Equivariant Symplectic Geometry of Cotangent Bundles

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EQUIVARIANT SYMPLECTIC GEOMETRY
OF COTANGENT BUNDLES

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Let a connected reductive complex algebraic group $G$ act on a smooth irreducible quasiaffine algebraic variety $X$. In this paper, we study the cotangent bundle $T^*X$ as a symplectic $G$-variety.

Following I. M. Gelfand and M. I. Graev [GG59], we define horospheres in $X$ as orbits of maximal unipotent subgroups of $G$ or, equivalently, as images under the action of $G$ of orbits of a fixed maximal unipotent subgroup $U$.

The set of all horospheres is not a good object. We consider a big piece of it consisting of horospheres “in general position” and having a natural structure of a smooth irreducible quasiaffine algebraic $G$-variety of the same dimension as $X$. We call it the variety of horospheres of $X$ and denote by Hor $X$.

The conormal bundle of any smooth subvariety of $X$ is a Lagrangian subvariety of $T^*X$. The conormal bundles of horospheres of the set Hor $X$ cover a dense subset of $T^*X$ and any point in general position of $T^*X$ belongs to finitely many of them. The disjoint union of these conormal bundles is naturally supplied with a structure of a smooth irreducible quasiaffine algebraic $G$-variety of the same dimension as $T^*X$. We call it the horospherical cotangent bundle of $X$ and denote by $HT^*X$. There is a natural $G$-equivariant morphism

$$HT^*X \rightarrow \text{Hor } X$$

whose fibers are the conormal bundles of horospheres.

Under the natural projection

$$HT^*X \rightarrow T^*X$$

the symplectic structure of $T^*X$ is lifted to an open subset of $HT^*X$. The intersections of the conormal bundles of horospheres with this subset constitute a Lagrangian fibration over Hor $X$. In this respect, the variety $HT^*X$ is similar to the cotangent bundle $T^* \text{Hor } X$ of Hor $X$.

On the other hand, there is a remarkable $G$-equivariant birational morphism

$$f: HT^*X \rightarrow T^* \text{ Hor } X$$

(see its definition in Section 5) such that the diagram

$$\begin{array}{ccc}
HT^*X & \xrightarrow{f} & T^* \text{ Hor } X \\
\downarrow & & \downarrow \\
\text{Hor } X & & 
\end{array}$$

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is commutative. It is natural to wonder if \( f \) is symplectic (on the open subset of \( H^\ast X \) where the symplectic structure is defined). In general it is not true. Our main result is that, for any symmetric space \( X \), the morphism \( f \) is symplectic.

It follows that, for a symmetric space \( X \), there is a \( G \)-equivariant symplectic rational covering

\[
T^\ast \text{Hor} X \to T^\ast X.
\]

Note that in this case the variety \( \text{Hor} X \) is a homogeneous space of the form \( G/S \) where \( S \supset U \). (A precise description of \( S \) see in Section 6.)

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Some notation

Algebraic groups are denoted by capital Latin letters; their tangent Lie algebras are denoted by the corresponding small Gothic letters. The connected component of an algebraic group \( G \) is denoted by \( G^0 \). The character group of \( G \) is denoted by \( \mathcal{X}(G) \); it is written additively, and any character is identified with its differential.

If a group \( G \) acts on a set \( X \), the stabilizer of a point \( x \in X \) is denoted by \( G_x \).

The set of fixed points of an element \( g \in G \) is denoted by \( X^g \); the set of fixed points of the whole group \( G \) is denoted by \( X^G \).

The homogeneous fibering over a homogeneous space \( G/H \) defined by an action \( H : Y \), is denoted by \( G \ast_H Y \). By definition, it is the quotient \( (G \times Y)/H \) with respect to the action of \( H \) defined by \( h(g, y) = (gh^{-1}, hy) \). The class of \( H \)-equivalence containing \( (g, y) \) is denoted by \([g, y]\). The action \( G : G \ast_H Y \) is defined by \( g'([g, y]) = [g'g, y] \). The projection \( p : G \ast_H Y \to G/H \) is defined by \( p([g, y]) = gH \).

Some more notation:

\( Z(H) \) (resp. \( N(H) \)): the centralizer (resp. the normalizer) of a subgroup \( H \) in a group;

\( Z(\mathfrak{h}) \) (resp. \( N(\mathfrak{h}) \)): the centralizer (resp. the normalizer) of a subalgebra \( \mathfrak{h} \) in a Lie algebra;

\( \mathbb{C}[X] \): the algebra of polynomial (= regular rational) functions on a quasiaffine algebraic variety \( X \);

\( \mathbb{C}(X) \): the field of rational functions on an irreducible algebraic variety \( X \);

\( V^\ast \): the dual space of a vector space \( V \);

\( \langle S \rangle \): the linear span of a subset \( S \) in a vector space.

