An interesting example: compact forms of the Ruijsenaars-Schneider system

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based on joint works with C. Klimcik [2012], T. Kluck [2014] and T.F. Görbe [2016]

We have already seen two real forms of the complex, trigonometric Ruijsenaars–Schneider (RS) system. Other interesting real forms are the hyperbolic system, defined by the Hamiltonian

$$H_{\text{hyp}-\text{RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j \neq k} \left[1 + \frac{\kappa^2}{\sinh^2(q_k - q_j)} \right]^{\frac{1}{2}},$$

and the compact(ified) trigonometric RS (III_b) system, locally given by

$$H_{\text{compact}-\text{RS}} = \sum_{k=1}^{n} (\cos p_k) \sqrt{\prod_{j \neq k} \left[1 - \frac{\sin^2 x}{\sin^2(q_k - q_j)} \right]}$$

Ruijsenaars (1995) studied the latter system assuming $0 < x < \pi/n$. He proved that (in the 'center of mass frame') the naive phase space (corresponding to 'particles' on the circle located at $e^{\mathrm{i}2q_k}$ subject to $x < q_{k+1} - q_k < (\pi - x)$, $\forall k = 1, \dots, n$) can be compactified to $\mathbb{CP}^{n-1}$. Then the flows are complete and the system is integrable.

We developed a reduction approach to this compact system: first for $0 < x < \pi/n$ with C. Klimcik, then for generic x with T. Kluck, and refined the results with T.F. Görbe.

Integrable systems from ‘point reduction’ of the quasi-Hamiltonian/quasi-Poisson double

For any reductive Lie group G, one can reduce the ‘phase space’

$$G \times G = \{(A,B)\} \text{ by imposing the constraint } ABA^{-1}B^{-1} = \mu_0$$

using any constant μ_0 and taking quotient by gauge transformations

$$(A,B) \longrightarrow (gAg^{-1}, gBg^{-1}), \quad g \in G \text{ with } g\mu_0 g^{-1} = \mu_0.$$

The reduced phase space is the moduli space of flat G-connections on the torus with a hole, such that the holonomy around the hole is constrained to the conjugacy class of μ_0. The group elements A and B are the holonomies along the standard cycles on the torus. **Their invariant functions generate two Abelian Poisson algebras.** In favourable circumstances, these reduce to the Hamiltonians of two (degenerate) integrable systems.

Plan of the presentation

- Quasi-Hamiltonian geometry and the internally fused double [following Alekseev-Malkin-Meinrenken 1998]

- Compact trigonometric RS systems from reduction: main results

- Sketch of the derivation of the results and their interpretation

- Direct construction of the trigonometric systems of type (i)

- Elliptic generalization of the type (i) systems

- References and concluding remarks

- An appendix (if time permits)

Quasi-Hamiltonian geometry [Alekseev, Malkin and Meinrenken 98]

Let G be a compact Lie group. The G-manifold M equipped with the invariant 2-form ω is called quasi-Hamiltonian if there exists a (moment) map $\mu : M \to G$ such that:

$$\mu(\Phi_g(m)) = g(\mu(m))g^{-1}, \qquad \forall m \in M, \quad \forall g \in G;$$

$$d\omega = \frac{1}{12}\langle \mu^{-1}d\mu, [\mu^{-1}d\mu, \mu^{-1}d\mu]\rangle;$$

$$\omega(\cdot, \zeta_M) = \frac{1}{2}\langle \mu^{-1}d\mu + d\mu\mu^{-1}, \zeta\rangle, \quad \forall \zeta \in \mathrm{Lie}(G);$$

$$\mathrm{Ker}(\omega_m) = \{\zeta_M(m) \,|\, \zeta \in \mathrm{Ker}(\mathrm{Id}_{\mathrm{Lie}(G)} + \mathrm{Ad}_{\mu(m)})\}, \quad \forall m \in M.$$

Here $\langle -, - \rangle$ denotes a G-invariant inner product on Lie(G), $\Phi_g : M \to M$ is the action of $g \in G$, ζ_M is the vector field on M that corresponds to $\zeta \in$ Lie(G) by $\zeta_M(m) = \frac{d}{dt}\Big|_{t=0}\Phi_{\exp(t\zeta)}(m)$. (These are the opposites of the 2-forms in [AMM]).

Quasi-Hamiltonian dynamical systems

The axioms of quasi-Hamiltonian geometry imply that for each G-invariant function h there exists a unique G-invariant ("quasi-Hamiltonian") vector field V_h on M verifying

$$\omega(\cdot, V_h) = dh \quad \text{and} \quad \mathcal{L}_{V_h}\mu = 0. \qquad (*)$$

Moreover, the formula

$$\{f, h\} := \omega(V_h, V_f) \qquad (**)$$

is G-invariant and this formula gives a Poisson bracket on the space of G-invariant functions on M. Thus, one obtains an honest Poisson structure on M/G, if M/G is an honest manifold.

The G-invariant functions on M are called Hamiltonians and, via $(*)$, they induce G-invariant evolution flows on M that descend to the symplectic leaves of M/G.

In conclusion, one can work with G-invariant functions on quasi-Hamiltonian manifolds in basically the same way as with arbitrary functions on symplectic manifolds.

Quasi-Hamiltonian reduction

With $h \in C^\infty(M)^G$, consider a quasi-Hamiltonian dynamical system (M, G, ω, μ, h), an element $\mu_0 \in G$, the isotropy subgroup $G_{\mu_0} < G$ of μ_0 with respect to the adjoint action and the 'constraint surface'

$$C_{\mu_0} = \mu^{-1}(\mu_0) = \{m \in M \,|\, \mu(m) = \mu_0\}.$$

We say that μ_0 is strongly regular if C_{μ_0} is an embedded submanifold of M and the quotient $M_{\text{red}}(\mu_0) \equiv C_{\mu_0}/G_{\mu_0}$ is a manifold such that the canonical projection $p : C_{\mu_0} \to C_{\mu_0}/G_{\mu_0}$ is a smooth submersion. For strongly regular μ_0 there is a **symplectic** form ω_{red} and a Hamiltonian h_{red} on $M_{\text{red}}(\mu_0)$ uniquely defined by

$$p^*\omega_{\text{red}} = \iota^*\omega, \quad p^*h_{\text{red}} = \iota^*h$$

with the tautological embedding $\iota : C_{\mu_0} \to M$.

The Hamiltonian vector field and its flow defined by h_{red} on the **reduced phase space** $M_{\text{red}}(\mu_0)$ can be obtained by first restricting the quasi-Hamiltonian vector field V_h and its flow to the constraint surface C_{μ_0} and then applying the projection p.

The internally fused double of G

A quasi-Hamiltonian manifold (D, G, ω, μ) is provided by direct product

$$D := G \times G = \{(A, B) \,|\, A, B \in G\}.$$

The group G acts on D by componentwise conjugation

$$\Phi_g(A, B) := (gAg^{-1}, gBg^{-1}), \qquad \forall g \in G.$$

The 2-form ω on D reads

$$\omega := -\frac{1}{2}\langle A^{-1}dA \overset{\wedge}{,} dBB^{-1}\rangle - \frac{1}{2}\langle dAA^{-1} \overset{\wedge}{,} B^{-1}dB\rangle$$

$$+\frac{1}{2}\langle (AB)^{-1}d(AB) \overset{\wedge}{,} (BA)^{-1}d(BA)\rangle,$$

and the G-valued moment map μ is defined by

$$\mu(A, B) = ABA^{-1}B^{-1}.$$

Quasi-Poisson structure [Alekseev, Kosmann-Schwarzbach and Meinrenken 2002]

Every quasi-Hamiltonian manifold carries also a unique G-invariant bivector P subject to the following properties:

The Schouten bracket satisfies $[P, P] = -\phi_M$, where $\phi = \frac{1}{12}\sum_{a,b,c}\langle[e_a, [e_b, e_c]\rangle e^a \wedge e^b \wedge e^c$ using dual bases of Lie(G), $\langle e_a, e^b\rangle = \delta_a^b$ (with the convention $[X, Y]_M = -[X_M, Y_M]$).

