Infinite-dimensional Geometry: Theory and Applications
Erwin Schrödinger International Institute
for Mathematics and Physics at the University of Vienna

13-17 January 2025, Vienna

Banach-Lie groupoids part I

Aneta Sliżewska
Faculty of Mathematics
University in Białystok

REFERENCES:

1. K. Mackenzie. *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge University Press, 2005.
2. A. Odzijewicz, A. Sliżewska. *Banach Lie groupoids associated to W^*-algebra*, J. Sympl. Geom. 14 (2016), no. 3, 687 - 736
3. A. Odzijewicz, G. Jakimowicz, A. Sliżewska. *Banach-Lie algebroids associated to the groupoid of partially invertible elements of a W^*-algebra*. J.Geom.Phys., 95 (2015) 108-126
4. A. Odzijewicz, G. Jakimowicz, A. Sliżewska. *Fibre-wise linear Poisson structures related to W*-algebras* , J. Geom. Phys. 123 (2018), 385-423
5. T. Goliński, G. Jakimowicz, A. Sliżewska. *Banach Lie groupoid of partial isometries over the restricted Grassmannian*. (to appear)

- The shortest definition:

A groupoid is a small category with invertible morphisms.

Groupoid $\mathcal{G} \rightrightarrows B$:

1. **source map s** : $\mathcal{G} \to B$ and **target map t** : $\mathcal{G} \to B$
2. **product** $m : \mathcal{G}^{(2)} \to \mathcal{G}$

$$m(g,h) =: gh,$$

defined on **the set of composable pairs**

$$\mathcal{G}^{(2)} := \{(g,h) \in \mathcal{G} \times \mathcal{G} : \ \mathbf{s}(g) = \mathbf{t}(h)\},$$

3. injective **identity section** $\varepsilon : B \to \mathcal{G}$,
4. **inverse map** $\iota : \mathcal{G} \to \mathcal{G}$, $\iota \circ \iota = id$,

which satisfy the following conditions:

$$\mathsf{s}(gh) = \mathsf{s}(h), \quad \mathsf{t}(gh) = \mathsf{t}(g), \tag{1}$$

$$k(gh) = (kg)h, \tag{2}$$

$$\varepsilon(\mathsf{t}(g))g = g = g\varepsilon(\mathsf{s}(g)), \tag{3}$$

$$\iota(g)g = \varepsilon(\mathsf{s}(g)), \qquad g\iota(g) = \varepsilon(\mathsf{t}(g)), \tag{4}$$

where $g, k, h \in \mathcal{G}$.

- A group is an example of groupoid.

Example 1 $X \times X \rightrightarrows X$

1. source map $\mathbf{s}((x,y)) = y$ and target map $\mathbf{t}((x,y)) = x$
2. product $(x,y)(y,z) = (x,z)$ for $\mathcal{G}^{(2)} = \{((x,y),(r,z)); \quad y = r, \quad x,y,r,z \in X\}$
3. identity section $\varepsilon(x) = (x,x)$
4. inverse map $\iota((x,y)) = (y,x)$

is a groupoid over X (called **pair groupoid**).

Example 2 (1st motivation - by Mackenzie)
For a vector bundle (E, q, M) we can take
the set $\Phi(E)$ of linear isomorphisms $\xi : E_x \to E_y$
between various fibres $E_x,\ E_y,\ x, y \in M$,

- Isomorphisms are invertible,
- and partially composable, that means we can multiply $\xi\eta$ only that isomorphisms $\xi : E_x \to E_y$, $\eta : E_{x'} \to E_{y'}$, for which $y' = x$,
- Identity section on $y \in M$ is given by the identity map on E_y,

the smooth structure on $\Phi(E)$ is induced from E.
$\Phi(E)$ is called the *frame groupoid* (or *linear frame groupoid*) *of the bundle* (E, q, M).