1. Preliminaries

Let \( B \) be a Borel subgroup of \( G \) and \( U \) the unipotent radical of \( B \) (which is a maximal unipotent subgroup of \( G \)).

Denote by \( \mathcal{X}_+(B) \) the semigroup of dominant characters of \( B \) and, for any \( \lambda \in \mathcal{X}_+(B) \), denote by \( \mathbb{C}[X]_\lambda \) the isotypic component of the natural linear representation \( G : \mathbb{C}[X] \), corresponding to the irreducible linear representation with highest weight \( \lambda \). The set

\[
\Lambda = \{ \lambda \in \mathcal{X}_+(B) : \mathbb{C}[X]_\lambda \neq 0 \}
\]

is a subsemigroup of \( \mathcal{X}_+(B) \). Denote by \( \mathbb{C}[X]^{(B)} \) the set of \( B \)-semi-invariant functions in \( \mathbb{C}[X] \).
Set
\[ P = \{ g \in G : g \varphi \in \langle \varphi \rangle \ \forall \varphi \in \mathbb{C}[X]^{(B)} \}, \]
\[ S = \{ g \in G : g \varphi = \varphi \ \forall \varphi \in \mathbb{C}[X]^{(B)} \}. \]

Obviously, \( P \) is a parabolic subgroup of \( G \) containing \( B \). Let \( U_P \) be its unipotent radical. Then
\[ U_P \subset S \subset P. \]

Moreover, any character \( \lambda \in \Lambda \) uniquely extends to a character of \( P \) (which we shall denote by the same letter) and
\[ S = \{ p \in P : \lambda(p) = 1 \ \forall \lambda \in \Lambda \} \tag{1} \]

It follows that \( S \) is a normal subgroup of \( P \) and the quotient \( A = P/S \) is a torus, whose character group is generated by \( \Lambda \).

The subgroup \( \Gamma = \Gamma(X, G) \subset \mathfrak{X}(B) \) generated by \( \Lambda \) is called the rank group of \( X \) (with respect to the action of \( G \)). Its rank is called the rank of \( X \) and is denoted by \( r(X) \) or, more precisely, by \( r(X, G) \).

**Proposition 1.** The rank group \( \Gamma \) and the groups \( P \) and \( S \) do not change, if \( X \) is replaced with a \( G \)-invariant open subvariety.

This follows from

**Lemma 1.** [VK78] Let a connected solvable algebraic group \( H \) act on an irreducible algebraic variety \( X \). Then any \( H \)-semi-invariant rational function \( \varphi \) on \( X \) is a quotient of two \( H \)-semi-invariant polynomial functions.

**Proof.** Let \( \varphi = \psi/\chi \), where \( \psi, \chi \in \mathbb{C}[X] \). The linear span of functions \( h\psi, \ h \in H \), is finite-dimensional and, hence, contains an \( H \)-semi-invariant function \( \psi_0 = \sum \epsilon_i c_i h_i \psi \) \((c_i \in \mathbb{C}, \ h_i \in H)\). Set \( \chi_0 = \sum \epsilon_i c_i h_i \chi \). Then \( \varphi = \psi_0/\chi_0 \) and the function \( \chi_0 \) is also \( H \)-semi-invariant \( \Box \)

Let \( L \) be a Levi subgroup (i.e. a maximal reductive subgroup) of \( P \) and \( L_0 = L \cap S \). We have the Levi decompositions:
\[ P = U_P \rtimes L, \quad S = U_P \rtimes L_0. \]

Moreover, \( L_0 \) contains the commutator subgroup of \( L \) and the quotient \( L/L_0 \) is naturally isomorphic to the torus \( A \).

Introduce in the Lie algebra \( \mathfrak{g} \) a \( G \)-invariant scalar product and denote by \( \mathfrak{a} \) the orthogonal subspace of \( L_0 \) in \( \mathfrak{l} \). Then \( \mathfrak{a} \) is a central subalgebra of \( \mathfrak{l} \), which is naturally identified with the tangent algebra of \( A \). Making use of the above scalar product restricted to \( \mathfrak{l} \), one can embed the group \( \mathfrak{X}(P) = \mathfrak{X}(L) \) into \( \mathfrak{l} \). Then \( \mathfrak{a} \) is the linear span of \( A \) and \( \mathfrak{l} \) coincides with the centralizer \( \mathfrak{z(\mathfrak{a})} \) of \( \mathfrak{a} \) in \( \mathfrak{g} \).

