Counting Level-1, Quaternionic Automorphic Representations on G_2

Rahul Dalal

Johns Hopkins

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Note on technical details

- Anything in gray is a technical detail not relevant to this particular topic
- Anything in orange is background material I will only explain intuitively and imprecisely due to time constraints

Outline

- Background: Quaternionic Automorphic Representations on G_2
- Background: Trace Formulas
- Background: Simple Trace Formula
- The computation
  - The spectral side
  - Stabilization, the geometric side, and some simplifying tricks

Details in [Dal21], *Counting Discrete, Level-*1*, Quaternionic Automorphic Representations on* G_2, ArXiv preprint

Quaternionic G_2 reps

Question: Can we find nice examples of automorphic representations π that don't correspond to forms which were discovered classically?

- Exceptional groups are good place to look
- Want to find nice class of π_∞—analogues to modular forms, not Maass forms

Simplest new example: $G = G_2$, π_∞ a quaternionic discrete series

- Quaternionic: puts a nice differential equation condition on functions, second-best to holomorphic$\leftrightarrow$simplest possible minimal K-types
- Discrete series: Relevance here: studyable with trace formula
- One quaternionic discrete series π_k for each weight $k \geq 2$.

Applications

Where do these come up?

- Fourier coefficients encode information about cubic rings [GGS02]
- Partition functions in certain quantum models of black holes [FGKP18, Chap. 15]
- More in the future?

Main Question

Question: How do we describe the quaternionic-G_2 automorphic representations?

Example: Can we count them with some local conditions?

Answers

We can do both without too much trouble at level-1...

- level-1: π^∞ has a (necessarily 1d) subspace fixed by hyperspecial K^∞.

...in terms of compact form G_2^c

- $\cong G_2$ at all finite places, compact at ∞. In particular, $G_2^c(\mathbb{Z})$ defined.
- V_λ: finite-dimensional rep of $G_2^c(\mathbb{R})$ with highest weight λ, matrix coefficients in $L^2(G_2^c(\mathbb{R}))$.

Notation: β is the highest root of G_2

Formula

Theorem

Let $k > 2$. The number of discrete (equiv. cuspidal) level-1, quaternionic automorphic representations on G_2 of weight k is

$$\dim\left({V_{(k-2)\beta}}^{G_2^c(\mathbb{Z})} \right) +$$

$$\begin{cases} \lfloor \frac{k}{4} \rfloor \left(\lfloor \frac{k}{12} \rfloor - 1 \right) & k \equiv 2 \pmod{12} \\ \lfloor \frac{k}{4} \rfloor \lfloor \frac{k}{12} \rfloor & k \equiv 0, 4, 6, 8, 10 \pmod{12} \\ -\left(\lfloor \frac{3k-1}{12} \rfloor - 1 \right) \left(\lfloor \frac{k+1}{12} \rfloor - 1 \right) & k \equiv 1 \pmod{12} \\ -\left(\lfloor \frac{3k-1}{12} \rfloor - 1 \right) \lfloor \frac{k+1}{12} \rfloor & k \equiv 5, 9 \pmod{12} \\ -\lfloor \frac{3k-1}{12} \rfloor \lfloor \frac{k+1}{12} \rfloor & k \equiv 3, 7, 11 \pmod{12} \end{cases}$$

A Jacquet-Langlands-style result

Theorem

Let $k > 2$. If k is even:

- *the discrete (equiv. cuspidal) level-1, weight k quaternionic representations of G_2 are the exactly the unramified representations of $G_2(\mathbb{A})$ with infinite component π_k and Satake parameters coming from weight $(k-2)\beta$ algebraic modular forms on G_2^c* **in addition to** *those coming from pairs of classical cupsidal newforms in $\mathcal{S}_{3k-2}(1) \times \mathcal{S}_{k-2}(1)$.*

If k is odd:

- *such representations of G_2 are the exactly those coming from weight $(k-2)\beta$ algebraic modular forms on G_2^c* **except for** *those also coming from pairs of classical cupsidal newforms in $\mathcal{S}_{3k-3}(1) \times \mathcal{S}_{k-1}(1)$.*

Table

Table: Counts of discrete, quaternionic automorphic representations of level 1 on G_2.