Example 5

For a principal bundle

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & M \end{array}$$

one defines the equivalence relation on $P \times P$

$$(x, y) \sim (x', y') \quad \Leftrightarrow \quad \exists g \in G \quad (xg, yg) = (x', y').$$

Then the set $\frac{P \times P}{G}$ of equivalence classes $\langle x, y \rangle$ with

1. source map $\mathbf{s}(\langle x, y \rangle) = \pi(y)$ and target map $\mathbf{t}(\langle x, y \rangle) = \pi(x)$
2. for $(\frac{P \times P}{G})^{(2)} = \{(\langle x, y \rangle, \langle r, z \rangle); \quad \exists g \in G \quad yg = r\}$ product is defined by $\langle x, y \rangle \langle r, z \rangle = \langle xg, z \rangle$
3. identity section $\varepsilon(\pi(x)) = \langle x, x \rangle$
4. inverse map $\iota(\langle x, y \rangle) = \langle y, x \rangle$

is a groupoid $\frac{P \times P}{G} \rightrightarrows P/G \cong M$ over P/G (called the **gauge groupoid of the principal bundle** P).

Groupoid action and action groupoid

For a groupoid $\mathcal{G} \rightrightarrows B$ and a set M we can take

$$\mu : M \to B,$$

and a set $\mathcal{G} *_l M := \{(g, r) \in \mathcal{G} \times M : \ \mathsf{s}(g) = \mu(r)\}$.
A left action of groupoid $\mathcal{G}$ *on* M is a map

$$\mathcal{G} *_l M \ni (g, r) \mapsto g \cdot r \in M,$$

which satisfy:

$$(gh) \cdot r = g \cdot (h \cdot r)$$
$$\mu(g \cdot r) = \mathbf{t}(g)$$
$$(\varepsilon\mu(r)) \cdot r = r$$

for $g, h \in G, \ \ r \in M$.

On the set $\mathcal{G} *_l M =: \tilde{G}$ can be defined a groupoid over M, where

1. $\tilde{\mathsf{s}}$, $\tilde{\mathsf{t}} : \tilde{G} \to M$ are given by:

$$\tilde{\mathsf{s}}(g,r) := r \qquad \tilde{\mathsf{t}}(g,r) := g \cdot r,$$

2. on the set of composable elements

$$\tilde{G}^{(2)} = \{((g,r),(h,n)) \in \tilde{G} \times \tilde{G}; \ \ \mathsf{t}(h) = \mathsf{s}(g)\}$$

the product $\tilde{m} : \tilde{G}^{(2)} \to \tilde{G}$ is defined by

$$\tilde{m}((g,r),(h,n)) := (gh,n),$$

3. the identity section $\tilde{\varepsilon} : M \to \tilde{G}$ by $\tilde{\varepsilon}(r) := (\varepsilon(\mu(r)), r)$,
4. the inverse map $\tilde{\iota} : \tilde{G} \to \tilde{G}$ by

$$\tilde{\iota}(g,r) := (\iota(g), g \cdot r).$$

Definition

A *Lie groupoid* is a groupoid $\mathcal{G}$ on base M together with smooth structures on $\mathcal{G}$ and M such that the maps **s** and **t** are surjective submersions, the identity section is smooth, and the (partial) product is smooth.

Facts

In a Lie groupoid
• $\mathcal{G}^{(2)}$ is a closed embedded submanifold of $\mathcal{G} \times \mathcal{G}$ since **s**, **t** are submersions,
• the inverse map is diffeomorphism.

Example 4 If $\mathcal{G} *_l M = G \times M \to M$ is a smooth action of a Lie group G on a manifold M ($\mu(m) = e$) then on the product manifold $G \times M$ can be defined a Lie groupoid structure on M with maps

$$\tilde{\mathbf{s}}(g,n) := n \qquad \tilde{\mathbf{t}}(g,n) := g \cdot n,$$

$$\tilde{m}((g,r),(h,n)) := (gh,n)$$

$$\tilde{\varepsilon}(r) := (e,r)$$

$$\tilde{\iota}(g,r) := (g^{-1}, g \cdot r).$$

Example 5 The *trivial groupoid*
$M \times G \times M$, where M - manifold, G- Lie group ;