2. Local structure theorem

The following theorem is a version of the so-called local structure theorem first proved by M. Brion, D. Luna, T. Vust [BLV86]. (A similar construction was considered by F. Grosshans [Gr85].) The idea of the proof given below belongs to F. Knop [Kn94], [Kn98].

We retain the assumptions and the notation of the previous section.

**Theorem 1.** There is an \( L \)-invariant subvariety \( Y \subset X \) such that
1) the map
\[ P \star L \ Y = U_P \times Y \to X, \quad [p, y] \mapsto py, \]

is a $P$-equivariant isomorphism onto an open subvariety $X_0 \subset X$;

2) the kernel of the action $L : Y$ is $L_0$.

Proof. 1) Any function
\[ \varphi \in \mathbb{C}[X]^{(B)} \cap \mathbb{C}[X]_{\lambda} \quad (\lambda \in \Lambda) \]
defines a $P$-equivariant morphism
\[ F : X_0 \cong \{ x \in X : \varphi(x) \neq 0 \} \to g^* \]
-taking any $x \in X_0$ to the linear function
\[ \xi \mapsto \frac{\xi \cdot \varphi(x)}{\varphi(x)} \quad (\xi \in g), \]

where the dot denotes the Lie derivative. The restriction of $F(x)$ to $p$ is equal to $\lambda$. So, if we identify $g^*$ with $g$ by means of a $G$-invariant scalar product and embed the group $X(P)$ into $I$ as above, then
\[ F(X_0) \subset \lambda + p^* = \lambda + u_P \quad (\lambda \in \mathfrak{a}). \]

Since $\mathfrak{a}(\mathfrak{a}) = I$ and $\mathfrak{a} = \langle \Lambda \rangle$, we can choose $\lambda \in \Lambda$ in such a way that $ad(\lambda)$ be non-degenerate on $u_P$. Then $\lambda + u_P$ is just one $P$-orbit. The stabilizers of its points are maximal reductive subgroups of $P$; in particular, $P_\lambda = L$. Set $Y = F^{-1}(\lambda)$. Then the map (2) is a $P$-equivariant isomorphism onto $X_0$.

2) The kernel of the action $L : Y$ is the intersection of the kernels of all irreducible linear representations of $L$ occurring in $\mathbb{C}[Y]$.

Note that $B \cap L$ is a Borel subgroup of $L$ and $B = U_P \times (B \cap L)$. The character groups of $B$ and $B \cap L$ are naturally identified.

The restriction to $Y$ of any $B$-semi-invariant polynomial function on $X$ is a $(B \cap L)$-semi-invariant polynomial function with the same weight. Conversely, identifying $X_0$ with $U_P \times Y$, one can uniquely extend any $(B \cap L)$-semi-invariant polynomial function $\psi$ on $Y$ to a $B$-semi-invariant polynomial function $\varphi$ on $X_0$ by the formula $\varphi(u, y) = \psi(y)$. It follows that
\[ \Gamma(Y, L) = \Gamma(X, G) = \Gamma. \]

Since all the characters in $\Gamma$ vanish on the commutator subgroup of $L$, we obtain that $\mathbb{C}[Y]$ decomposes into a sum of one-dimensional representations of $L$ whose weights generate $\Gamma$. The intersection of their kernels is just $L_0$. \qed

Clearly, the variety $Y$ is quasias affine, irreducible, and smooth. The action of $L$ on it reduces to an action of the torus $A$.

Let $\varphi_1, \ldots, \varphi_r \in \mathbb{C}[Y]$ be $A$-semi-invariant functions whose weights $\lambda_1, \ldots, \lambda_r$ constitute a basis of the group $X(A) = \Gamma$. Set
\[ C = \{ y \in Y : \varphi_i(y) = 1 \text{ for } i = 1, \ldots, r \}. \]

Then the map
\[ A \times C \to Y, \quad (a, c) \mapsto ac, \]
is an $A$-equivariant isomorphism onto the open subvariety
\[ Y_0 = \{ y \in Y : \varphi_i(y) \neq 0 \text{ for } i = 1, \ldots, r \} \subset Y. \]
(It is meant that \( A \) acts on \( A \times C \) by shifts of the first factor.)

Replacing \( Y \) with \( Y_0 \), we shall assume that the map (3) is an isomorphism onto the whole of \( Y \). In particular, this will imply that \( A \) acts on \( Y \) freely.

3. The variety of horospheres

Let us first study the \( U \)-orbits lying in \( X_0 \). Obviously, they are parametrized by the points of \( Y \).

