Intrinsically interacting symmetry protected phases of fermions

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(work with A. Vishwanath and M. Metlitski)
Setting:

- equilibrium zero temperature phases of many body quantum systems

- many body quantum system = (interacting) Hamiltonian model of fermions on a lattice

- require gap in many body spectrum => studying physics of the ground state
Plan:

1) Topological insulators

2) Symmetry Protected Topological (SPT) phases = topological insulators with interactions

3) 3d SPT surface states: surface topological order

4) symmetry fractionalization

5) surface topological order for new intrinsically interacting 3d SPT
Symmetry breaking order

- Landau-Ginzburg-Wilson paradigm
- Universal signature: local order parameter
2d quantum spin Hall system (HgTe):

- distinct quantum phases preserving time reversal symmetry

3d topological insulator (Bi2Se3):

- cannot be understood with conventional symmetry breaking picture
Band insulators

Hamiltonians can be smoothly connected, with no phase transition
Band insulators

- Band insulators with unbroken symmetry

- Trivial band insulator
- Topological band insulator

Any path that connects the two must encounter phase transition or break symmetry
Quantized index distinguishing TIs

can occur over every point in the Brillouin zone
Why ‘topological’ band insulator?

Interpret filled and empty states geometrically as vectors:

Focus on filled state subspace as you scan across Brillouin zone:

Why ‘topological’ band insulator?
Add interactions

band insulators with unbroken symmetry

band insulators
Add interactions:

- Single phase
  - Interacting insulators continuously connected to band (free fermion) insulators
  - Phase with no free fermion description: intrinsically interacting fermionic SPT
interacting insulators continuously connected to band (free fermion) insulators

Add interactions:

single phase

phase with no free fermion description: intrinsically interacting fermionic SPT
Intrinsically interacting fermion SPT: Example

Cheng, Tantivasadakarn, Wang, arXiv:1705.08911

- 3d, $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ internal symmetry

(no free fermion TIs in this symmetry class)

- exactly solved model

- non-trivial quantized response by coupling to $G$-gauge field

$\Delta L = \theta \vec{E} \cdot \vec{B}$, with $\theta$

quantized to $\theta = 0, \pi$ by $T$ symmetry in an ordinary 3d TI)

(analagous to $\Delta L = \theta \vec{E} \cdot \vec{B}$, with $\theta$)
Coupling to G-gauge field (bosonic case)

- 2d: statistics of symmetry fluxes
  \( \omega(f, g, h) \in U(1) \)  
  (Chen, Wen et al; Levin, Gu)

- 3d: 3-loop braiding
  \( \Omega(f, g, h, k) \in U(1) \)  
  (Levin, Wang, Lin, Liu)
Coupling to G-gauge field:

**Bosonic case:**
- in general dimension d, response given by $U(1)$-valued function of $d+1$ group variables
  - constraints
  - gauge redundancies

\[ \Rightarrow H^{d+1}(G, U(1)) \]

(Dijkgraaf, Witten 1989)

**Fermionic case:**
- worked out recently
- new intrinsically interacting phase has quantized responses not realized in any free fermion system

Kapustin, Thorngren arXiv:1701.08264
Wang, Gu arXiv:1703.10937
Cheng, Tantivasadakarn, Wang, arXiv:1705.08911
- Understanding surface states of SPTs is important:
  - physical signature
  - 't Hooft anomalies (e.g. parity anomaly for 3d TI)

- Surface states of intrinsically interacting fermion SPTs still poorly understood
Surface states of SPTs: new possibilities in 3d

- 3d Topological insulator:

  (1) Dirac cone:

  ![Diagram of Dirac cone](image)

  (2) Surface topological order:

  Anyons at the surface

  ![Diagram of surface topological order](image)

  (Chen, Fidkowski, Vishwanath)
  (Senthil, Wang)
  (Bonderson, Qi, Nayak)
  (Fisher, Kane, Metlitski)
Surface states of SPTs: new possibilities in 3d

- proposal for a topologically ordered fermionic surface state for 3d intrinsically interacting fermionic SPT, and a general bulk / surface ’t Hooft anomaly matching condition
1d bosonic SPTs:

Haldane phase: $G = SO(3)$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$
Symmetry fractionalization

1d bosonic SPTs:

General G:

\[ \hat{U}^g_{\text{L}} = \hat{U}^g_{\text{L}} \hat{U}^g_{\text{R}} \]

\[ \hat{U}^{gh}_{\text{L}} = e^{i\phi(g,h)} \hat{U}^g_{\text{L}} \hat{U}^h_{\text{L}} \]

irremovable Berry’s phase
1+1 dimensional bosonic SPTs

- Algebraic properties of $\phi(g, h)$:

1) $d\phi(g, h, k) \equiv \\
\phi(h, k) - \phi(gh, k) + \phi(g, hk) - \phi(g, h) = 0 \mod 2\pi$

2) $\phi(g, h) \sim \phi(g, h) + \alpha(g) + \alpha(h) - \alpha(gh)$

$[\phi] \in H^2(G, U(1))$

- Interpretation as action for 1+1d G-gauge field: bulk/boundary correspondence

(Dijkgraaf, Witten)
Surface topological order

information about topological phase encoded in braiding statistics of anyons:

\[
\{1, a, b, c\}
\]

anyon types

- braiding information:

\[
S_{ab}, T_{ab}
\]

\[
- a \times b = \sum_c N_{c}^{a,b} c
\]