Example 6 The *fundamental groupoid*
$\Pi(M)$ - the set of homotopy classes of smooth paths $\gamma : [0,1] \rightarrow M$ in manifold M; if M is connected, then its **s**-fibres are the universal covering spaces of M;

A C^*-algebra $\mathfrak{M}$ is called W^*-**algebra** (von Neumann algebra) if there exists a Banach space $\mathfrak{M}_*$ such that

$$(\mathfrak{M}_*)^* = \mathfrak{M}, \tag{5}$$

where $(\mathfrak{M}_*)^*$ is the Banach space dual to $\mathfrak{M}_*$. For the W^*-algebra $\mathfrak{M}$ there exists the unique Banach space $\mathfrak{M}_*$ with property (5) called **predual space of** $\mathfrak{M}$.

1. An element $p \in \mathfrak{M}$ is called a (orthogonal) **projection** if $p^* = p = p^2$. We will denote the set of projections of the W^*-algebra $\mathfrak{M}$ by $\mathcal{L}(\mathfrak{M})$.
2. An element $u \in \mathfrak{M}$ is called a **partial isometry** if uu^* (or equivalently u^*u) is a projection. We will denote the set of partial isometries of the W^*-algebra $\mathfrak{M}$ by $\mathcal{U}(\mathfrak{M})$.

Motivation (by Odzijewicz): **In quantum system**

- $\mathfrak{M}$ - W^*-algebra, e.g. $\mathfrak{M} = L^\infty(\mathcal{H})$ and $(L^1(\mathcal{H}))^* = L^\infty(\mathcal{H})$

- Space of states $\mathcal{S}$:
$\mathcal{S} := \mathfrak{M}_*^+$, where $(\mathfrak{M}_*)^* = \mathfrak{M}$

- Space of observables $\mathcal{O}$:
$\mathcal{O} := \{space\ of\ spectral\ measures\ E : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathfrak{M})\}$,
where $\mathcal{L}(\mathfrak{M})$ is the orthocomplement lattice of the orthogonal projections, i.e. $p \in \mathcal{L}(\mathfrak{M}) \subset \mathfrak{M}$ iff $p = p^2 = p^*$.

- $\mathcal{B}(\mathbb{R}) \ni \Delta \to \langle \rho, E(\Delta) \rangle$ - is the probability that a measurement of the observable $\hat{X} = \int_{\mathbb{R}} \lambda dE(\lambda)$ will be in $\Delta \subset \mathbb{R}$ iff the system is in the state $\rho \in \mathcal{S}$, where $||\rho||_* = 1$

V.S.Varadarajan. *Geometry of quantum theory*, Springer-Verlag,1968.
J.M.Jauch.*Foundations of quantum mechanics*, Addison-Wesley Publishing Company, 1973
C.Piron.*Foundations of quantum physics*, 1976
G.W.Mackey.*Mathematical foundations of quantum mechanics*, Dover Publications, 2004
A. Odzijewicz.*Coherent states and geometric quantization*, Commun. Math. Phys. 150 (1992)

A. Odzijewicz, T. Ratiu. *Banach Lie-Poisson spaces and reduction*, Comm. Math.

Phys., 243 (2003)

Theorem

The set of partial isometries $\mathcal{U}(\mathfrak{M}) \subset \mathfrak{M}$ with

1. source and target maps $\mathsf{s}, \mathsf{t} : \mathcal{U}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$ given by
$$\mathsf{s}(u) := u^*u, \qquad \mathsf{t}(u) := uu^*,$$
2. the product $m : \mathcal{U}(\mathfrak{M})^{(2)} \to \mathcal{U}(\mathfrak{M})$ defined on the set $\mathcal{U}(\mathfrak{M})^{(2)} := \{(v,u) \in \mathcal{U}(\mathfrak{M}) \times \mathcal{U}(\mathfrak{M}) : \quad uu^* = v^*v\}$ as
$$m(v,u) := vu,$$
3. the identity section $\varepsilon : \mathcal{L}(\mathfrak{M}) \hookrightarrow \mathcal{U}(\mathfrak{M}), \quad \varepsilon(p) := p,$
4. the inverse map $\iota : \mathcal{U}(\mathfrak{M}) \to \mathcal{U}(\mathfrak{M}), \quad \iota(u) := u^*,$

is the groupoid over $\mathcal{L}(\mathfrak{M})$.