The quasi-Poisson bracket defined by $\{f, h\} := P(df, dh)$ and the moment map μ verify the identity

$$\{f, F \circ \mu\} = \frac{1}{2}\sum_a \mu^*(e_a^L + e_a^R)[F])(e^a)_M[f], \qquad \forall f \in C^\infty(M),\ F \in C^\infty(G),$$

where e_a^L and e_a^R are the usual left-invariant and right-invariant vector fields on G.

The 2-form ω and the bivector P obey the compatibility condition

$$P^\sharp \circ \omega^\flat = \mathrm{id}_{TM} + \frac{1}{4}\sum_a (e^a)_M \otimes \langle e_a, \mu^{-1}d\mu - d\mu\mu^{-1}\rangle.$$

Remark: The firs two properties define the notion of *Hamiltonian quasi-Poisson G-manifold* in general. Not every such structure comes from a quasi-Hamiltonian structure. When it does, the two definitions of the Poisson brackets of the invariant functions agree, and P has the 'non-degeneracy' property that for all $m \in M$

$$T_m M = \mathrm{span}_{\mathbb{R}}\{P_m^\sharp(\alpha_m), (\xi_M)_m \mid \alpha_m \in T_m^* M \text{ and } \xi \in \ker(\mathrm{Id}_{\mathrm{Lie}(G)} + \mathrm{Ad}_{\mu(m)})\}.$$

Two quasi-Poisson structures on $G \times G$

The quasi-Poisson bivector, P_D, of the quasi-Hamiltonian double (D, G, ω, μ) has the form

$$P_D = \frac{1}{2} \sum_a \left(e_a^{1,R} \wedge e_a^{1,L} - e_a^{2,R} \wedge e_a^{2,L} + e_a^{1,L} \wedge (e_a^{2,L} + e_a^{2,R}) + e_a^{1,R} \wedge (e_a^{2,L} - e_a^{2,R}) \right),$$

where we use an orthonormal basis of the Lie algebra of G, and $e_a^{1,L}$, $e_a^{2,L}$ denote the left-invariant vector fields depending on the first and second factors of $G \times G$, respectively. This bivector yields a bijection between the cotangent and tangent spaces over a dense open subset of D.

One has another G-invariant bivector on the same manifold $D = G \times G$, denoted P_D^c, which is provided by

$$P_D^c = \frac{1}{2} \sum_a \left(e_a^{1,R} \wedge e_a^{1,L} - e_a^{2,R} \wedge e_a^{2,L} - (e_a^{1,L} - e^{1,R}) \wedge (e_a^{2,L} - e_a^{2,R}) \right).$$

This comes from the direct fusion of the standard quasi-Poisson structure on G and its opposite on the second factor. This second structure admits the moment map

$$\mu^c(g_1, g_2) = g_1 g_2^{-1},$$

but it does not correspond to a quasi-Hamiltonian structure (it does not have the 'non-degeneracy' property).

Unless said otherwise, D will denote the double $G \times G$ equipped with the structure (D, G, ω, μ, P_D).

Mapping class group action on D and on D_{red}

Consider the (orientation-preserving) mapping class group of the "one-holed torus" Σ,

$$\mathrm{MCG}^{+}(\Sigma) \equiv \pi_0(\mathrm{Diff}^{+}(\Sigma)) \simeq SL(2,\mathbb{Z}),$$

which is generated by two elements S and T subject to

$$S^2 = (ST)^3, \quad S^4 = 1.$$

As concrete matrices, one may take $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

In association to S and T, define diffeomorphisms S_D and T_D of D:

$$S_D(A,B) := \Phi_B(B^{-1},A) = (B^{-1}, BAB^{-1}), \quad T_D(A,B) := (AB,B)$$

In fact, S_D and T_D are automorphisms of the double D:

$$S_D^* \omega = \omega, \quad S_D \circ \Phi_g = \Phi_g \circ S_D, \quad \mu \circ S_D = \mu, \quad \text{and similar for} \quad T_D.$$

Moreover, S_D and T_D satisfy

$$S_D^2 = (S_D \circ T_D)^3, \quad S_D^4 = Q,$$

where Q is the following 'universal central automorphism' of D:

$$Q(A, B) := \Phi_{\mu(A,B)^{-1}}(A, B).$$

S_D and T_D descend to maps S_{red} and T_{red} on the reduced phase space $D_{\mathrm{red}}(\mu_0)$, and these maps generate an $SL(2,\mathbb{Z})$ action on $D_{\mathrm{red}}(\mu_0)$.
Indeed, Q descends to the trivial identity map on $D_{\mathrm{red}}(\mu_0)$, and therefore

$$S_{\mathrm{red}}^2 = (S_{\mathrm{red}} \circ T_{\mathrm{red}})^3, \quad S_{\mathrm{red}}^4 = \mathrm{id}.$$

The $SL(2,\mathbb{Z})$ action preserves (stratified) symplectic structure on $D_{\mathrm{red}}(\mu_0)$ as well as the Poisson structure on D/G.

This is a concrete description of the standard mapping class group action on the moduli space $\mathrm{Hom}(\pi_1(\Sigma), G)/G$.

'Free' Hamiltonians on the double

For any $\mathcal{H} \in C^\infty(G)^G$, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the invariant functions on D given by $\mathcal{H}_1(A,B) := \mathcal{H}(A)$ and $\mathcal{H}_2(A,B) := \mathcal{H}(B)$. Then $\{\mathcal{H}_1\}$ and $\{\mathcal{H}_2\}$ form two Abelian Poisson algebras on D. One can easily write down the corresponding quasi-Hamiltonian flows on D.

Define the Lie(G) valued derivative $\nabla\mathcal{H}$ of $\mathcal{H} \in C^\infty(G)^G$ by

$$\langle X, \nabla\mathcal{H}(g)\rangle = \frac{d}{dt}\Big|_{t=0} \mathcal{H}(e^{tX} g), \quad \forall X \in \mathrm{Lie}(G).$$

Then, the integral curve of $\mathcal{H}_1$ through the initial value (A_0, B_0) is given by $(A(t), B(t)) = (A_0, B_0 \exp(-t\nabla\mathcal{H}(A_0)))$ and the integral curve of $\mathcal{H}_2$ reads $(A(t), B(t)) = (A_0 \exp(t\nabla\mathcal{H}(B_0)), B_0)$.

By reduction, one obtains two Abelian Poisson algebras on D/G and on each reduced phase space $D_{\mathrm{red}}(\mu_0)$. These Abelian algebras are interchanged under the action of the mapping class generator S_{red}.

Compact Ruijsenaars–Schneider systems from reduction: The results

We take $G := SU(n)$ carrying the inner product $\langle \eta, \zeta \rangle := -\frac{1}{2}\mathrm{tr}\,(\eta\zeta)$ and impose the moment map constraint $ABA^{-1}B^{-1} = \mu_0$ on the double $D = G \times G = \{(A, B)\}$ with

$$\mu_0 := \mu_0(x) := \mathrm{diag}\left(e^{2\mathrm{i}x}, \ldots, e^{2\mathrm{i}x}, e^{-2\mathrm{i}(n-1)x}\right),$$

where $0 < x < \pi$ and $e^{2\mathrm{i}xm} \neq 1$ for all $m = 1, 2, \ldots, n$. [This is the compact analogue of $\Delta(y)$ from Lecture 2.]

The action of $G_{\mu_0(x)}/Z(G)$ is free on $\mu^{-1}(\mu_0(x))$ and we obtain a Liouville integrable system on the compact, connected, smooth reduced phase space $D_{\mathrm{red}}(\mu_0(x))$ of dimension $2(n-1)$.

