Generators and Relations of the
Affine Coordinate Rings of
Connected Semisimple Algebraic Groups

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Vienna, Preprint ESI 972 (2000)  
December 15, 2000

Supported by Federal Ministry of Science and Transport, Austria
Available via http://www.esi.ac.at
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Abstract. Let $G$ be a connected semisimple algebraic group over an algebraically closed field of characteristic zero. A canonical presentation by generators and relations of the algebra of regular functions on $G$ is found. The conjectures of D.Flath and J.Towber, [FT], are proved.

(1) Let $G$ be a connected semisimple algebraic group over an algebraically closed field $k$ of characteristic zero. We obtain here a canonical presentation of the algebra $k[G]$ of regular functions on $G$ (i.e., the affine coordinate ring of $G$) by means of generators and relations. In the simplest case when $G = SL_2$ one has the isomorphism

$$
\mu : k[A, B, C, D]/(AD - BC - 1) \rightarrow k[G],
$$

where

$$
\mu(A)\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = a, \ldots,
$$

and our presentation is obtained from (1.1) by rewriting it in the form

$$
k[G] \simeq (S \otimes_k S^-)/I,
$$

where $S = k[\mu(A), \mu(C)] \simeq k[A, C], S^- = k[\mu(B), \mu(D)] \simeq k[B, D]$ and $I = (\mu(A) \otimes \mu(D) - \mu(B) \otimes \mu(C) - 1)$. The meaning of (1.2) is that $S$ is the algebra of invariants of

1991 Mathematics Subject Classification. 20G65.

Key words and phrases. Semisimple algebraic group, maximal torus, Borel subgroup, weight, Weyl module, orbit, invertible sheaf.

Research partly supported by Grant # MQZ000 from the International Science Foundation, by the Max-Planck-Institut für Mathematik, and The Erwin Schrödinger International Institute for Mathematical Physics (Vienna, Austria).

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the maximal unipotent subgroup of $\text{SL}_2$ consisting of all upper triangular matrices and acting on $G$ by right translation, and $S^-$ is the algebra of invariants of the opposite maximal unipotent subgroup. We show that there is an analogue of (1.2) for any connected semisimple algebraic group $G$. The algebras $S$ and $S^-$ and the ideal $I$ case can be described quite explicitly. This gives the canonical presentation of the algebra $k[G]$ by generators and relations. The canonical generators of the ideal $I$ are of degree 2 and not homogeneous. If the monoid of dominant weights of $G$ is freely generated, the algebras $S$ and $S^-$ are quadratic. Thus, for instance, if $G$ is simply connected, we obtain an explicit canonical presentation of the algebra $k[G]$ by generators and relations, in which all relations have degree 2. This gives a canonical presentation of the group variety $G$ as an intersection of quadrics in an affine space (as a parallel, recall that an abelian variety can be canonically presented as an intersection of quadrics in a projective space given by the Riemann equations, [K], [LB]). In a nonsimply connected case one has to combine this presentation for the simply connected covering of $G$ with the corresponding action of the fundamental group of $G$ which is easy to control.

For $G = \text{SL}_n, \text{GL}_n, \text{SO}_n$ and $\text{Sp}_n$ the analogue of (1.2) was obtained by D.Flath and J.Towber in [FT] by means of the direct bulky computations of the corresponding Laplace decompositions, minors and the algebraic identities between them. They formulated two general conjectures which we prove here, cf. Section (4). The problem of describing the ideal $I$ in general case was formulated in [F].

The algebras of regular functions on the linear algebraic groups are the basic objects of the theory of quantum groups. I believe that the canonical character of our presentation of $k[G]$ is a true indication that there exists a nice general procedure of quantization of $k[G]$ based on this presentation. I also think that the results similar to the ones proven here can be obtained for the infinite dimensional groups as well, cf. [KP]. Finally, I would like to mention that our results have the applications in the theory of coherent tensor operators, cf. [F].

In this preprint, the proofs (and even some formulations, see Remark in Section (13)) are heavily dependent on the assumption that the ground field has characteristic zero. It is remarkable however that all main results remain true without this assumption. In particular, this gives an approach to quantization of $k[G]$ in positive characteristic, the case in which only a little is known at the moment. Of course the characteristic free proofs (and some formulations) become more complicated. This material will appear in a forthcoming final version of this paper.