Left support $l(x) \in \mathcal{L}(\mathfrak{M})$ (**right support** $r(x) \in \mathcal{L}(\mathfrak{M})$) of $x \in \mathfrak{M}$ is the least projection in $\mathfrak{M}$, such that

$$l(x)x = x \qquad (\text{resp. } x\ r(x) = x). \tag{6}$$

If $x \in \mathfrak{M}$ is selfadjoint, then **support** $s(x)$

$$s(x) := l(x) = r(x).$$

Polar decomposition for $x \in \mathfrak{M}$

$$x = u|x|, \tag{7}$$

where $u \in \mathfrak{M}$ is partial isometry and $|x| := \sqrt{x^*x} \in \mathfrak{M}^+$. Then

$$l(x) = s(|x^*|) = uu^*, \qquad r(x) = s(|x|) = u^*u.$$

Let $G(p\mathfrak{M}p)$ be the group of all invertible elements in W^*-subalgebra $p\mathfrak{M}p \subset \mathfrak{M}$.
We define the set $\mathcal{G}(\mathfrak{M})$ of **partially invertible** elements in $\mathfrak{M}$

$$\mathcal{G}(\mathfrak{M}) := \{x \in \mathfrak{M}; \quad |x| \in G(p\mathfrak{M}p), \text{ where } p = s(|x|)\}$$

Remark: $\mathcal{G}(\mathfrak{M}) \subsetneq \mathfrak{M}$ in a general case.

Proposition

The set $\mathcal{G}(\mathfrak{M})$ with

1. the source and target maps $\mathbf{s}, \mathbf{t} : \mathcal{G}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$

$$\mathbf{s}(x) := r(x), \qquad \mathbf{t}(x) := l(x),$$

2. the product defined as the product in $\mathfrak{M}$ on the set

$$\mathcal{G}(\mathfrak{M})^{(2)} := \{(x, y) \in \mathcal{G}(\mathfrak{M}) \times \mathcal{G}(\mathfrak{M}); \;\; \mathbf{s}(x) = \mathbf{t}(y)\},$$

3. the identity section $\varepsilon : \mathcal{L}(\mathfrak{M}) \hookrightarrow \mathcal{G}(\mathfrak{M})$ as the inclusion map,
4. the inverse map $\iota : \mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ defined by

$$\iota(x) := |x|^{-1} u^*,$$

is a groupoid over $\mathcal{L}(\mathfrak{M})$.

• Inner action $I : \mathcal{U}(\mathfrak{M}) * \mathcal{L}(\mathfrak{M}) \to \mathcal{L}(\mathfrak{M})$

$$I_x p := xp\iota(x), \quad \mathsf{s}(x) = p$$

on the lattice $\mathcal{L}(\mathfrak{M})$ gives the equivalence relation:

$$p \sim q \quad \Leftrightarrow \quad q \in \mathcal{O}_p.$$

• The equivalence class $[p]$ of p in sense of Murray-von Neumann is the orbit $\mathcal{O}_p$ of $p \in \mathcal{L}(\mathfrak{M})$.

Remark

The Murray-von Neumann classification of W^*-algebras directly corresponds to the classification of orbits of the inner action of $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ (or $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$) on the lattice of projections $\mathcal{L}(\mathfrak{M})$.