k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$	k	$\lvert\mathcal{Q}_k(1)\rvert$
3	0	13	5	23	76	33	478	43	1792
4	0	14	13	24	126	34	610	44	2112
5	0	15	8	25	121	35	637	45	2250
6	**1**	16	23	26	175	36	807	46	2619
7	0	17	17	27	173	37	849	47	2790
8	2	18	37	28	248	38	1037	48	3233
9	1	19	30	29	250	39	1097	49	3447
10	4	20	56	30	341	40	1332	50	3938
11	1	21	50	31	349	41	1412	51	4201
12	9	22	83	32	460	42	1686	52	4780

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Method

First trick to try for studying subreps: look at traces

- Assume for a moment

$$L^2(G(F)\backslash G(\mathbb{A}_F), \chi) = \bigoplus_{\pi \text{ d.a.}} \pi$$

- Then if R is an operator on L^2

$$\operatorname{tr}_{L^2} R = \sum_{\pi \text{ d.a.}} \operatorname{tr}_\pi R$$

- Source of R? Convolution: f cmpct. support, smooth/$G(\mathbb{A})$:

$$f(v) := R_f(v) = \int_{G(\mathbb{A})} f(g) g \cdot v \, dg$$

Test Functions Example

Want: f such that

$$\mathrm{tr}_{L^2}(f) = \#\{G_2\text{-quat, lv. 1, wt. } k\}$$

Idea: $f = \prod_v f_v$ so $\mathrm{tr}_\pi(f) = \prod_v \mathrm{tr}_{\pi_v}(f_v)$

- Find f_∞ so that

$$\mathrm{tr}_{\pi_\infty}(f_\infty) = \mathbf{1}_{\pi_\infty \text{ is the weight-}k\text{, quaternionic discrete series}}$$

- If K^∞ is a maximal compact in $G_2(\mathbb{A}^\infty)$ note that

$$\mathrm{tr}_{\pi^\infty}(\mathbf{1}_{K^\infty}) = \mathrm{vol}(K^\infty)\mathbf{1}_{\pi^\infty \text{ is unramified}}$$

Therefore, plug in $f = f_\infty \mathbf{1}_{K^\infty}$

Trace Formula

How do we compute $\mathrm{tr}_{L^2}(f)$?

- Tool: Arthur-Selberg trace formula

$$\sum_{\pi \in \mathcal{AR}(G)} m_\pi \, \mathrm{tr}_\pi(f) \approx$$

$$\sum_{\gamma \in [G(F)]} \mathrm{Vol}(G_\gamma(F) \backslash G_\gamma(\mathbb{A})) \int_{G_\gamma(\mathbb{A}) \backslash G(\mathbb{A})} f(g^{-1}\gamma g) \, dg$$

- Interested in spectral side I_{spec}: averages over aut. reps.
- Try to compute geometric side I_{geom}
  - rational conjugacy classes, volumes of adelic quotients, orbital integrals

Discrete Series

Infinite component discrete series $\implies$ make the $\approx$ explicit:

- Discrete series: appear discretely in $L^2(G(F_\infty))$.
- Classified into L-packets Π_λ
  - L-packets parameterized by dominant weights λ ($\Pi_\lambda \mapsto$inf. char. $\lambda + \rho$)
  - Regular when λ is.
  - Π_λ parameterized by K-dominant Weyl-translates of $\lambda + \rho$: Harish-Chandra parameter.
- Quaternionic discrete series on G_2
  - $\pi_k \in \Pi_{(k-2)\beta}$
  - Harish-Chandra parameter in chamber adjacent to long compact root
  - Equiv: minimal K-type trivial on one SU_2-factor

"Simple" trace formula

Theorem ([Art89])

Let G/F be a cuspidal reductive group and let Π_λ be a regular discrete series L-packet. Let $\mathcal{A}_\lambda$ be the set of automorphic representations π of G with $\pi_\infty \in \Pi_\lambda$. Then for any compactly supported, smooth test function f on $G(\mathbb{A}^\infty)$

$$\sum_{\pi \in \mathcal{A}_\lambda} \operatorname{tr}_{\pi^\infty} f = \sum_{M \text{ std. Levi}} (-1)^{[G:M]} \frac{|\Omega_M|}{|\Omega_G|} \sum_{\gamma \in [M(F)]_{ell}} a_\gamma \Phi_M^G(\gamma) O_\gamma^{M,\infty}(f_M)$$

- "Conjugacy classes" counted with principle of inclusion-exclusion
- "Volume term"
- "Orbital integral" factored into infinite and finite places