In terms of suitable coordinates on a dense open submanifold of $D_{\mathrm{red}}(\mu_0(x))$, the 'main Hamiltonian' coming from $\Re\,(\mathrm{tr}\,(A))$ takes the RS form of $\mathrm{III_b}$ type:

$$H_{\mathrm{compact-RS}} = \sum_{k=1}^{n} (\cos p_k) \sqrt{\prod_{j \neq k} \left[1 - \frac{\sin^2 x}{\sin^2(q_k - q_j)}\right]}$$

The 'position variables' $e^{2\mathrm{i}q_k}$ arise from the eigenvalues of B and the 'action variables' arise from the eigenvalues of A (or vice versa). The domains of these two variables are the same and depend on x.

Two types of compact RS systems

The analysis requires finding the spectra of B for all (A, B) in the constraint surface $\mu^{-1}(\mu_0(x))$, where $ABA^{-1}B^{-1} = \mu_0(x)$. $/e^{2\mathrm{i}xm} \neq 1$ for all $m = 1, 2, \ldots, n/$

In principle, two qualitatively different types of cases can occur:

- Type (i): the constraint surface satisfies $\mu^{-1}(\mu_0(x)) \subset G_{\text{reg}} \times G_{\text{reg}}$.

- Type (ii): the relation $\mu^{-1}(\mu_0(x)) \subset G_{\text{reg}} \times G_{\text{reg}}$ does not hold.

The reduced phase space $D_{\text{red}}(\mu_0(x))$ is naturally a Hamiltonian toric manifold if and only if $\mu^{-1}(\mu_0(x)) \subset G_{\text{reg}} \times G_{\text{reg}}$, i.e., in the type (i) cases. In other words, one obtains $(n-1)$ globally smooth, independent action variables generating an effective torus action.

Indeed, in the type (i) cases certain "spectral functions" on G that are smooth on G_{reg} but only continuous at G_{sing} descend to smooth action variables and position variables when applied to A and B with $(A, B) \in \mu^{-1}(\mu_0(x))$.

In the type (ii) cases the particles can collide and the action variables become non-differentiable at singular points, where the $(n-1)$ commuting smooth Hamiltonians lose their independence.

Basic "spectral functions" on $SU(n)$

Simplex: $\Delta := \left\{ (\xi_1, ..., \xi_{n-1}) \in \mathbb{R}^{n-1} \,\middle|\, \xi_j \geq 0, \; j = 1, ..., n-1, \; \sum_{j=1}^{n-1} \xi_j \leq \pi \right\}$

$n \times n$ matrices: $\Lambda_k := \sum_{j=1}^{k} E_{j,j} - \frac{k}{n} \mathbf{1}_n, \quad k = 1, ..., n-1$

Any element of $G = SU(n)$ is conjugate to $\delta(\xi) := \exp\left(-2\mathrm{i} \sum_{k=1}^{n-1} \xi_k \Lambda_k\right)$ for unique $\xi \in \Delta$. Hence, we can define conjugation invariant functions Ξ_k on G by setting

$$\Xi_k(\delta(\xi)) := \xi_k, \quad \forall \xi \in \Delta, \quad k = 1, ..., n-1.$$

The '**spectral functions**' Ξ_k are only continuous at G_{sing}, but their restrictions to G_{reg} belong to $C^\infty(G_{\mathrm{reg}})^G$. G_{reg} is mapped onto the **interior** of Δ by $(\Xi_1, \ldots, \Xi_{n-1})$.

Crucial fact: the invariant functions $\alpha_k(A, B) := \Xi_k(A)$ and $\beta_k(A, B) := \Xi_k(B)$ **generate 2π-periodic flows on the regular part of the double $D = G \times G$.**

Then, the reductions of $(\alpha_1, \ldots, \alpha_{n-1}) : D \to \Delta$ and $(\beta_1, \ldots, \beta_{n-1}) : D \to \Delta$ yield **toric moment maps** on $D_{\mathrm{red}}(\mu_0(x))$ whenever $\mu^{-1}(\mu_0(x)) \subset G_{\mathrm{reg}} \times G_{\mathrm{reg}}$. We found the values of the parameter $0 < x < \pi$ for which this happens.

Classification of the 'coupling parameter'

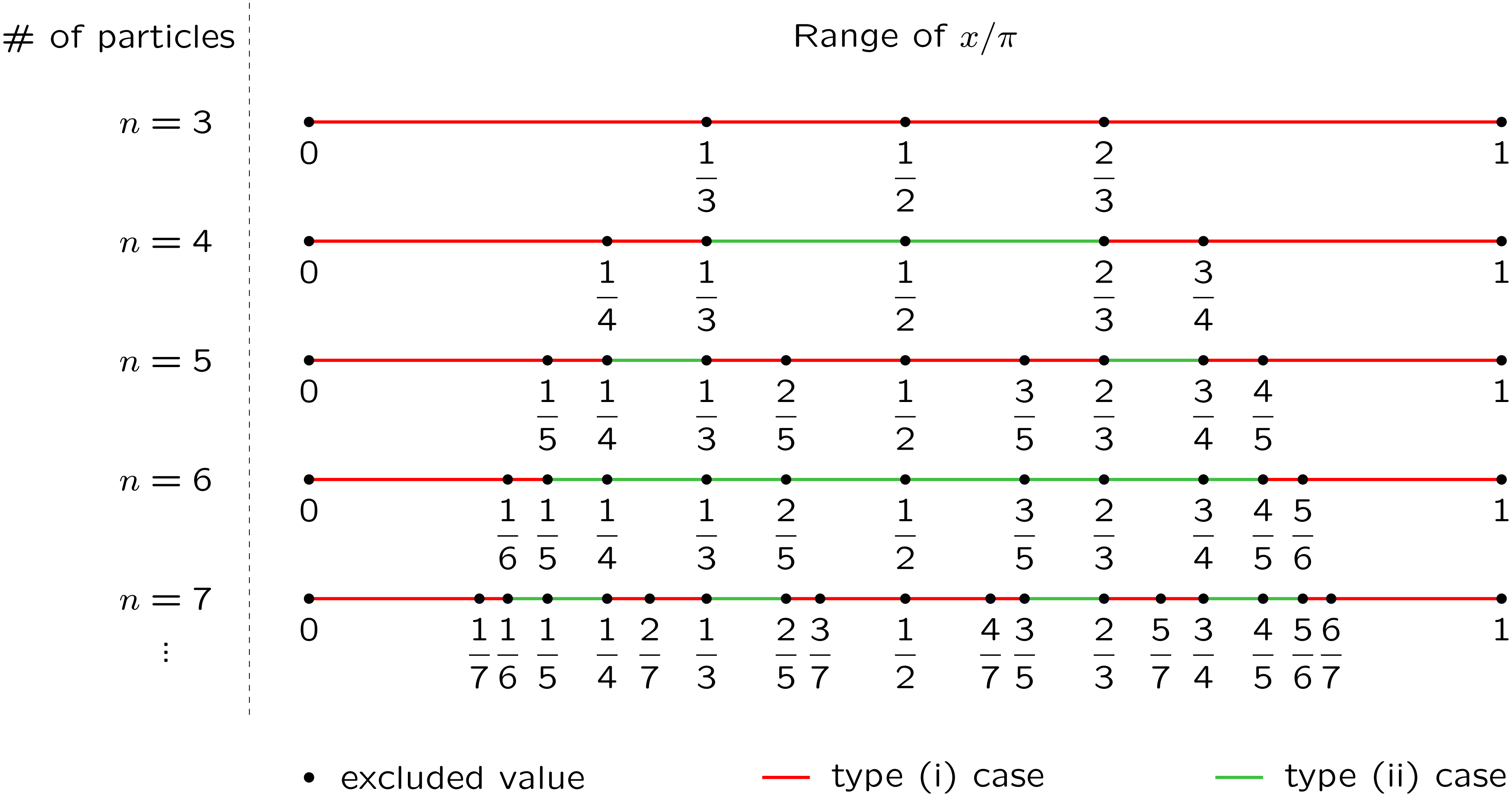
Main Theorem of `[L.F.- T. Kluck]`**:**
The type (i) cases are precisely those for which the coupling parameter $0 < x < \pi$ (subject to $e^{2\mathrm{i}xm} \neq 1$ for all $m = 1, 2, \ldots, n$) belongs to an interval of the form

$$\pi\left(\frac{c}{n} - \frac{1}{nd}, \frac{c}{n} + \frac{1}{(n-d)n}\right)$$

with integers c, d satisfying $1 \leq c, d \leq (n-1)$, $\gcd(n, c) = 1$ and $cd = 1 \mod n$.
In these cases the reduced phase space $D_{\text{red}}(\mu_0(x))$ is symplectomorphic to $\mathbb{CP}^{n-1}$ endowed with a multiple of the Fubini-Study symplectic structure.