We consider the following locally convex topologies on $\mathfrak{M}$:

$$\sigma\text{−topology} \prec s\text{−topology} \prec s^*\text{−topology} \prec \text{uniform topology}$$

where for $\omega \in \mathfrak{M}_*^+$

1. σ-topology is defined by a family of semi-norms $\|x\|_\sigma := |\langle x, \omega\rangle|$,
2. s-topology is defined by a family of semi-norms $\|x\|_\omega := \sqrt{\langle x^*x, \omega\rangle}; \quad x \in \mathfrak{M}$;
3. s^*-topology is defined by a family of semi-norms $\{\|\cdot\|_\omega, \|\cdot\|_\omega^* : \ \omega \in \mathfrak{M}_*^+\}$ where $\|x\|_\omega^* := \sqrt{\langle xx^*, \omega\rangle}; \quad x \in \mathfrak{M}$.

Proposition

The groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is not a topological groupoid with respect to any natural topology in $\mathfrak{M}$.

Proof: Let us take $p \in \mathcal{L}(\mathfrak{M})$ and define $x_n \in \mathcal{G}(\mathfrak{M})$ by $x_n = p + \frac{1}{n}(1-p), \;\; n \in \mathbb{N}$. One has

$$s(x_n) = t(x_n) = 1 \quad \text{and} \quad s(p) = t(p) = p.$$

Since the uniform limit of x_n is $p = \lim_{n\to\infty} x_n$, we see that source and target maps of $\mathcal{G}(\mathfrak{M})$ are not continuous. Thus we obtain that $\mathcal{G}(\mathfrak{M})$ is not a topological groupoid. Note that the above consideration does not depend on the choice of topology on $\mathfrak{M}$.

Proposition

The groupoid $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is topological with respect to the uniform topology and $s^*(\mathcal{U}(\mathfrak{M}), \mathfrak{M}_*)$-topology.

$\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ as a complex Banach-Lie groupoid

For $p \in \mathcal{L}(\mathfrak{M})$ let us define the subset of $\mathcal{L}(\mathfrak{M})$

$$\Pi_p := \{q \in \mathcal{L}(\mathfrak{M}) : \quad \mathfrak{M} = q\mathfrak{M} \oplus (1-p)\mathfrak{M}\}$$

then $q \wedge (1-p) = 0, \qquad q \vee (1-p) = 1$
and $\qquad p = x_p - y_p \ \in q\mathfrak{M} \oplus (1-p)\mathfrak{M}$.
Define the maps

$$\sigma_p : \Pi_p \to q\mathfrak{M}p, \qquad \varphi_p : \Pi_p \tilde{\to} (1-p)\mathfrak{M}p$$

by

$$\sigma_p(q) := x_p, \qquad \varphi_p(q) := y_p.$$

The map φ_p is a bijection of Π_p onto the Banach space $(1-p)\mathfrak{M}p$.

In order to find the transitions maps

$$\varphi_p \circ \varphi_{p'}^{-1} : \varphi_{p'}(\Pi_p \cap \Pi_{p'}) \to \varphi_p(\Pi_p \cap \Pi_{p'})$$

in the case $\Pi_p \cap \Pi_{p'} \neq \emptyset$, let us take for $q \in \Pi_p \cap \Pi_{p'}$ the following splittings

$$\begin{array}{l} \mathfrak{M} = q\mathfrak{M} \oplus (1-p)\mathfrak{M} = p\mathfrak{M} \oplus (1-p)\mathfrak{M} \\ \mathfrak{M} = q\mathfrak{M} \oplus (1-p')\mathfrak{M} = p'\mathfrak{M} \oplus (1-p')\mathfrak{M}. \end{array} \tag{8}$$

and we obtain

$$y_{p'} = (\varphi_{p'} \circ \varphi_p^{-1})(y_p) = (b + dy_p)\iota(a + cy_p),$$

where $a = p'p$, $b = (1-p')p$, $c = p'(1-p)$ and $d = (1-p')(1-p)$.

$\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ as a complex Banach-Lie groupoid

Theorem

The family of maps

$$(\Pi_p, \varphi_p) \quad p \in \mathcal{L}(\mathfrak{M})$$

defines a complex analytic atlas on a $\mathcal{L}(\mathfrak{M})$. This atlas is modeled by the family of Banach spaces $(1-p)\mathfrak{M}p$, where $p \in \mathcal{L}(\mathfrak{M})$.