**Theorem 2.** [Kn93], [Kn94] The stabilizer of any \( U \)-orbit lying in \( X_0 \) coincides with \( S \). Moreover, if \( gU \cdot y_1 = U \cdot y_2 \) (\( y_1, y_2 \in Y \)), then \( g \in P \).

**Proof.** For any \( y \in Y \),

\[
S_y = U \cdot y = U_P \cdot y.
\]

In particular, the stabilizer of \( U \cdot y \) contains \( S \). Denote it by \( \Tilde{S} \).

Let \( \Tilde{U}_P \) be the unipotent radical of \( \Tilde{S} \). Clearly, \( \Tilde{U}_P \subset U \subset S \) and, hence, \( \Tilde{U}_P \subset U_P \). We have

\[
\Tilde{S} = \Tilde{S}_y \cdot U_P, \quad \Tilde{S}_y \cap U_P = \{ e \}
\]

whence the variety \( \Tilde{S}/U_P \cong \Tilde{S}_y \) is affine. Since \( \Tilde{S}/U_P = (\Tilde{S}/U_P)/(U_P/U_P) \) and the group \( \Tilde{S}/U_P \) is reductive, it follows that \( \Tilde{U}_P = U_P \). But then \( \Tilde{S} \subset N(U_P) = P \) and, since the torus \( A = P/S \) acts on \( Y \) freely, \( \Tilde{S} = S \).

Let now \( gU \cdot y_1 = U \cdot y_2 \) (\( y_1, y_2 \in Y \)). Then the stabilizers of \( U \cdot y_1 \) and \( U \cdot y_2 \) are conjugate by means of \( g \), whence \( g \in N(S) = P \). \( \square \)

In an analogous way, one can prove that the stabilizer of any \( B \)-orbit lying in \( X_0 \) coincides with \( P \) and, moreover, if \( gB \cdot y_1 = B \cdot y_2 \) (\( y_1, y_2 \in Y \)), then \( g \in P \).

Obviously, the \( B \)-orbits lying in \( X_0 \) are parametrized by the points of \( C \). Recall that the codimension of a generic \( B \)-orbit is called the complexity of \( Y \) (with respect to the action of \( G \)) and is denoted by \( c(Y) \) or, more precisely, by \( c(X, G) \). It follows from the above that

\[
(4) \quad \dim C = c(X), \quad \dim Y = c(X) + r(X).
\]

Denote by \( \text{Hor} X \) the set of horospheres \( G \)-equivalent to \( U \)-orbits lying in \( X_0 \). Due to Theorem 2, the map

\[
(5) \quad G \ast \text{Hor} Y = G/S \times C \rightarrow \text{Hor} X, \quad [g, y] \mapsto gU \cdot y,
\]

is bijective. (It is meant here that the action \( P : Y \) is defined by the action \( L : Y \) by means of the natural homomorphism \( P \rightarrow L \cong P/U_P \).) Thereby \( \text{Hor} X \) is supplied with a structure of an algebraic variety.

Clearly, the variety \( \text{Hor} X \) is irreducible and smooth. Moreover, it is quasiaffine, as follows from

**Proposition 2.** The variety \( G/S \) is quasiaffine.

**Proof.** It follows from the definition of \( S \) that the group \( \mathfrak{X}(A) = \Gamma \) is generated by dominant weights. Let \( \lambda_1, \ldots, \lambda_r \) be some dominant weights generating \( \Gamma \). Then \( G/S \) is the orbit of the sum of highest weight vectors of irreducible representations with highest weights \( \lambda_1, \ldots, \lambda_r \) [VP72]. \( \square \)

To calculate the dimension of \( \text{Hor} X \), let us note that \( \dim G/P = \dim U_P \). Therefore,

\[
\dim \text{Hor} X = \dim G/P + \dim Y = \dim U_P + \dim Y = \dim X.
\]
Remark 1. The action $G$: Hor $X$ can be considered as a contraction of the action $G$: $X$ twisted by means of a Weyl involution of $G$ [Kn60]. (A Weyl involution is an involutory automorphism acting as inversion on a maximal torus of $G$.)

4. The horospherical cotangent bundle

Let $\pi$ be the canonical projection of $T^*X$ to $X$. Consider the variety

$$HT^*X = \{(a, H) \in T^*X \times \text{Hor } X : \pi(a) \in H \text{ and } a = 0 \text{ on } T_\pi(a)H\}.$$ 

Denote by $p$ and $q$ the projections of $HT^*X$ to $T^*X$ and Hor $X$, respectively.

With respect to $q$, $HT^*X$ is a fibering over Hor $X$, whose fibers are the conormal bundles of the corresponding horospheres.