- The result was obtained by determining the possible spectra of the matrix B satisfying $ABA^{-1}B^{-1} = \mu_0(x)$.
- In the type (i) cases we found that the "Delzant polytope" is a simplex.
- The existence of type (ii) cases was not anticipated before our work.

Illustration of type (i) and type (ii) cases



How we found the classification? To start, it is convenient to map $\mathbb{R}^{n-1}$ onto the hyperplane

$$E := \{\xi \in \mathbb{R}^n \mid \xi_1 + \cdots + \xi_n = \pi\} \text{ that contains } \Delta = \{\xi \in E \mid \xi_\ell \geq 0,\ \forall \ell = 1, \ldots, n\}.$$

Then, with $\delta(\xi) = \mathrm{diag}(\delta_1(\xi), \ldots, \delta_n(\xi)) := \exp\big(-2\mathrm{i} \sum_{k=1}^{n-1} \xi_k \Lambda_k\big)$, we define

$$z_\ell(\xi, x) := \frac{\sin(x)}{\sin(nx)} \prod_{\substack{j=1 \\ j \neq \ell}}^{n} \frac{e^{-\mathrm{i}x}\delta_j - e^{\mathrm{i}x}\delta_\ell}{\delta_j - \delta_\ell} = \frac{\sin(x)}{\sin(nx)} \prod_{j=\ell+1}^{\ell+n-1} \frac{\sin(\sum_{k=\ell}^{j-1} \xi_k - x)}{\sin(\sum_{k=\ell}^{j-1} \xi_k)}, \quad \forall \xi \in \Delta^{\mathrm{reg}}.$$

Basic lemma. For $\xi \in \Delta^{\mathrm{reg}}$, there exists a solution of

$$ABA^{-1} = \mu_0(x) B \quad \text{with} \quad B \sim \delta(\xi) \tag{$*$}$$

if and only if $z_\ell(\xi, x) \geq 0$ for all $\ell = 1, \ldots, n$.

Sketch of proof. If we have (*), then there exists $g \in G$ such that

$$A^g \delta(\xi) (A^g)^{-1} = (g \mu_0(x) g^{-1}) \delta(\xi) \quad \text{with} \quad A^g = g A g^{-1},\ \delta(\xi) = g B g^{-1}.$$

Due to the form of $\mu_0(x)$, we have $g\mu_0(x)g^{-1} = e^{2\mathrm{i}x}\mathbf{1}_n + (e^{2\mathrm{i}(1-n)x} - e^{2\mathrm{i}x}) v v^\dagger$, where $v \in \mathbb{C}^n$ is the last column of g. For the characteristic polynomials, we then get

$$\prod_{j=1}^{n} (\delta_j(\xi) - \lambda) = \prod_{j=1}^{n} (\delta_j(\xi) e^{2\mathrm{i}x} - \lambda) + (e^{2\mathrm{i}(1-n)x} - e^{2\mathrm{i}x}) \sum_{k=1}^{n} \Big(|v_k|^2 \delta_k(\xi) \prod_{\substack{j=1 \\ j \neq k}}^{n} (\delta_j(\xi) e^{2\mathrm{i}x} - \lambda) \Big).$$

By evaluating this at the n *distinct values* $\lambda = \delta_\ell(\xi) e^{2\mathrm{i}x}$, we obtain $|v_\ell|^2 = z_\ell(\xi, x)$. The argument can be turned backwards, and we can explicitly write down all solutions of the moment map constraint for which $B \sim \delta(\xi)$ with $\xi \in \Delta^{\mathrm{reg}}$.

The "configuration space"

We call 'configuration space' the β-image of the reduced phase space. The α-image is actually the same since (for any moment map value)

$$(A,B) \in \mu^{-1}(\mu_0) \Longleftrightarrow (B^{-1}, BAB^{-1}) \in \mu^{-1}(\mu_0).$$

We proved that the "configuration space", denoted, $\mathcal{A}_x$ is provided by the closure of

$$\mathcal{A}_x^{\text{reg}} = \{\xi \in \Delta^{\text{reg}} \mid z_\ell(\xi, x) \geq 0,\ \ell = 1, \ldots, n\}.$$

Then, the configuration space was found by inspection of the signs of the functions $z_\ell(\xi, x)$. These functions vanish on certain hyperplanes inside E. It useful to note that $z_\ell(\xi^*, x) > 0$ for all ℓ, where $\xi_k^* = \frac{\pi}{n}$.

Using periodic convention $\xi_j = \xi_{j+n}$ ($\forall j \in \mathbb{Z}$), we obtained that for

$$c\frac{\pi}{n} < x < (c+1)\frac{\pi}{n}, \quad (c = 0, \ldots, n-1)$$

$$\mathcal{A}_x = \{\xi \in E \mid \xi_\ell + \cdots + \xi_{\ell+c-1} \leq x,\ \forall \ell = 1, \ldots, n\} \cap \{\xi \in E \mid \xi_\ell + \cdots + \xi_{\ell+c} \geq x,\ \forall \ell = 1, \ldots n\}$$

$\mathcal{A}_x$ is the intersection of two polyhedra, which is contained in the closed simplex Δ. (If $c = 0$ or $c = n-1$ then only one polyhedron occurs, and it lies inside Δ.)

In type (i) cases $\mathcal{A}_x$ does not reach the boundary of Δ. This happens when x is near enough to $\pi\frac{c}{n}$ for $\gcd(c,n) = 1$. In these cases one of the two polyhedra is a simplex, which is contained in the other polyhedron and inside Δ^{reg}. Note that $\mathcal{A}_x = \mathcal{A}_{\pi - x} = \mathcal{A}_{-x}$.

Characterization of the reduced system on a dense open submanifold

Pick any x for which $e^{2\mathrm{i}xm} \neq 1$ for all $m = 1, 2, \ldots, n$. Consider domain $\mathcal{A}_x^+$ containing those regular ξ for which $z_\ell(\xi, x) > 0$ for all $\ell = 1, \ldots, n$.