Fact

If $p' \in \mathcal{O}_p$ then $(1-p)\mathfrak{M}p \cong (1-p')\mathfrak{M}p'$.

For projections $\tilde{p}, p \in \mathcal{L}(\mathfrak{M})$ we define the set

$$\Omega_{\tilde{p}p} := \mathbf{t}^{-1}(\Pi_{\tilde{p}}) \cap \mathbf{s}^{-1}(\Pi_p)$$

and the map

$$\psi_{\tilde{p}p} : \Omega_{\tilde{p}p} \to (1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$$

in the following way

$$\psi_{\tilde{p}p}(x) := (\varphi_{\tilde{p}}(\mathbf{t}(x)), \iota(\sigma_{\tilde{p}}(\mathbf{t}(x)))x\sigma_p(\mathbf{s}(x)), \varphi_p(\mathbf{s}(x))) = (y_{\tilde{p}}, z_{\tilde{p}p}, y_p)\,.$$

Theorem

The family of maps

$$(\Omega_{\tilde{p}p}, \psi_{\tilde{p}p}) \quad \tilde{p}, p \in \mathcal{L}(\mathfrak{M})$$

defines a complex analytic atlas on the groupoid $\mathcal{G}(\mathfrak{M})$. The complex Banach manifold structure of $\mathcal{G}(\mathfrak{M})$ has type $\mathfrak{G}$, where $\mathfrak{G}$ is the set of Banach spaces

$$(1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$$

indexed by the pair of equivalent projections of $\mathcal{L}(\mathfrak{M})$.

$\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ as a complex Banach-Lie groupoid

Proposition

The underlying topology of complex Banach groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is Hausdorff.

Corollary

$\mathcal{L}(\mathfrak{M})$ is a submanifold of $\mathcal{G}(\mathfrak{M})$ then the underlying topology of $\mathcal{L}(\mathfrak{M})$ is also Hausdorff.

The groupoid $\mathcal{U}(\mathfrak{M})$ is the set of fixed points of the automorphism $J : \mathcal{G}(\mathfrak{M}) \to \mathcal{G}(\mathfrak{M})$ defined by

$$J(x) := \iota(x^*).$$

Since $J^2(x) = x$ for $x \in \mathcal{U}(\mathfrak{M})$ one has

$$(DJ(x))^2 = \mathbf{1}$$

for $DJ(x) : T_x\mathcal{G}(\mathfrak{M}) \to T_x\mathcal{G}(\mathfrak{M})$.
Thus one obtains a splitting of the tangent space

$$T_x\mathcal{G}(\mathfrak{M}) = T_x^+\mathcal{G}(\mathfrak{M}) \oplus T_x^-\mathcal{G}(\mathfrak{M}) \tag{9}$$

defined by the Banach space projections

$$P_\pm(x) := \frac{1}{2}\left(\mathbf{1} \pm DJ(x)\right). \tag{10}$$

Proposition

The groupoid $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ of partial isometries has a natural structure of the real Banach manifold of the type $\mathfrak{G}$, where the family $\mathfrak{G}$ consist of the real Banach spaces

$$(1-\tilde{p})\mathfrak{M}\tilde{p} \oplus i\; p\mathfrak{M}^h p \oplus (1-p)\mathfrak{M}p$$

parameterized by the pairs $(\tilde{p}, p) \in \mathcal{L}(\mathfrak{M}) \times \mathcal{L}(\mathfrak{M})$ of equivalent projections.

In order to investigate the Banach-Lie groupoid structure of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ it is enough to restrict ourselfs to

$$\mathcal{L}_{p_0}(\mathfrak{M}) := \{p \in \mathcal{L}(\mathfrak{M}) : \ p \sim p_0\} = \mathcal{O}_p$$

$$\mathcal{G}_{p_0}(\mathfrak{M}) := \mathbf{t}^{-1}(\mathcal{L}_{p_0}(\mathfrak{M})) \cap \mathbf{s}^{-1}(\mathcal{L}_{p_0}(\mathfrak{M})).$$

Then $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$ is a Banach-Lie subgroupoid of $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$.