The morphism (2) naturally extends to a $P$-equivariant morphism

$$U_P \times T^*Y \to T^*X,$$

whose image is the (disjoint) union of the conormal bundles of $U$-orbits (or, equivalently, $U_P$-orbits) lying in $X_\pi$. The morphism (6) gives rise to a $G$-equivariant map

$$G \ast_P (U_P \times T^*Y) \to HT^*X,$$

which is bijective by Theorem 2. This bijection provides $HT^*X$ with a structure of a smooth irreducible (quasiaffine) algebraic variety. We shall call this variety the horospherical cotangent bundle of $X$.

The projections $p$ and $q$ are induced by the morphism (6) and the canonical projection of $U_P \times T^*Y$ to $Y$, respectively, and thereby they are morphisms of algebraic varieties.

Considering $HT^*X$ as a fibering over Hor $X$, we see that

$$\dim HT^*X = \dim \text{Hor } X + \dim X = \dim T^*X.$$

Theorem 3. [Kn94] The morphism

$$p: HT^*X \to T^*X$$

is dominant.

Since $\dim HT^*X = \dim T^*X$, this means that $p$ is a “rational covering”, that is the field $\mathbb{C}(HT^*X)$ is a finite extension of $\mathbb{C}(T^*X)$.

Proof. Let us denote by $Z$ the image of the morphism (6). Let $U^-_P$ be the unipotent radical of the parabolic subgroup, opposite to $P$ and containing $L$. Note that

$$\text{codim}_{T^*X} Z = \dim U_P = \dim U^-_P.$$

Since $Z$ is $P$-invariant and $u^-_P$ is a complementary subspace of $p$ in $\mathfrak{g}$, the assertion of theorem is equivalent to existence of a point $z \in Z$ such that

$$\{\eta \in u^-_P : \eta z \in T_z Z\} = 0$$

(and, hence, $T_z T^*X = T_z Z \oplus u^-_P z$).

Consider the moment map

$$\Phi: T^*X \to \mathfrak{g}^*, \quad a \mapsto (\xi \mapsto a(\xi \pi(a))).$$

Since the subgroup $S$ preserves each $U$-orbit lying in $X_\pi$, $\Phi(Z)$ lies in the annihilator of $\mathfrak{z}$. If we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of a $G$-invariant scalar product, then

$$\Phi(Z) \subset \mathfrak{z}^\perp = \mathfrak{a} + u_P.$$
Moreover, considering the action of \( P \) on \( Z \), one can see that \( \Phi(Z) \) projects onto \( \mathfrak{a} \). Since the map \( \Phi \) is \( G \)-equivariant, the set \( \Phi(Z) \) is \( P \)-invariant. We know that
\[
\mathfrak{a}^{\text{reg}} \triangleq \{ \lambda \in \mathfrak{a} : [\lambda, u_P] = u_P \} \neq \emptyset
\]
and, for any \( \lambda \in \mathfrak{a}^{\text{reg}} \), the plane \( \lambda + u_P \) is one \( P \)-orbit. It follows that
\[
\Phi(Z) \supset \mathfrak{a}^{\text{reg}} + u_P.
\]
Let \( z \in Z \) be such that \( \Phi(z) = \lambda \in \mathfrak{a}^{\text{reg}} \). Suppose that \( \eta z \in T_z(z) \) for some \( \eta \in u_P \). Then
\[
\Phi(\eta z) = [\eta, \lambda] \in \mathfrak{a} + u_P
\]
whence \( [\eta, \lambda] = 0 \) (since \( [\eta, \lambda] \in u_P \)). But \( \text{ad}(\lambda) \) is non-degenerate on \( u_P \) and, hence, it is also non-degenerate on \( u_P \). It follows that \( \eta = 0 \), q.e.d.

Due to Theorem 3, the morphism \( p \) permits to lift the canonical symplectic structure of \( T^*X \) to an open subvariety \( \Omega \) of \( HT^*X \). Since the canonical bundle of any smooth subvariety of \( X \) is a Lagrangian subvariety of \( T^*X \), the intersections of fibers of \( q \) with \( \Omega \) constitute a Lagrangian fibering of \( \Omega \) over \( \text{Hor} X \). In this respect, \( HT^*X \) is similar to \( T^* \text{Hor} X \). We shall see that, indeed, in some cases \( HT^*X \) is birationally isomorphic to \( T^* \text{Hor} X \) as a symplectic \( G \)-variety.