- in fermionic theories one of the anyons is a transparent local fermion \( f \)

- this is a mathematical structure \( \mathcal{A} \) called a ‘topological quantum field theory’ (TQFT)
Surface topological order and symmetry

- Extra data characterizing action of symmetry:

  1) the way symmetry *permutes* anyon types

\[ G \rightarrow \text{Aut}(\mathcal{A}) \]

  2) the way symmetry *fractionalizes* on anyons

- mere fact that symmetry fractionalizes on anyons does not indicate anomaly; but, certain fractionalization patterns are anomalous in a more subtle way

(Tarantino, Lindner, Fidkowski, NJP 18, 2016, also Barkeshli, Bonderson, Cheng, Wang; Teo, Fradkin, Hughes)
Symmetry fractionalization on anyons

- Symmetry fractionalization on surface anyons:

\[ U^a_{gh} = e^{i\phi_a(g,h)} U^a_g U^a_h \]

- consistency requirement:

\[ \phi_a(g,h) + \phi_b(g,h) = \phi_c(g,h) \mod 2\pi \]

whenever \( N^{a,b}_c > 0 \)

- fractionalization encoded in \( \{\phi_a(g,h)\} \)

- are all fractionalization patterns realizable in 2d? No.
Symmetry fractionalization on anyons

- in 2d, can introduce dynamical pi fluxes:

\[ A \subset \mathcal{A}' \]

original set of anyons

set of anyons including pi fluxes

- must be able to extend:

\[ \{ \phi_a(g, h) \}_{a \in \mathcal{A}} \rightarrow \{ \phi_{a'}(g, h) \}_{a' \in \mathcal{A}'} \]
Symmetry fractionalization on anyons

- There may be an *obstruction* to performing this extension

- Can encode fractionalization data in \( \{ \omega(g, h) \} \), with

\[
\phi_a(g, h) = S_{a,\omega(g,h)}
\]

- however, \( \omega(g, h) \) *not unique*:

\[
\omega(g, h) \rightarrow \omega(g, h) \times f
\]
Symmetry fractionalization on anyons

- Also, because for all $a$ we have

$$
\phi_{g^{-1}a}(h, k) - \phi_a(gh, k) + \phi_a(g, hk) - \phi_a(g, h) = 0 \mod 2\pi
$$

it must be that

$$
d\omega(g, h, k) \equiv g \cdot \omega(h, k) - \omega(gh, k) + \omega(g, hk) - \omega(g, h) \in \{1, f\}
$$

- it may be impossible to gauge $d\omega$ away by modifications of the form

$$
\omega(g, h) \rightarrow \omega(g, h) \times f
$$

- $d\omega \in H^3(G, \mathbb{Z}_2)$ is then a possibly non-zero cohomology class. This signals an anomaly, since in this case symmetry action not associative on pi fluxes.
Symmetry fractionalization on anyons

- however, we can realize this symmetry fractionalization pattern on the surface of 3d SPT, because pi flux no longer pointlike:

- matches bulk invariant in $H^3(G, \mathbb{Z}_2)$
3+1d fermionic SPT surface: example

Bulk SPT: (Wang, Gu arXiv:1703.10937)

\[ G = \mathbb{Z}_2 \times \mathbb{Z}_4 \]  (total symmetry \( \mathbb{Z}_2^f \times \mathbb{Z}_2 \times \mathbb{Z}_4 \))

\[ [\rho] = [\alpha][\beta] \]

\[ [\alpha] \in H^1(\mathbb{Z}_4, \mathbb{Z}_2) \]
\[ [\beta] \in H^2(\mathbb{Z}_2, \mathbb{Z}_2) \]

Surface topological order \( = \mathcal{A} \times \{1, f\} \)

upward

bosonic \( \mathbb{Z}_4 \) gauge theory
3+1d fermionic SPT surface: example

Denote surface anyons by \([j, k, \mu] \equiv m^j e^k f^\mu\)

\(\mathbb{Z}_4\) flux \(\uparrow\)

\(\mathbb{Z}_4\) charge \(\downarrow\)

\[j, k = 0, \ldots, 3; \quad \mu = 0, 1\]

Denote group elements by \((g_1, g_2)\)

\(\downarrow\)

in \(\mathbb{Z}_2\)

in \(\mathbb{Z}_4\)
3+1d fermionic SPT surface: *example*

Permutation action of $G$ on anyons:

$$[j, k, \mu] \rightarrow [j, k + 2g_2j, \mu + g_2j]$$

Symmetry fractionalization:

$$\omega(g, h) = [g_1h_1, g_2h_1, 0]$$

Then $[d\omega] \in H^3(G, \mathbb{Z}_2)$ is non-zero, and equal to $[\rho]$. 
Conclusions:

- Connection between anomalous 2+1d symmetry enriched fermionic topological orders and 3+1d SPTs

- Surface topological order for interacting fermionic SPTs with no free fermion realization

- Future directions: gapless parent surface state / effective field theory description? beyond supercohomology SPTs?