Proposition

The Banach-Lie groupoid $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$ is a disjoint union of Banach-Lie subgroupoids $\mathcal{G}_{p_0}(\mathfrak{M}) \rightrightarrows \mathcal{L}_{p_0}(\mathfrak{M})$, $p_0 \in \mathcal{L}(\mathfrak{M})$, which are its closed-open Banach subgroupoids.

Fact

Every transitive Lie groupoid is isomorphic with a gauge groupoid of some principal bundle.

We consider $P_0 := \mathsf{s}^{-1}(\mathcal{L}_{p_0}(\mathfrak{M}))$ as the total space of the G_0-principal bundle $\pi_0 := \mathbf{t}|_{P_0} : P_0 \to \mathcal{L}_{p_0}(\mathfrak{M})$, where G_0 is the Banach-Lie group $G(p_0\mathfrak{M}p_0)$ of the invertible elements of the W^*-subalgebra $p_0\mathfrak{M}p_0$. The free right actions of G_0 on P_0 and on $P_0 \times P_0$ are defined by

$$\kappa : P_0 \times G_0 \ni (\eta, g) \mapsto \eta g \in P_0 \tag{11}$$

and by

$$\kappa_2 : P_0 \times P_0 \times G_0 \ni (\eta, \xi, g) \mapsto (\eta g, \xi g) \in P_0 \times P_0, \tag{12}$$

respectively. The above allows us to define the quotient groupoid $\frac{P_0 \times P_0}{G_0} \rightrightarrows P_0/G_0$ of the pair groupoid $P_0 \times P_0 \rightrightarrows P_0$, which by definition is the gauge groupoid associated to the G_0-principal bundle $\pi_0 : P_0 \to P_0/G_0 \cong \mathcal{L}_{p_0}(\mathfrak{M})$.

The complex analytic maps

$$\phi : \frac{P_0 \times P_0}{G_0} \ni \langle \eta, \xi \rangle \mapsto \eta \xi^{-1} \in \mathcal{G}_{p_0}(\mathfrak{M}) \tag{13}$$

$$\varphi : P_0/G_0 \ni \langle \eta \rangle \mapsto \eta \eta^{-1} \in \mathcal{L}_{p_0}(\mathfrak{M}) \tag{14}$$

define the isomorphism

$$\begin{array}{ccc} \frac{P_0 \times P_0}{G_0} & \xrightarrow{\phi} & \mathcal{G}_{p_0}(\mathfrak{M}) \\ \downarrow\downarrow & & \mathbf{t}\ \downarrow\downarrow\ \mathbf{s} \\ P_0/G_0 & \xrightarrow{\varphi} & \mathcal{L}_{p_0}(\mathfrak{M}) \end{array} , \tag{15}$$

of Banach-Lie groupoids.

EXAMPLE

Let $\mathfrak{M} = L^\infty(\mathcal{H})$, where $\mathcal{H}$ is a separable complex Hilbert space with a fixed orthonormal basis $\{|e_k\rangle\}_{k=0}^{\infty}$. Assuming $p_0 = |e_0\rangle\langle e_0|$ we find

1. $\mathfrak{M}p_0 = \{|\vartheta\rangle\langle e_0| : \quad \vartheta \in \mathcal{H}\} \cong \mathcal{H}$,
2. $P_0 = \{|\eta\rangle\langle e_0| : \quad \eta \in \mathcal{H} \setminus \{0\}\} \cong \mathcal{H} \setminus \{0\}$,
3. $G(p_0\mathfrak{M}p_0) = \mathbb{C} \setminus \{0\}$,
4. $\mathcal{L}_{p_0}(\mathfrak{M}) = \left\{ \frac{|\eta\rangle\langle\eta|}{\langle\eta|\eta\rangle} : \quad \eta \in \mathcal{H} \setminus \{0\} \right\} \cong \mathbb{CP}(\mathcal{H})$,
5. $\mathcal{G}_{p_0}(\mathfrak{M}) = \left\{ \frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle} : \quad \eta, \xi \in \mathcal{H} \setminus \{0\} \right\}$.