Then take $v_\ell(\xi, x) := \sqrt{z_\ell(\xi, x)}$, and using $v \equiv v(\xi, x)$ introduce the matrix $g := g_x(\xi)$ having the elements

$$g_{nn} := v_n, \;\; g_{jn} := -g_{nj} := v_j, \;\; g_{jl} := \delta_{jl} - \frac{v_j v_l}{1 + v_n}, \;\; \forall j, l = 1, \ldots, n-1.$$

Its satisfies $g_x(\xi)\mu_0(x)g_x(\xi)^{-1} = e^{2\mathrm{i}x}\mathbf{1}_n + (e^{2\mathrm{i}(1-n)x} - e^{2\mathrm{i}x})vv^\dagger$. Finally, with $(e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_{n-1}}) \in \mathbb{T}^{n-1}$ prepare

$$\varrho := \mathrm{diag}(e^{-\mathrm{i}\theta_1}, e^{\mathrm{i}(\theta_1 - \theta_2)}, e^{\mathrm{i}(\theta_2 - \theta_3)}, \ldots, e^{\mathrm{i}(\theta_{n-2} - \theta_{n-1})}, e^{\mathrm{i}\theta_{n-1}}).$$

Then, we define the unitary 'local RS Lax matrix'

$$\mathcal{L}_x^{\mathrm{loc}}(\xi, \theta)_{j\ell} = \frac{\sin(nx)}{\sin(x)} \frac{e^{\mathrm{i}x} - e^{-\mathrm{i}x}}{e^{\mathrm{i}x}\delta_j(\xi)\delta_\ell(\xi)^{-1} - e^{-\mathrm{i}x}} v_j(\xi, x) v_\ell(\xi, \pi - x) \varrho(\theta)_\ell.$$

Local Theorem. For any generic x, the set

$$\left\{\Big(g_x(\xi)^{-1}\mathcal{L}_x^{\rm loc}(\xi,\theta)g_x(\xi), g_x(\xi)^{-1}\delta(\xi)g_x(\xi)\Big)\,\Big|\,(\xi,e^{\mathrm{i}\theta})\in\mathcal{A}_x^+\times\mathbb{T}^{n-1}\right\}\subset G\times G$$

defines a cross-section of the orbits of $G_{\mu_0(x)}$ in the open submanifold $\beta^{-1}(\mathcal{A}_x^+)\cap\mu^{-1}(\mu_0(x))$ of the constraint surface. The parametrization by $(\xi,e^{\mathrm{i}\theta})\in\mathcal{A}_x^+\times\mathbb{T}^{n-1}$ induces Darboux coordinates on corresponding submanifold of reduced phase space: we have $\omega^{\rm loc}=\sum_{k=1}^{n-1}\mathrm{d}\xi_k\wedge\mathrm{d}\theta_k$.

On this submanifold, which is **dense** in the full reduced phase space, the Poisson commuting reduced Hamiltonians descending from the class functions of A in $(A,B)\in G\times G$ become the class functions of $\mathcal{L}_x^{\rm loc}(\xi,\theta)$. The reduction of the function $\Re(\mathrm{tr}\,(A))$ provides the RS Hamiltonian

$$H_x^{\rm loc}(\xi,\theta)=\sum_{j=1}^{n}\cos(\theta_j-\theta_{j-1})\sqrt{\prod_{k=j+1}^{j+n-1}\left[1-\frac{\sin^2 x}{\sin^2(\sum_{m=j}^{k-1}\xi_m)}\right]}$$

The α and β images of reduced phase space give the closure of $\mathcal{A}_x^+\subset\Delta$.

(Here we employed the conventions $\theta_0=\theta_n=0$, $\xi_n=\pi-\xi_1-\cdots-\xi_{n-1}$ and $\xi_{k+n}=\xi_k$.)

For interpretation, put $\delta_k = e^{2\mathrm{i}q_k}$, $\varrho_k = e^{-\mathrm{i}p_k}$, $\quad q_{k+1} - q_k = \xi_k$, $\quad \Big(\prod_{k=1}^n \delta_k = \prod_{k=1}^n \varrho_k = 1\Big)$.

Then $H_x^{\mathrm{loc}}(q,p) = \sum_{j=1}^n \cos(p_j)\sqrt{\prod_{k\neq j}\Big[1 - \frac{\sin^2 x}{\sin^2(q_j - q_k)}\Big]}$ and, after conjugation by $g_x(\xi)$, the local Lax matrix becomes

$$L_x^{\mathrm{loc}}(q,p)_{j,\ell} = \frac{\sin(nx)}{\sin(x)} \frac{\sin(x)}{\sin(q_j - q_\ell + x)} v_j(\xi, x) v_\ell(\xi, \pi - x)\varrho_\ell$$

$$v_j(\xi, x) = \Big[\frac{\sin(x)}{\sin(nx)} \prod_{\substack{k=1\\k\neq j}}^n \frac{\sin(q_k - q_j - x)}{\sin(q_k - q_j)}\Big]^{\frac{1}{2}} = \Big[\frac{\sin(x)}{\sin(nx)} \prod_{k=j+1}^{j+n-1} \frac{\sin(\sum_{m=j}^{k-1} \xi_m - x)}{\sin(\sum_{m=j}^{k-1}\xi_m)}\Big]^{\frac{1}{2}}$$

Regarding the type (i) cases: Let us fix integers $1 \leq c, d \leq (n-1)$ such that $\gcd(c,n) = \gcd(d,n) = 1$ and $cd = 1$ modulo n. Then the parameter x can vary as

$$\Big(\frac{c}{n} - \frac{1}{nd}\Big)\pi < x < \frac{c\pi}{n} \qquad \text{or} \qquad \frac{c\pi}{n} < x < \Big(\frac{c}{n} + \frac{1}{(n-d)n}\Big)\pi.$$

Defining $M := c\pi - nx$, we have $M > 0$ or $M < 0$, respectively in the two cases, and $\xi \in \mathcal{A}_x$ satisfies

$$\mathrm{sgn}(M)(\xi_j + \cdots + \xi_{j+c-1} - x) \geq 0, \quad j = 1, \dots, n.$$

This means that, for $M > 0$ and $M < 0$, the '*distances of the c-th neighbours*' are subject to $q_{j+c} - q_j \geq x$ and respectively to $q_{j+c} - q_j \leq x$, $\forall j$, where $q_{k+n} := q_k + \pi$. The 'angular separations' $2\xi_k = 2q_{k+1} - 2q_k$ have minima and maxima in all cases.

Explicit embedding of the dense open 'local phase space' into $\mathbb{CP}^{n-1}$

For this, we introduce the mapping $\mathcal{E}\colon \mathcal{A}_x^+ \times \mathbb{T}^{n-1} \to \mathbb{C}^n, \quad (\xi, e^{\mathrm{i}\theta}) \mapsto (u_1, \dots, u_n)$ with the complex coordinates having the squared absolute values

$$|u_j|^2 = \mathrm{sgn}(M)(\xi_j + \cdots + \xi_{j+c-1} - x), \quad j = 1, \dots, n,$$

and arguments $\arg(u_j) = -\,\mathrm{sgn}(M) \sum_{k=1}^{n-1} W_{j,k}\theta_k$, $j = 1, \dots, n-1$, $\arg(u_n) = 0$. We have

$$|u_j|^2 = \begin{cases} \mathrm{sgn}(M)\big(\sum_{k=1}^{n-1} T_{j,k}\xi_k - x\big), & \text{if } 1 \le j \le n-p, \\ \mathrm{sgn}(M)\big(\sum_{k=1}^{n-1} T_{j,k}\xi_k - x + \pi\big), & \text{if } n-p < j \le n-1 \end{cases}$$

with an integer matrix $T \in \mathrm{SL}(n-1, \mathbb{Z})$, and take W to be inverse-transpose of T. (We determined T and T^{-1} explicitly.) The image of $\mathcal{E}$ lies in

$$S_{|M|}^{2n-1} = \{(u_1, \dots, u_n) \in \mathbb{C}^n \mid |u_1|^2 + \cdots + |u_n|^2 = |M|\},$$

which engenders $\mathbb{CP}^{n-1} = S_{|M|}^{2n-1}/\mathrm{U}(1)$, with projection $\pi_{|M|}\colon S_{|M|}^{2n-1} \to \mathbb{CP}^{n-1}$.

As $\mathcal{E}^*\left(\mathrm{i} \sum_{j=1}^{n} d\bar{u}_j \wedge du_j\right) = \sum_{k=1}^{n-1} d\xi_k \wedge d\theta_k$ holds, we obtained **symplectic embedding**

$$\pi_{|M|} \circ \mathcal{E} : \mathcal{A}_x^+ \times \mathbb{T}^{n-1} \to \mathbb{CP}^{n-1}$$

with respect to $\omega^{\mathrm{loc}} = \sum_{j=1}^{n-1} d\xi_j \wedge d\theta_j$ and the re-scaled Fubini-Study form $|M|\omega_{\mathrm{FS}}$. The image is the dense, open submanifold where no homogeneous coordinate can vanish.