Test Function At Infinity

- Discrete Series π come with pseudocoefficients φ_π. For ρ a standard module, $\mathrm{tr}_\rho(\varphi_\pi) = \mathbf{1}_{\pi=\rho}$
- η_λ Euler-Poincaré function

$$\eta_\lambda = \frac{1}{|\Pi_{\mathrm{disc}}(\lambda)|} \sum_{\pi \in \Pi_\lambda} \varphi_\pi$$

- When λ regular, for ρ any unitary representation: $\mathrm{tr}_\rho(\eta_\lambda) = |\Pi_{\mathrm{disc}}(\lambda)|^{-1} \mathbf{1}_{\rho \in \Pi_\lambda}$
- Simple trace formula: use Euler-Poincaré's as infinite component of test function: $\eta_\lambda f^\infty$, the above computes spectral side, geometric side harder

This doesn't quite work for us

Problem 1: counts all reps with $\pi_\infty \in \Pi_{(k-2)\beta}$ instead of all with $\pi_\infty = \pi_k$

- Solution Idea: Use pseudocoefficient at ∞ instead of EP-function.
- Geometric side doesn't simplify nicely then!
- Stabilization resolves this

Problem 2: $(k-2)\beta$ not regular!

- Spectral side may not simplify nicely w/ $f_\infty = \eta_{(k-2)\beta}$ or φ_{π_k}.
- Solution: Facts from real representation theory $\implies$ not an issue for specifically φ_{π_k}, i.e., for quaternionic representations

Problem 3: Terms on geometric side explicit but very hard

- Solution: Chenevier/Renard have tricks to simplify—level 1

Goal

What we want:

Lemma (Spectral Goal)

Let f^∞ be a compactly supported, smooth test function $G_2(\mathbb{A}^\infty)$. Then, for any weight $k > 2$, the spectral side of Arthur's invariant trace formula simplifies:

$$I_{\text{spec}}(\varphi_{\pi_k} f^\infty) = \sum_{\pi \in \mathcal{AR}_{\text{disc}}(G_2)} \mathbf{1}_{\pi_\infty = \pi_k} \operatorname{tr}_{\pi^\infty}(f^\infty).$$

Step 1: Trace Formula Work

Since φ_{π_k} is cuspidal, we still have

$$I_{\mathrm{spec}}(\varphi_{\pi_k} f^\infty) = \sum_{\pi \in \mathcal{AR}_{\mathrm{disc}}(G_2)} \mathrm{tr}_{\pi_\infty}(\varphi_{\pi_k})\,\mathrm{tr}_{\pi^\infty}(f^\infty)$$

Problem reduces to

Lemma (Spectral Goal')

Let σ be an arbitrary unitary representation of $G_2(\mathbb{R})$ and weight $k > 2$. Then

$$\mathrm{tr}_\sigma(\varphi_{\pi_k}) = \mathbf{1}_{\sigma=\pi_k}.$$

Fact: Suffices to check on cohomological representations of weight $(k-2)\beta$—classified by Vogan-Zuckerman.

Step 2: Quaternionic case-specific computations

$k > 2 \implies$ only one non-discrete series cohomological rep of weight $(k-2)\beta$: π_{bad}

- The A-packet of π_{bad} is an Adams-Johnson Packet: $\{\pi_{\text{bad}}, \pi_k\}$ (computation in [Mun20]).
- Lemma in Adams-Johnson paper $\implies$ π_k appears in Grothendieck Group expansion of exactly one member of the A-packet

Conclusion: π_k doesn't appear in Grothendieck Group expansion of $\pi_{\text{bad}} \implies \text{tr}_{\pi_{\text{bad}}}(\varphi_{\pi_k}) = 0$.

Geometric Side: Endoscopy and Stabilization

Goal:

- Invariant terms not good enough—need stably invariant instead

How?

- G has elliptic endoscopic groups $H \in \mathcal{E}_{\text{ell}}(G)$ if G^{der} simply connected
  - (H, s, η): $\widehat{H} = Z_{\widehat{G}}(s)$, $\eta : {}^L H \hookrightarrow {}^L G$
- f on G has a transfer f^H on H
  - κ-orbital integral identity locally: $O^{\kappa_H}_{\gamma_G}(f) = SO_{\gamma_H}(f^H)$
- For $S_\star$ stably-invariant:

$$I^G_\star(f) = \sum_{H \in \mathcal{E}_{\text{ell}}(G)} \iota(G, H) S^H_\star(f^H)$$

How to Use

How to compute I_{geom}?

