)_T]/\Gamma^3.$$

Obstruction:

$$\mathrm{d}n_1 = 0,$$
$$\mathrm{d}n_2 = \omega_2 \smile n_1 + s_1 \smile n_1 \smile n_1$$
$$\mathrm{d}\nu_3 = \mathcal{O}_4[n_2],$$

Additional couboundary from fermion SLU transformation:

$$\Gamma^2 = \{\omega_2 \in H^2(G_b, \mathbb{Z}_2)\},$$
$$\Gamma^3 = \{(-1)^{\omega_2 \smile n_1} \in H^3(G_b, U(1)_T) | n_1 \in H^1(G_b, \mathbb{Z}_2)\}$$

Comparing with Fermionic SPT in 1D

$$n_1 \in H^1(G_b, \mathbb{Z}_2),$$
$$\nu_2 \in C^2(G_b, U(1)_T)/B^2(G_b, U(1)_T)/\Gamma^2$$
$$\mathrm{d}n_1 = 0,$$
$$\mathrm{d}\nu_2 = (-1)^{\omega_2 \smile n_1}$$

$$\Gamma^2 = \{(-1)^{\omega_2} \in H^2(G_b, U(1)_T)\}$$

Classifying Fermionic SPT in 3D

$$n_1 \in H^1(G_b, \mathbb{Z}_T),$$
$$n_2 \in C^2(G_b, \mathbb{Z}_2)/B^2(G_b, \mathbb{Z}_2)/\Gamma^2,$$
$$n_3 \in C^3(G_b, \mathbb{Z}_2)/B^3(G_b, \mathbb{Z}_2)/\Gamma^3,$$
$$\nu_4 \in C^4(G_b, U(1)_T)/B^4(G_b, U(1)_T)/\Gamma^4.$$

- The first layer: p+ip topological superconductor decoration.
- The second layer: Majorana chain decoration
- The third layer: complex fermion decoration

Obstruction:

$$\mathrm{d}n_2 = \omega_2 \smile n_1 + s_1 \smile n_1 \smile n_1,$$
$$\mathrm{d}n_3 = \omega_2 \smile n_2 + n_2 \smile n_2 + s_1 \smile (n_2 \smile_1 n_2),$$
$$\mathrm{d}\nu_4 = \mathcal{O}_5[n_3]$$
$$\mathcal{O}_5[n_3] = \mathcal{O}_5^{\mathrm{symm}}[n_3] \cdot \mathcal{O}_5^{c}[n_3] \cdot \mathcal{O}_5^{c\gamma}[\mathrm{d}n_3] \cdot \mathcal{O}_5^{\gamma}[\mathrm{d}n_3].$$

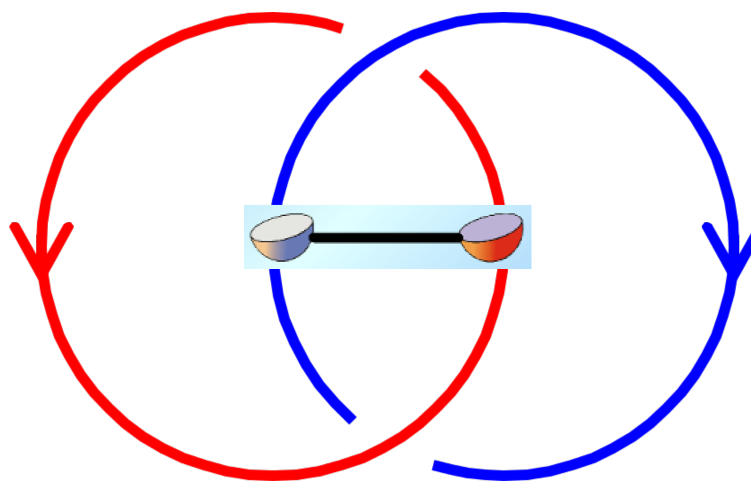
Additional couboundary from fermion SLU transformation:

$$\Gamma^2 = \{\omega_2 \smile n_0 \in H^2(G_b, \mathbb{Z}_2) | n_0 \in H^0(G_b, \mathbb{Z}_T)\},$$
$$\Gamma^3 = \{\omega_2 \smile n_1 + s_1 \smile n_1 \smile n_1 + (\omega_2 \smile_1 \omega_2) \lfloor n_0/2 \rfloor \in H^3(G_b, \mathbb{Z}_2) | n_1 \in H^1(G_b, \mathbb{Z}_2), n_0 \in H^0(G_b, \mathbb{Z}_T)\},$$
$$\Gamma^4 = \{\mathcal{O}_4[n_2] \in H^4(G_b, U(1)_T) | n_2 \text{ for some } n_1 \in H^1(G_b, \mathbb{Z}_2)\} \cup \Gamma^4_{n_0 \neq 0}.$$

Examples of 3D fermionic SPT phase:

$G_b \backslash d_{sp}$	0	1	2	3
$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_1$
$\mathbb{Z}_{2k+1}$	$\mathbb{Z}_{4k+2}$	$\mathbb{Z}_1$	$\mathbb{Z}_{2k+1}$	$\mathbb{Z}_1$
$\mathbb{Z}_{2k}$	$\mathbb{Z}_{2k} \times \mathbb{Z}_2$	$\mathbb{Z}_2$	$\begin{cases} \mathbb{Z}_{4k} \times \mathbb{Z}_2, & k \text{ even} \\ \mathbb{Z}_{8k}, & k \text{ odd} \end{cases}$	$\mathbb{Z}_1$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(\mathbb{Z}_2)^3$	$(\mathbb{Z}_2)^3$	$(\mathbb{Z}_8)^2 \times \mathbb{Z}_4$	$(\mathbb{Z}_2)^2$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_4 \times (\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^3$	$(\mathbb{Z}_8)^2 \times (\mathbb{Z}_2)^3$	$\mathbb{Z}_4 \times \mathbb{Z}_2$
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$(\mathbb{Z}_4)^2 \times \mathbb{Z}_2$	$(\mathbb{Z}_2)^2 \times \mathbb{Z}_4$	$(\mathbb{Z}_8)^2 \times \mathbb{Z}_4 \times (\mathbb{Z}_2)^3$	$(\mathbb{Z}_4)^2 \times \mathbb{Z}_2$
$\mathbb{Z}_2 \times \mathbb{Z}_8$	$\mathbb{Z}_8 \times (\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^3$	$\mathbb{Z}_{16} \times \mathbb{Z}_8 \times (\mathbb{Z}_2)^3$	$\mathbb{Z}_8 \times \mathbb{Z}_2$

- Gauging the global symmetry for SPT states with Majorana chain decoration leads to (Non-Abelian) Ising three loop braiding process, with (minimal) simplest example $G_b = Z_4 * Z_4$ or $G_b = Z_2 * Z_8$ (Jinren Zhou Qingrui Wang, Chenjie Wang, Z C Gu, 2019)



Fermionic SPT with space group symmetry

By applying the fermionic crystalline equivalence principle, we can classify all space group FSPT phase

Example of wall-paper group FSPT for spin-1/2 electron:

No	Name	Majorana-chain	Complex-fermion	Bosonic	Total
1	p1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	8
2	p2	$3\mathbb{Z}_2$	$4\mathbb{Z}_2$	$4\mathbb{Z}_2$	2048
3	p1m1	$\mathbb{Z}_2$	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	32
4	p1g1	$2\mathbb{Z}_2$	$\mathbb{Z}_2$	0	8
5	c1m1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	8
6	p2mm	0	0	$8\mathbb{Z}_2$	256
7	p2mg	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$	$3\mathbb{Z}_2$	256
8	p2gg	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	$2\mathbb{Z}_2$	64
9	c2mm	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$5\mathbb{Z}_2$	128
10	p4	$2\mathbb{Z}_2$	$3\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus 2\mathbb{Z}_4$	1024
11	p4mm	0	0	$6\mathbb{Z}_2$	64
12	p4gm	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$2\mathbb{Z}_2 \oplus \mathbb{Z}_4$	64
13	p3	0	$\mathbb{Z}_2$	$3\mathbb{Z}_3$	54
14	p3m1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	4
15	p31m	0	$\mathbb{Z}_2$	$\mathbb{Z}_6$	12
16	p6	$\mathbb{Z}_2$	$2\mathbb{Z}_2$	$2\mathbb{Z}_6$	288
17	p6mm	0	0	$4\mathbb{Z}_2$	16

Fermionic SPT with Lie group symmetry

- By replacing G as BG and using the Borel cohomology, our results for finite group can be generalized into Lie group symmetry as well.

  Example of 10-fold way:

class	TCS	G_f	0+1D free	0+1D inter.	1+1D free	1+1D inter.	2+1D free	2+1D inter.	3+1D free	3+1D inter.
A	000	$U(1)^f$	$\mathbb{Z}$	$\mathbb{Z}$	0	0	$\mathbb{Z}$	$\mathbb{Z}^2$	0	0
AIII	00+	$U(1)^f \times \mathbb{Z}_2^T$	0	0	$\mathbb{Z}$	$\mathbb{Z}_4$	0	0	$\mathbb{Z}$	$\mathbb{Z}_8 \times \mathbb{Z}_2$
AI	+00	$U(1)^f \rtimes \mathbb{Z}_2^T$	$\mathbb{Z}$	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	0	0	$\mathbb{Z}_2$
BDI	+++	$\mathbb{Z}_2^f \times \mathbb{Z}_2^T$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_8$	0	0	0	0
D	0+0	$\mathbb{Z}_2^f$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	0
DIII	−++	$\mathbb{Z}_4^{Tf}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_{16}$
AII	−00	$\frac{U(1)^f \rtimes \mathbb{Z}_4^T}{\mathbb{Z}_2}$	$\mathbb{Z}$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
CII	−−+	$SU(2)^f \times \mathbb{Z}_2^T$	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
C	0−0	$SU(2)^f$	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^2$	0	0
CI	+−+	$\frac{SU(2)^f \times \mathbb{Z}_4^T}{\mathbb{Z}_2}$	0	0	0	$\mathbb{Z}_2$	0	0	$\mathbb{Z}$	$\mathbb{Z}_4 \times \mathbb{Z}_2$

- Domain wall decoration = (Atiyah-Hirzebruch) spectral sequence (Qingrui Wang, Shangqiang Ning, Meng Cheng, to appear)

Summary and future works

- We construct and classify SPT states for interacting fermions in 3D with arbitrary symmetry $G_{f.}$ (also works for Lie group)

- All 3D topological phases in interacting fermion systems can be realized by gauging certain FSPT states. (Tian Lan and X G Wen, 2018)

- According to the crystalline equivalence principle, our scheme can also classify point/space group SPT.

- Conjecture: Elementary particle can be regarded as tiny linked loop attached by Majorana zero modes.

A natural explanation for three generations and mass mixing matrix!

(Z.-C. Gu, arXiv:1308.2488, accepted by PRR)