Result about direct construction of trigonometric RS systems on $\mathbb{CP}^{n-1}$

Theorem. Define the diagonal unitary matrix $D = \mathrm{diag}(D_1, \dots, D_{n-1}, 1)$ with

$$D_j = \exp\left(\mathrm{i} \sum_{k=1}^{n-1} W_{j,k}\theta_k\right), \quad j = 1, \dots, n-1.$$

Then, in every type (i) case, there exists a smooth function $L^x : \mathbb{CP}^{n-1} \to \mathrm{SU}(n)$ that satisfies the following relation:

$$(L^x \circ \pi_{|M|} \circ \mathcal{E})(\xi, \theta) = D(\theta) L_x^{\mathrm{loc}}(\xi, \theta) D(\theta)^{-1}, \quad \forall (\xi, e^{\mathrm{i}\theta}) \in \mathcal{A}_x^+ \times \mathbb{T}^{n-1},$$

which means that L^x is an extension of the local Lax matrix $D L_x^{\mathrm{loc}} D^{-1}$ to $\mathbb{CP}^{n-1}$.

Corollary. The symmetric functions of the *global Lax matrix* L^x define an integrable system on $\mathbb{CP}^{n-1}$, whose main Hamiltonian extends the local RS Hamiltonian H_x^{loc}.

We have this extension in explicit form as well. Next I sketch the crux of the proof, and then give the analogous result in the elliptic case.

The direct construction was inspired by Ruijsenaars' work [RIMS 95], which dealt with the case $0 < x < \pi/n$, and his remarks on the corresponding elliptic case.

To explain the crux, first note that $C^\infty(\mathbb{CP}^{n-1}) = C^\infty(S^{2n-1}_{|M|})^{\mathrm{U}(1)}$. Thus the squared absolute values $|u_j|^2$ give rise to smooth functions on $\mathbb{CP}^{n-1}$, and the same is true for the components ξ_k, which be written as affine combinations of the $|u_j|^2$.

Consider 'building block' $v_j(\xi, x)$ of local Lax matrix. We have $v_j(\xi,x) = |u_j| w_j(\xi,x)$, where $w_j(\xi,x)$ is the function

$$w_j(\xi,x) = \left[\frac{\sin(|u_j|^2)}{|u_j|^2} \frac{\mathrm{sgn}(M)\sin(x)}{\sin(nx)\sin(\sum_{k=j}^{j+p-1}\xi_k)} \prod_{\substack{m=j+1\\(m\neq j+p)}}^{j+n-1} \frac{\sin(\sum_{k=j}^{m-1}\xi_k - x)}{\sin(\sum_{k=j}^{m-1}\xi_k)} \right]^{\frac{1}{2}}.$$

The point to notice is that w_j extends to a smooth function on $\mathbb{CP}^{n-1}$. Inspecting all building blocks, we find that the local Lax matrix exhibits the following structure:

$$L_x^{\mathrm{loc}}(\xi,0)_{j,\ell} = \begin{cases} \Lambda^x_{j,j+p}(\xi), & \text{if } 1 \le j \le n-p,\ \ell = j+p \\ \Lambda^x_{j,j-(n-p)}(\xi), & \text{if } n-p < j \le n,\ \ell = j-(n-p), \\ |u_j||u_{\ell-p+n}|\Lambda^x_{j,\ell}(\xi), & \text{if } 1 \le j \le n,\ 1 \le \ell \le p,\ \ell \neq j-(n-p), \\ |u_j||u_{\ell-p}|\Lambda^x_{j,\ell}(\xi), & \text{if } p < \ell \le n,\ \ell \neq j+p. \end{cases}$$

where the $\Lambda^x_{j,\ell}(\xi)$ extend to smooth functions on $\mathbb{CP}^{n-1}$. The absolute values are not smooth functions (at the origin), but they appear quadratically. Everything will be fine if we can "engineer" replacements like $|u_j||u_{\ell-p}| \longrightarrow \bar{u}_j u_{\ell-p}$ since on the r.h.s we have a U(1) invariant smooth function on $S^{2n-1}_{|M|}$. This is precisely what is achieved by conjugating $L_x^{\mathrm{loc}}(\xi,\theta) = L_x^{\mathrm{loc}}(\xi,0)\varrho(\theta)$ by the phase matrix $D(\theta)$.

Elliptic preparations

Let ω, ω' stand for the half-periods of the Weierstrass elliptic function $\wp$,

$$\wp(z;\omega,\omega') = z^{-2} + \sum_{\substack{m,m'=-\infty \\ (m,m')\neq(0,0)}}^{\infty} \left((z-\Omega_{m,m'})^{-2} - \Omega_{m,m'}^{-2}\right),$$

with $\Omega_{m,m'} = 2m\omega + 2m'\omega'$. We choose $\omega, -\mathrm{i}\omega' \in (0,\infty)$, which ensures that $\wp$ is positive on the real axis. Next, introduce the "s-function" by the formula

$$\mathrm{s}(z;\omega,\omega') = a^{-1}\sin(az)\prod_{m=1}^{\infty}\left(1 - \frac{\sin^2(az)}{\sin^2(2am\omega')}\right)$$

with $a = \pi/(2\omega)$. An important identity connecting $\wp$ and s is

$$\frac{\mathrm{s}(z+y)\,\mathrm{s}(z-y)}{\mathrm{s}^2(z)\,\mathrm{s}^2(y)} = \wp(y) - \wp(z).$$

The s-function is odd, satisfies $\mathrm{s}(\pi/a - z) = \mathrm{s}(z)$, has simple zeros at $\Omega_{m,m'}$, $m, m' \in \mathbb{Z}$ and enjoys the scaling property

$$\mathrm{s}(tz;t\omega,t\omega') = t\,\mathrm{s}(z;\omega,\omega'),$$

which permits to work with $a = 1$ $(\omega = \pi/2)$. In the trigonometric limit,

$$\lim_{-\mathrm{i}\omega'\to\infty} \wp(z;\pi/2,\omega') = \frac{1}{\sin^2(z)} - \frac{1}{3}, \qquad \lim_{-\mathrm{i}\omega'\to\infty} \mathrm{s}(z;\pi/2,\omega') = \sin(z).$$

Type (i) compact forms of the elliptic RS system

Since $\mathsf{s}(z)$ and $\sin(z)$ have matching properties, the following variant Ruijsenaars' [1986] elliptic Lax matrix is well-behaved on the type (i) local phase space $\mathcal{A}_x^+ \times \mathbb{T}^{n-1}$:

$$L_x^{\mathrm{loc}}(\xi,\theta|\lambda)_{j,\ell} = \frac{\mathsf{s}(nx)}{\mathsf{s}(x)}\frac{\mathsf{s}(x)}{\mathsf{s}(\lambda)}\frac{\mathsf{s}(q_j - q_\ell + \lambda)}{\mathsf{s}(q_j - q_\ell + x)} v_j(\xi,x)v_\ell(\xi,-x)\varrho(\theta)_\ell,$$

where $\lambda \in \mathbb{C} \setminus \{\Omega_{m,m'} : m, m' \in \mathbb{Z}\}$ is a spectral parameter, $v_\ell(\xi,\pm x) = \sqrt{z_\ell(\xi,\pm x)}$ with

$$z_\ell(\xi,x) = \frac{\mathsf{s}(x)}{\mathsf{s}(nx)} \prod_{m=\ell+1}^{\ell+n-1} \frac{\mathsf{s}(\sum_{k=\ell}^{m-1}\xi_k - x)}{\mathsf{s}(\sum_{k=\ell}^{m-1}\xi_k)} = \frac{\mathsf{s}(x)}{\mathsf{s}(nx)} \prod_{\substack{m=1\\ m\neq\ell}}^{n} \frac{\mathsf{s}(q_m - q_\ell - x)}{\mathsf{s}(q_m - q_\ell)}$$

Theorem. There exists a smooth, spectral parameter dependent elliptic Lax matrix $L^x(\cdot|\lambda)$ on $\mathbb{CP}^{n-1}$ which is an extension of $L_x^{\mathrm{loc}}(\xi,\theta|\lambda)$ since it satisfies

$$L^x(\pi_{|M|} \circ \mathcal{E}(\xi,\theta)|\lambda) = D(\theta)L_x^{\mathrm{loc}}(\xi,\theta|\lambda)D(\theta)^{-1}, \quad \forall(\xi, e^{\mathrm{i}\theta}) \in \mathcal{A}_y^+ \times \mathbb{T}^{n-1},$$

where D and $\mathcal{E} \circ \pi_{|M|} : \mathcal{A}_x^+ \times \mathbb{T}^{n-1} \to \mathbb{CP}^{n-1}$ are the same as in the trigonometric case.