5. Comparison of \( HT^*X \) and \( T^* \text{Hor} X \)

Let \( \rho \) be the canonical projection of \( T^* \text{Hor} X \) to \( \text{Hor} X \). In this section, we shall construct a \( G \)-equivariant birational morphism
\[
(\text{\textbf{9}}) \quad f: HT^*X \rightarrow T^* \text{Hor} X
\]
such that the diagram
\[
\begin{array}{ccc}
HT^*X & \xrightarrow{f} & T^* \text{Hor} X \\
\downarrow q & & \downarrow \rho \\
\text{Hor} X & & \text{Hor} X
\end{array}
\]
is commutative.

Due to isomorphism (3), we have the following natural identifications:
\[
(\text{\textbf{11}}) \quad G \ast_P (U_P \times T^*Y) = G \ast_P (U_P \times T^*A \times T^*C) \\
= G \ast_S (U_P \times T^*A \times T^*C) = G \ast_S (U_P \times \mathfrak{a}) \times T^*C.
\]
Here the action \( S : (U_P \times \mathfrak{a}) \) reduces to the action \( S : U_P \) defined as follows: \( U_P \) acts by left shifts, while \( L_\alpha \) acts by conjugations.

On the other hand, we have the following natural identifications:
\[
(\text{\textbf{12}}) \quad T^*(G/S \times C) = T^*(G/S) \times T^*C \\
= G \ast_S (g/S)^* \times T^*C = G \ast_S (\mathfrak{a} + u_P) \times T^*C.
\]
Here \( S \) acts on \( \mathfrak{a} + u_P \) via the adjoint representation of \( G \).

Consider the morphism
\[
\varphi_0 : U_P \times \mathfrak{a} \rightarrow \mathfrak{a} + u_P, \quad (u, \lambda) \mapsto \text{Ad}(u)\lambda.
\]
Since $[\mathfrak{u} P, \mathfrak{a}] = \mathfrak{u} P$ for a generic $\lambda \in \mathfrak{a}$, the morphism $\varphi_0$ is birational. Moreover, it is easy to see that it is $S$-equivariant. Hence it induces a $G$-equivariant birational morphism

$$
\varphi: G \times S (U_P \times \mathfrak{a}) \times T^* C \to G \times S (\mathfrak{a} + u_P) \times T^* C
$$

by the formula

$$
\varphi([g, z], \gamma) = ([g, \varphi_0(z)], \gamma) \quad (z \in U_P \times \mathfrak{a}, \gamma \in T^* C).
$$

Due to (11), (12) and isomorphisms (7), (5) this gives rise to a $G$-equivariant birational morphism

$$
f: H T^* X \to T^* \text{Hor} X.
$$

In the same sense, the projections $q$ and $\rho$ are induced by the trivial maps of $U_P \times \mathfrak{a}$ and $\mathfrak{a} + u_P$ to a one-point set (and by the canonical projection of $T^* C$ to $C$). Hence, the diagram (10) is commutative.

To find out, whether or not $f$ is symplectic, let us prove the following general lemma concerning invariant symplectic structures on homogeneous fiberings.

Let $(M, \omega)$ be a (smooth irreducible) symplectic variety. Denote by $\mathcal{P}(M)$ the Lie algebra of functions on $M$ with respect to the Poisson bracket defined by $\omega$. For any function $\varphi$, define, as usually, its skew gradient $\text{grad} \varphi$ by

$$
\omega(\text{grad} \varphi, \eta) = \partial_\eta \varphi \quad \forall \eta \in TM.
$$

A symplectic action $G: M$ is called Poissonian, if there is a homomorphism

$$
g \to \mathcal{P}(M), \quad \xi \mapsto H_\xi,
$$

such that the velocity vector field on $M$ corresponding to any $\xi \in g$, coincides with $\text{grad} H_\xi$. The function $H_\xi$ is called the Hamiltonian of $\xi$.

Let now $M = G \times S F$, were $S \subset G$ is an algebraic subgroup and $F$ is a (smooth irreducible) $S$-variety.

**Lemma 2.** If the action $G: M$ is Poissonian, then the symplectic structure $\omega$ on $M$ is uniquely defined by the following data:

1) the restriction of $\omega$ to $F$;
2) the restriction of functions $H_\xi, \xi \in g$, to $F$.

(We identify $F$ with the fiber $[e, F]$ of $M$.)

**Proof.** It suffices to reconstruct the values of $\omega$ at points of $F$. Note that, for any $p \in F$, the vector space $T_p M$ is generated by $T_p F$ and the velocities of the action $G: M$. If the restriction of $\omega$ to $T_p F$ is known, it suffices to determine the values $\omega(\xi p, \eta)$ for $\xi \in g$ and $\eta \in T_p M$.