- $I_{\mathrm{geom}}(f_\infty f^\infty)$ simplifies if f_∞ linear combination of η_λ's.
- Try: write $I_{\mathrm{geom}}(\varphi_{\pi_0} f^\infty)$ in terms of $I_{\mathrm{geom}}(\eta_\lambda f^\infty)$'s

Lemma

If $\pi_0 \in \Pi_{\mathrm{disc}}(\lambda)$, φ_{π_0} has the same stable orbital integrals as η_λ. Furthermore, all endoscopic transfers $(\varphi_{\pi_0})^H$'s can be taken to be linear combinations of η_λ's.

- Therefore stabilization will help!

Application to G_2

Recall our test function is $f = \varphi_{\pi_\infty} \mathbf{1}_{K^\infty}$.

- Compute: For this test function, all endoscopic terms vanish except $H = \mathrm{SL}_2 \times \mathrm{SL}_2 / \pm 1$.
- Compute: $I^H(f^H) = S^H(f^H)$—uses finite component $\mathbf{1}_{K^\infty}$!

Trick: G_2^c the compact form of G_2. Note:

$$(\eta_\lambda^{G_2^c})^{G_2} = \eta_\lambda^{G_2},$$

so can compare corresponding endoscopic expansions:

$$I^{G_2}(\varphi_{\pi_k} \mathbf{1}_{K^\infty}) = I^{G_2^c}(\eta^{G_2^c}_{(k-2)\beta} \mathbf{1}_{K^\infty_{G_2^c}}) - \frac{1}{2} I^H((\eta^{G_2^c}_{(k-2)\beta})^H \mathbf{1}_{K^\infty_H}) - \frac{1}{2} I^H((\varphi_{\pi_k})^H \mathbf{1}_{K^\infty_H})$$

The G_2^c-term

How do we compute $I^{G_2^c}(\eta^{G_2^c}_{(k-2)\beta}\mathbf{1}_{K^\infty_{G_2^c}})$?

- Spectrally: this counts algebraic modular forms on G_2^c of weight $(k-2)\beta$ and level 1.
- $V_{(k-2)\beta}$: finite dimenisonal rep of G_2^c with that weight
- Can show: this count is

$$\dim\left(V_{(k-2)\beta}{}^{G_2^c(\mathbb{Z})}\right)$$

- A computer can compute—table in appendices to [CR15]

The H-terms: Computing transfers

How do we compute the two I^H terms?
Step 1: compute transfers.

- If ϵ_1, ϵ_2 the fundamental weights of the SL_2 factors in $\mathrm{SL}_2 \times \mathrm{SL}_2/\pm 1$:

$$(\varphi_{\pi_k})^H = -\eta^H_{3(k-1)\epsilon_1+(k-1)\epsilon_2} + \eta^H_{(3k-2)\epsilon_1+(k-2)\epsilon_2} - \eta^H_{2(k-1)\epsilon_2}$$
$$(\eta^{G_2^c}_{(k-2)\beta})^H = \eta^H_{3(k-1)\epsilon_1+(k-1)\epsilon_2} - \eta^H_{(3k-2)\epsilon_1+(k-2)\epsilon_2} - \eta^H_{2(k-1)\epsilon_2}$$

- Hardest part is the $\pm$ signs, exact formula depends on a lot of choices of positive Weyl chambers, etc.

The H-terms: Modular form interpretation

How do we compute terms of the form $I^H(\eta_\lambda \mathbf{1}_{K_H^\infty})$?

- Spectrally: Counting level-1 discrete automorphic representations of a given weight on H
- Idea: Up to center details, $H \approx \mathrm{GL}_2 \times \mathrm{GL}_2$ so term $\approx$ counts of pairs of classical modular forms.

Lemma

(vague) The idea works exactly at level 1.

- Proven through various techniques Chenevier and Taïbi developed to do computations on level-1 representations for classical groups.

Papers Mentioned

- James Arthur, *The L^2-Lefschetz numbers of Hecke operators*, Invent. Math. **97** (1989), no. 2, 257–290. MR 1001841
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- Philipp Fleig, Henrik P. A. Gustafsson, Axel Kleinschmidt, and Daniel Persson, *Eisenstein series and automorphic representations*, Cambridge Studies in Advanced Mathematics, vol. 176, Cambridge University Press, Cambridge, 2018, With applications in string theory. MR 3793195
- Wee Teck Gan, Benedict Gross, and Gordan Savin, *Fourier coefficients of modular forms on G_2*, Duke Math. J. **115** (2002), no. 1, 105–169. MR 1932327
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Contact info: dalal@jhu.edu