We have $\mathrm{sgn}(\mathsf{s}(nx))\Re(\mathrm{tr}\, L_x^{\mathrm{loc}}(\xi,\theta)) = H_x^{\mathrm{loc}}(\xi,\theta)$ with the elliptic RS ($\mathrm{IV_b}$) Hamiltonian:

$$H_x^{\mathrm{loc}}(\xi,\theta) = \sum_{j=1}^{n} \cos(\theta_j - \theta_{j-1}) \sqrt{\prod_{m=j+1}^{j+n-1} \Big(\mathsf{s}(x)^2(\wp(x) - \wp(\textstyle\sum_{k=\ell}^{m-1}\xi_k))\Big)}.$$

References for the lecture on compact RS systems

L.F. and C. Klimcik: Self-duality of the compactified Ruijsenaars-Schneider system from quasi-Hamiltonian reduction, Nucl.Phys. B860, 464-515 (2012)

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Related papers on the quantization of the type (i) systems

T.F. Görbe and M.A. Hallnäs: Quantisation of compact forms of the trigonometric Ruijsenaars-Schneider system, J. Int. Sys., xyy015 (2018)

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Final remarks:

- We can also construct a degenerate integrable system on the Poisson manifold $D_*/G \subset D/G$, for any compact Lie group G, quite similarly to the cases of the cotangent bundle and the Heisenberg double discussed in lectures 1 and 2.

- It is a very interesting open problem to investigate the structure of the type (ii) compact RS systems that we constructed. One should also explore the Hamiltonian systems living on the symplectic leaves of D_*/G in the general case.

In this Appendix, we sketch a generalization of the trigonometric Gibbons–Hermsen model. For this, recall the GH model is obtained by Hamiltonian reduction from

$$T^*U(n) \times \mathbb{C}^{n\times d}.$$

The second factor encodes nd ($d \geq 2$) copies of the symplectic vector space $\mathbb{R}^2$. Denote the general element of $\mathbb{C}^{n\times d}$ as the matrix S_{aj}, and let (g, J) stand for the general element of the cotangent bundle, trivialized by right-translations. Then the following formula gives a Poisson map into $\mathfrak{u}(n) \simeq \mathfrak{u}(n)^*$,

$$\Phi(g, J, S) = J - g^{-1}Jg + \mathrm{i}SS^\dagger.$$

This is the moment map for the Hamiltonian action of $U(n)$ given by

$$A_\eta : (g, J, S) \mapsto (\eta g \eta^{-1}, \eta J \eta^{-1}, \eta S), \quad \forall \eta \in U(n).$$

Now, reduce by imposing the moment map constraint $\Phi(g, J, S) = \mathrm{i}c\mathbf{1}_n$, with $c > 0$. On a dense open subset, one can employ the partial gauge fixing where $g = \exp(\mathrm{i}q) \in \mathcal{T}^n_{\mathrm{reg}}$ with the maximal torus $\mathbb{T}^n < U(n)$. Then, one gets

$$J_{ab} = \mathrm{i}p_a\delta_{ab} - \mathrm{i}(1-\delta_{ab})\frac{S_aS_b^\dagger}{1-\exp(\mathrm{i}(q_b-q_a))}, \quad \text{with arbitrary} \quad p_a \in \mathbb{R}.$$

In this gauge, the 'free' Hamiltonian gives $H = -\frac{1}{2}\mathrm{tr}\,(J^2) = \frac{1}{2}\sum_{a=1}^n p_a^2 + \frac{1}{8}\sum_{a\neq b}\frac{|S_aS_b^\dagger|^2}{\sin^2\frac{q_a-q_b}{2}}$, and the moment map constraint implies $S_aS_a^\dagger = c$. The residual gauge transformations are given by the torus $\mathbb{T}^n$ and by the permutation group S_n, and the pertinent open dense subset of the full reduced phase space can be identified as

$$\left(T^*\mathbb{T}^n_{\mathrm{reg}} \times (\mathbb{CP}^{d-1} \times \cdots \times \mathbb{CP}^{d-1})\right)/S_n,$$

with n-copies of the complex projective space. (If $d = 1$, then one gets the spinless Sutherland model.)

For generalization, take the unreduced phase space $\mathcal{M} := GL(n,\mathbb{C}) \times \mathbb{C}^{n\times d}$, where the real manifold $GL(n,\mathbb{C}) \simeq U(n) \times \mathfrak{P}(n)$ is the Heisenberg double of the Poisson–Lie group $U(n)$ and the d columns of $\mathbb{C}^{n\times d}$ carry a $U(n)$ covariant Poisson structure,

$$\{w_k, w_l\} = \mathrm{i}\,\mathrm{sgn}(k-l) w_k w_l, \qquad \forall 1 \le k,l \le n,$$
$$\{w_k, \overline{w}_l\} = \mathrm{i}\,\delta_{kl}(2 + |w|^2) + \mathrm{i} w_k \overline{w}_l + \mathrm{i}\,\delta_{kl} \sum_{r=1}^{n} \mathrm{sgn}(r-k)|w_r|^2\,,$$

which is due to Zakrzewski (1996), and is actually symplectic. Consider the following Iwasawa decompositions of $K \in GL(n,\mathbb{C})$ and the factorization of $(\mathbf{1}_n + ww^\dagger) \in \mathfrak{P}(n)$:

$$K = g_L b_R^{-1} = b_L g_R^{-1}, \quad \mathbf{1}_n + ww^\dagger = \mathbf{b}(w)\mathbf{b}(w)^\dagger$$

where $g_L, g_R \in U(n)$ and $b_L, b_R, \mathbf{b}(w) \in B(n)$: the upper-triangular subgroup of $GL(n,\mathbb{C})$ with positive diagonal. Then, define the Poisson map $\Lambda : \mathcal{M} \to B(n) \equiv U(n)^*$ by

$$\Lambda(K, w^1, \ldots, w^d) := b_L b_R \mathbf{b}(w^1)\mathbf{b}(w^2)\cdots\mathbf{b}(w^d), \quad \text{with} \quad (w^1, w^2, \ldots, w^d) \in \mathbb{C}^{n\times d}.$$

This generates an action of the Poisson–Lie group $U(n)$ on $\mathcal{M}$, and we obtain the reduced phase space

$$\mathcal{M}_{\text{red}} = \Lambda^{-1}(e^\gamma \mathbf{1}_n)/U(n),$$

which is a smooth symplectic manifold for any $\gamma > 0$.

The unreduced phase space carries the commuting Hamiltonians

$$H_j := \mathrm{tr}\,(L^j) \quad \text{with} \quad L := b_R b_R^\dagger, \quad j = 1, \ldots, n.$$

They have very simple flows and yield an integrable system on $\mathcal{M}$, quite similar to the cotangent bundle case.