Due to the formula

$$
g H_\xi = H_{\lambda \delta_0 g \xi} \quad (g \in G, \xi \in g),
$$

the restrictions of functions $H_\xi$ to $F$ completely define these functions. Then the values $\omega(\xi p, \eta)$ can be found by

$$
\omega(\xi p, \eta) = \partial_\eta H_\xi.
$$

\[\square\]

For any action $G: Z$, the action $G: T^* Z$ is Poissonian with

$$
H_\xi(\gamma) = \gamma(\xi \tau(\gamma)),
$$

(14)
where $\tau: T^*Z \to Z$ is the canonical projection. Obviously, a covering of any Poissonian action is also Poissonian. Hence, considering $HT^*X$ and $T^* \text{Hor} X$ as homogeneous fiberings over $G/S$, we can apply Lemma 2 to determine if the morphism $f$ is symplectic.

Let us assume for simplicity that the action $G: X$ is spherical (i.e. the Borel subgroup $B$ has an open orbit). Then $X_0$ coincides with this orbit, and $C$ reduces to one point $o$ (whose $L$-orbit is $Y$), so we have the following commutative diagram:

$$
\begin{array}{ccc}
G \ast_S (U_P \times a) & \xrightarrow{\phi} & G \ast_S (a + u_P) \\
\downarrow s & & \downarrow s \\
G/S & \xrightarrow{\varepsilon} & G/S \\
\downarrow f & & \downarrow f \\
HT^*X & \xrightarrow{q} & T^* \text{Hor} X \\
\downarrow \rho & & \downarrow \rho \\
\text{Hor} X & & \text{Hor} X \\
\end{array}
$$

where the vertical arrows are the isomorphisms described above. Since the fibers of $q$ and $\rho$ are Lagrangian, we need only to compare the Hamiltonians $\overline{H}_\xi$ and $\overline{H}_\xi$ of the actions $G: HT^*X$ and $G: T^* \text{Hor} X$.

A generic point of $G \ast_S (U_P \times a)$ is $G$-equivalent to a point of the subvariety $\{e\} \times a \subset U \times a$, while a generic point of $G \ast_S (a + u_P)$ is $G$-equivalent to a point of the subvariety $a \subset a + u_P$. By definition, $\phi((e, \lambda)) = \lambda$ for $\lambda \in a$. It follows that the morphism $f$ is symplectic if and only if

$$
\overline{H}_\xi(\delta(e, \lambda)) = \overline{H}_\xi(\varepsilon(\lambda))
$$

for any $\xi \in \mathfrak{g}$ and $\lambda \in a$.

According to (14),

$$
\overline{H}_\xi(\varepsilon(\lambda)) = (\lambda, \xi).
$$

On the other hand, $\delta(\{e\} \times a)$ is naturally identified with $N^*_e(U_o) \subset T^*X$ and, hence,

$$
H_\xi(\delta(e, \lambda)) = (\lambda, \dot{\xi}),
$$

where $\dot{\xi}$ is defined by

$$
\dot{\xi} = \eta \circ \xi = \hat{\xi} \circ \xi (\eta \in u_P, \xi \in a).
$$

For $\xi \in a$, we have $(\lambda, \dot{\xi}) = 0$ and $\dot{\xi} = 0$, so (16) holds. For $\xi \in a$, we have $\dot{\xi} = \xi$, so (16) holds as well. For $\xi \in u_P$, we have $(\lambda, \dot{\xi}) = 0$, so (16) holds for any $\lambda \in a$ if and only if $\xi = 0$. Thus, we obtain

**Proposition 3.** Let the action $G: X$ be spherical. Then the morphism $f$ is symplectic if and only if $u_P \subset u_P$.

**Remark 2.** In the general case, the morphism $f$ is symplectic if and only if $u_P \subset u_P$ for any $c \in C$. 
6. The case of a symmetric space

Let \( X = G/K \) be a symmetric space, that is \( k = \mathfrak{g}^\sigma \) for some involutory automorphism \( \sigma \) of \( G \). It is known [Vu74] (and we shall see it later) that in this case the action \( G \cdot X \) is spherical, so we can apply Proposition 3.

Our main theorem is

**Theorem 4.** For a symmetric space \( X = G/K \), one can choose the subvariety \( Y \) so that the \( G \)-equivariant birational morphism \( f: HT^*X \to T^* \text{Hor} \ X \) defined in Section 5, is symplectic.