The target map $l : \mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \to \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$ restricted to P_0 defines the complex Hopf principal bundle

$$\begin{tikzcd} \mathbb{C} \setminus \{0\} \arrow[r] & \mathcal{H} \setminus \{0\} \arrow[d, "l"] \\ & \mathbb{CP}(\mathcal{H}) \end{tikzcd} \tag{16}$$

where

$$
\begin{aligned}
l\left(\frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle}\right) &= \frac{|\eta\rangle\langle\eta|}{\langle\eta|\eta\rangle} \\
r\left(\frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle}\right) &= \frac{|\xi\rangle\langle\xi|}{\langle\xi|\xi\rangle} \\
\left(\frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle}\right)^{-1} &= \frac{|\xi\rangle\langle\eta|}{\langle\eta|\eta\rangle}
\end{aligned}
\tag{17}
$$

and the product of $\frac{|\eta\rangle\langle\xi|}{\langle\xi|\xi\rangle}$ and $\frac{|\xi\rangle\langle\lambda|}{\langle\lambda|\lambda\rangle}$ is equal to $\frac{|\eta\rangle\langle\lambda|}{\langle\lambda|\lambda\rangle}$.

We conclude that the groupoid $\mathcal{G}_{p_0}(L^\infty(\mathcal{H})) \rightrightarrows \mathcal{L}_{p_0}(L^\infty(\mathcal{H}))$ is isomorphic to the gauge groupoid of the Hopf principal bundle $P(\mathbb{CP}(\mathcal{H}), \mathbb{C} \setminus \{0\}, l)$.

The orthonormal projection $p_k := |e_k\rangle\langle e_k|,\ k \in \mathbb{N} \cup \{0\}$, defines a complex analytic atlas $\varphi_k : \Pi_k \to (1 - p_k)(L^\infty(\mathcal{H}))p_k$, where

$$\Pi_k = \left\{ q = \frac{|\xi\rangle\langle\xi|}{\xi\xi}; \quad \xi_k \neq 0, \text{where } \xi = \sum_{k=o}^{\infty} \xi_k |e_k\rangle \right\}, \tag{18}$$

and

$$\varphi_k(q) = \frac{1}{\xi_k} |\xi\rangle\langle e_k| - |e_k\rangle\langle e_k| = y_k \tag{19}$$

$$\sigma_k(q) = \frac{1}{\xi_k} |\xi\rangle\langle e_k| . \tag{20}$$

Let us note here that

$$y_k = \sum_{l \neq k} \frac{\xi_l}{\xi_k} |e_l\rangle\langle e_k|. \tag{21}$$

So, $\frac{\xi_l}{\xi_k} =: y_k^l$, where $k \neq l \in \mathbb{N} \cup \{0\}$, are the homogeneous coordinates of $q \in \Pi_k$.

The atlas for groupoid $\mathcal{G}_{p_0}(L^\infty(\mathcal{H}))$ is the following

$$\psi_{km} := l^{-1}(\Pi_k) \cap r^{-1}(\Pi_m) \to$$

$$\to (1 - p_k)(L^\infty(\mathcal{H}))p_k \oplus p_k(L^\infty(\mathcal{H}))p_m \oplus (1 - p_m)(L^\infty(\mathcal{H}))p_m,$$

where

$$\psi_{km}(g) = \left(\varphi_k(l(g)), (\sigma_k(l(g)))^{-1} g \sigma_m(r(g)), \varphi_m(r(g))\right) =$$

$$= (y_k, z_{km}, y_m) \,.$$

The coordinates y_k and y_m are defined in (19) and

$$z_{km} = \frac{\eta_k}{\xi_m} \left| e_k \right\rangle \left\langle e_m \right| . \tag{22}$$

So, as one should expect the complex analytic manifold structure of $\mathcal{G}_{p_0}(L^\infty(\mathcal{H}))$ is consistent with the complex analytic structure of the Hopf bundle (16).

THANK YOU