We can go to the gauge slice where g_R becomes a diagonal matrix, $Q \in \mathbb{T}^n_{\text{reg}}$. Decomposing $b \in B(n)$ as $b = b_0 b_+$, with diagonal and unipotent factors, we write

$$b_R = b_0 b_+ \quad \text{and} \quad \mathbf{b}(w^1)\mathbf{b}(w^2)\cdots\mathbf{b}(w^d) =: S(W) =: S_0(W)S_+(W).$$

Then the moment map condition becomes equivalent to the following constraints:

$$S_0(W) = e^{\gamma}\mathbf{1}_n \quad \text{and} \quad b_+ S_+(W) = Q^{-1} b_+ Q.$$

The first equation constraints $W = (w^1, \ldots, w^d)$ only, while the second one permits us to express b_+ in terms of $Q = e^{\mathrm{i}q} \in \mathbb{T}^n_{\text{reg}}$ and $S_+(W) \in \mathbb{C}^{n\times d}$. (This is the same eq. as $b_+\lambda = Q^{-1}b_+Q$ that we encountered before.)

$Q \in \mathbb{T}^n_{\text{reg}}$ and $b_0 \equiv \exp(p)$, with $p = \text{diag}(p_1, \ldots, p_n)$, are arbitrary, and a dense open subset of the reduced phase space is parametrized by Q, p and the constrained 'primary spins', W, up to the usual residual gauge transformations.

The reduction of the spectral invariants of $L = b_R b_R^\dagger$ yields an integrable system.

To connect our reduced system with the Gibbons–Hermsen model, we introduce a positive 'scaling parameter' ϵ and make the replacements

$$p \to \epsilon p, \quad W \to \epsilon^{\frac{1}{2}} W, \quad Q \to Q, \quad \Omega_{\mathcal{M}} \to \epsilon^{-1}\Omega_{\mathcal{M}}, \quad \gamma \to \epsilon\gamma,$$

where $\Omega_{\mathcal{M}}$ is the symplectic form on $\mathcal{M}$. With $L := b_R b_R^\dagger$ and $b_R = \exp(\epsilon p) b_+(Q, \epsilon^{\frac{1}{2}} W)$, writing $Q = \mathrm{diag}(e^{\mathrm{i}q_1}, \dots, e^{\mathrm{i}q_n})$ and letting w_i denote the i-th row of $W \in \mathbb{C}^{n\times d}$, we get

$$\lim_{\epsilon\to 0} \frac{1}{8\epsilon^2}(\mathrm{tr}\,(L) + \mathrm{tr}\,(L^{-1}) - 2n) = \frac{1}{2}\mathrm{tr}\,(p^2) + \frac{1}{32}\sum_{i\neq j} \frac{|w_i^\alpha w_j^\dagger|^2}{\sin^2 \frac{q_i - q_j}{2}},$$

$$\lim_{\epsilon\to 0} \epsilon^{-1}(\Omega_{\mathrm{red}}) = \sum_{j=1}^n dp_j \wedge dq_j + \frac{\mathrm{i}}{2}\sum_{j=1}^n \sum_{\alpha=1}^d dw_j^\alpha \wedge d\overline{w}_j^\alpha,$$

reproducing the Hamiltonian and symplectic form of the Gibbons–Hermsen model.

Details are explained in Fairon, L.F. and Marshall: *Trigonometric real form of the spin RS model of Krichever and Zabrodin*, [arXiv:2007.08388].

Our construction is a 'real form' of earlier reduction treatments of the holomorphic spin RS models of Krichever–Zabrodin (1995), which are due to Chalykh and Fairon [arXiv:1811.08727] and to Arutyunov and Olivucci [arXiv:1906.02619]. The connection to the Gibbons–Hermsen model was not noticed in those papers.

We finish by sketching the degenerate integrability of the reduced system. For this, we exhibit the sufficient number of integrals of motion. To do this, we introduce the new 'spin vectors' $v(1),\dots,v(d)$ with $v(\alpha) := b_R \mathbf{b}(w^1)\cdots \mathbf{b}(w^{\alpha-1}) w^\alpha$, which transform nicely under $\mathrm{U}(n)$. Then, we consider the polynomial subalgebra of $C^\infty(\mathcal{M})^{\mathrm{U}(n)}$:

$$\mathcal{I}_L = \mathbb{R}[\mathrm{tr} L^k, \Re(I^k_{\alpha\beta}), \Im(I^k_{\alpha\beta}) \mid 1 \leq \alpha,\beta \leq d,\ k \geq 0], \quad I^k_{\alpha\beta} := \mathrm{tr}\left(v(\alpha) v(\beta)^\dagger L^k\right).$$

This is closed under the Poisson bracket and its center contains

$$\mathfrak{H}_{\mathrm{tr}} := \mathbb{R}[\mathrm{tr} L^k, k \geq 0].$$

Explicitly, we have

$$\begin{aligned}\{I^M_{\alpha\beta}, I^N_{\gamma\epsilon}\} = {} & 2\mathrm{i}\delta_{\alpha\epsilon} I^{M+N+1}_{\gamma\beta} - 2\mathrm{i}\delta_{\gamma\beta} I^{M+N+1}_{\alpha\epsilon} \\ & + \mathrm{i}(\delta_{\alpha\epsilon} - \delta_{\gamma\beta}) I^M_{\alpha\beta} I^N_{\gamma\epsilon} + 2\mathrm{i}\delta_{\alpha\epsilon} \sum_{\mu<\alpha} I^N_{\gamma\mu} I^M_{\mu\beta} - 2\mathrm{i}\delta_{\gamma\beta} \sum_{\lambda<\beta} I^M_{\alpha\lambda} I^N_{\lambda\epsilon} \\ & + \mathrm{i}\,\mathrm{sgn}(\gamma - \alpha) I^M_{\gamma\beta} I^N_{\alpha\epsilon} - \mathrm{i}\,\mathrm{sgn}(\epsilon - \beta) I^N_{\gamma\beta} I^M_{\alpha\epsilon} \\ & + \mathrm{i}\left(\sum_{b=0}^{M-1} + \sum_{b=0}^{N-1}\right)\left(I^b_{\gamma\beta} I^{M+N-b}_{\alpha\epsilon} - I^{M+N-b}_{\gamma\beta} I^b_{\alpha\epsilon}\right)\end{aligned}$$

and the reality property $\{\overline{I^M_{\alpha\beta}}, \overline{I^N_{\gamma\epsilon}}\} = \overline{\{I^M_{\alpha\beta}, I^N_{\gamma\epsilon}\}}$.

Our Hamiltonian reduction actually works in the real-analytic category, and $\mathfrak{H}_{\mathrm{tr}}$ and $\mathcal{I}_L$ descend to polynomial Poisson algebras on the connected, real-analytic reduced symplectic manifold $(\mathcal{M}_{\mathrm{red}}, \Omega_{\mathrm{red}})$.

Theorem. The reduced polynomial algebras of functions $\mathfrak{H}_{\mathrm{tr}}^{\mathrm{red}}$ and $\mathcal{I}_L^{\mathrm{red}}$ inherited from $\mathfrak{H}_{\mathrm{tr}}$ and $\mathcal{I}_L$ have functional dimension n and $2nd - n$, respectively. In particular, on the phase space $\mathcal{M}_{\mathrm{red}}$ of dimension $2nd$, the Abelian Poisson algebra $\mathfrak{H}_{\mathrm{tr}}^{\mathrm{red}}$ yields a real-analytic, degenerate integrable system with integrals of motion $\mathcal{I}_L^{\mathrm{red}}$

Concretely, for any $d > 1$, we proved that the $2n(d-1)$ integrals of motion:

$$\mathrm{tr}(L^k), \quad I_{1,1}^k, \quad \Re[I_{\alpha,1}^k], \quad \Im[I_{\alpha,1}^k]$$

with $k = 1, \ldots, n$ and $\alpha = 2, \ldots, d-1$, are independent after reduction, and n further integrals of motion may be selected from the real and imaginary parts of the functions $I_{d,1}^k$ in such a way that all in all these provide a set of $2nd - n$ independent functions.

In the $d = 1$ case $\mathfrak{H}_{\mathrm{tr}}^{\mathrm{red}} = \mathcal{I}_L^{\mathrm{red}}$ and one has (only) Liouville integrability.