*Proof.* Let \( \mathfrak{m} \) be the orthogonal subspace of \( \mathfrak{k} \) in \( \mathfrak{g} \), and \( \mathfrak{a} \) a maximal diagonalizable subalgebra in \( \mathfrak{m} \). Let \( \mathcal{L} = \mathcal{Z}(\mathfrak{a}) \) be the centralizer of \( \mathfrak{a} \) in \( G \) and \( \mathcal{L}_0 = \mathcal{L} \cap K \). Then \( \mathcal{L} = \mathcal{L}_0 + \mathfrak{a} \). Suppose that the Borel subgroup \( B \) is chosen so that \( \mathfrak{b} \supseteq \mathfrak{a} \).

Let us first assume that \( G \) is semisimple and \( X \) is simply connected. Then it is known (see, e.g., [Hel84]) that an irreducible representation of \( G \) occurs in \( \mathbb{C}[X] \) if and only if its highest weight vector is fixed by \( \mathcal{L}_0 \). It follows that \( P = B\mathcal{L}_0 \supseteq \mathcal{L} \), so we can choose \( \mathcal{L} \) as \( L \). Then \( \mathcal{L}_0 = \mathcal{L}_0 \) and \( a = \mathfrak{a} \).

Let \( \Delta \) be the root system of \( X \) with respect to \( a \) and, for any \( \alpha \in \Delta \), let \( \mathfrak{g}_\alpha \) denote the corresponding root subspace. Then

\[
\mathfrak{p} = \mathfrak{k} + \sum_{\alpha \neq 0} \mathfrak{g}_\alpha, \quad \mathfrak{u}_P = \sum_{\alpha \neq 0} \mathfrak{g}_\alpha, \quad \mathfrak{u}_P^+ = \sum_{\alpha < 0} \mathfrak{g}_\alpha
\]

(with respect to the ordering of \( \Delta \) defined by \( B \)).

Since \( \sigma(a) = -a \) for any \( \alpha \in \Delta \), then

\[
\sigma(\mathfrak{u}_P) = \mathfrak{u}_P^+.
\]

It follows that

\[
\mathfrak{k} + \mathfrak{b} = \mathfrak{k} + \mathfrak{a} + \mathfrak{u}_P = \mathfrak{g}.
\]

Hence, the point \( \hat{\mathfrak{g}} = eK \) lies in an open orbit of \( B \). Since it is fixed by \( \mathcal{L}_0 \), we can take it for \( o \). Then (20) implies that \( \mathfrak{u}_P^+ o = \mathfrak{u}_P o \), so the morphism \( f \) is symplectic by Proposition 3.

If \( X = G \) is a torus, then \( \text{Hor} \ X = X \), \( HT^*X = T^*X = T^* \text{Hor} \ X \), and the theorem is trivial. Together with the case considered above, this proves the theorem in the case when \( X \) is a direct product of a simply connected symmetric space and a torus.

A symmetric space of the latter type can be represented in the form \( X = G/K \), where \( G \) is a direct product of a torus \( Z \) and a simply connected semisimple group \( H \), and \( K \) is a connected subgroup of \( H \) (coinciding with \( H^\sigma \)). (The involution \( \sigma \) acts on \( Z \) as inversion.)

Any symmetric space can be represented in the form

\[
X' = G/K' = X/F,
\]

where \( X = G/K \) is a symmetric space of the above type, \( K' \subset N(K) \) is a finite extension of \( K \), and \( F = K'/K \) is a finite subgroup of \( N(K)/K \) acting on \( X = G/K \) by right shifts.

The group \( F \) naturally \( G \)-equivariantly acts on \( \text{Hor} \ X = X/S \). Since the group of \( U \)-automorphisms of a \( U \)-orbit is unipotent, it does not contain non-trivial finite subgroups. Hence, the action \( F : \text{Hor} \ X \) is effective. This gives rise to a natural isomorphism \( F \cong S'/S \), where \( S' \subset N(S) \) is a finite extension of \( S \) playing the role of \( S \) for \( X' \).
All the morphisms of the diagram (15) are $F$-equivariant. Factorizing this diagram by $F$, we obtain the corresponding diagram for the space $X'$. It follows that the corresponding morphism $f'$ is also symplectic. □

Remark 3. As it follows from the above proof, it is true for any symmetric space $X$, that an irreducible representation of $G$ occurs in $\mathbb{C}[X]$ if and only if highest weight vector is fixed by $L_0$ (or, equivalently, by $S$).

Remark 4. For many (but not for all) symmetric spaces $X$, the subvariety $Y$ satisfying the conditions of Theorem 1, is unique.

Remark 5. Theorem 5 is not extended to any spherical varieties, as the example $X = (\text{SL}_2 \times \text{SL}_2 \times \text{SL}_2)/\text{SL}_2$ shows.

References