(2) We fix the following notation:

- $B$ a Borel subgroup of $G$;
- $T$ a maximal torus of $B$;
- $U$ the unipotent radical of $B$;
- $\mathcal{X}(H)$ the group of characters of an algebraic group $H$ written additively;
- $t^\lambda$ the value of $\lambda \in \mathcal{X}$ at $t \in T$;
- $\mathcal{P}_{++} \subseteq \mathcal{X}$ the monoid of the highest weights of the simple $G$-modules;
\( R(\lambda) \) a simple \( G \)-module with the highest weight \( \lambda \in P_{++} \);
\( \lambda^* \) the highest weight of \( R(\lambda)^* \);
\( U^- \) the maximal unipotent subgroup of \( G \) opposite to \( U \);
\( w_0 \) an element of the normalizer of \( T \) such that \( w_0 U w_0^{-1} = U^- \);
\( \mathfrak{g} \) the Lie algebra of \( G \);
\( U(\mathfrak{g}) \) the universal enveloping algebra of \( \mathfrak{g} \);
\( \mathfrak{g}_\alpha \) the root line in \( \mathfrak{g} \) corresponding to a root \( \alpha \);
\( \mathfrak{t} \) the Lie algebra of \( T \);
\( V_\lambda \) the weight space of a \( T \)-module \( V \) corresponding to a weight \( \lambda \);
\( M^H \) the fixed point set of an action of a group \( H \) on a set \( M \);
\( H \cdot x \) the \( H \)-orbit of a point \( x \);
\( H \cdot X := \{ H \cdot x \mid x \in X \} \);
\( H_x \) the \( H \)-stabilizer of a point \( x \);
\( k[X] \) the algebra of regular functions of an algebraic variety \( X \);
\( k(\lambda) \) the field of of rational functions of an algebraic variety \( X \).

(3) We consider \( k[G] \) as the \( G \)-module with respect to the action defined by the left translations. We set

\[
S = \{ f \in k[G] \mid f(gu) = f(g), \ g \in G, u \in U \}
\]

and, for any \( \lambda \in P_{++} \),

\[
S_\lambda = \{ f \in S \mid f(gb) = b^\lambda f(g), \ g \in G, b \in B \}.
\]

It is well known that \( S_\lambda \) is a simple submodule of \( k[G] \) with the highest weight \( \lambda^* \) with respect to \( B \) and that the submodules \( S_\lambda \) give a \( P_{++} \)-grading of \( S \):

\[
S = \bigoplus_{\lambda \in P_{++}} S_\lambda, \quad S_\lambda \cdot S_\gamma = S_{\lambda + \gamma},
\]

Algebra

\[
S^- = \{ f \in k[G] \mid f(du) = f(g), \ g \in G, u \in U^- \}
\]

is obtained from \( S \) by the right translation by \( w_0 \). It is a submodule of \( k[G] \). If \( S^-_\lambda \) is the right translation of \( S_\lambda \) by \( w_0 \), then \( S^-_\lambda \) is a simple submodule of \( k[G] \) with the highest weight \( \lambda^* \) with respect to \( B \) and

\[
S^-_\lambda = \{ f \in S^- \mid f(gt) = t^{w_0 \lambda} f(g), \ g \in G, t \in T \}.
\]

The submodules \( S^-_\lambda \) give a \( P_{++} \)-grading of \( S^- \):

\[
S^- = \bigoplus_{\lambda \in P_{++}} S^-_\lambda, \quad S^-_\lambda \cdot S^-_\gamma = S^-_{\lambda + \gamma}.
\]

(4) Consider the homomorphism of algebras

\[
\mu : S \otimes_k S^-, \ f \otimes h \mapsto fh.
\]

The conjectures of D.Flath and J.Towber, [FT], are formulated as follows:
(Sur) \( \mu \) is surjective;
(Ker) \( \ker \mu \) is generated by \( (\ker \mu)^G \).

We shall prove (Sur) in Section (6) and (Ker) in Section (7). In Section (12) we describe the algebra structure of \( (\ker \mu)^G \) and find the invariant generators of \( \ker \mu \). Combining this with the presentations of \( S \) and \( S^- \) by generators and relations described in Section (13), we obtain a presentation of \( k[G] \) by generators and relations for any connected semisimple algebraic group \( G \) such that the monoid \( P_++ \) is freely generated. In particular this gives a presentation of \( k[G] \) for any simply connected semisimple algebraic group \( G \). If \( G \) is not simply connected semisimple group and \( \pi : \tilde{G} \to G \) is its universal covering, then \( k[\tilde{G}] \) is obtained from \( k[G] \) by taking the invariants of the action of the finite central subgroup \( \ker \pi \) of \( G \) on \( k[G] \). The structure of this action is quite explicit with respect to our presentation of \( k[\tilde{G}] \) and, if necessary, can be used in the concrete cases for obtaining a presentation of \( k[G] \) as well.

(5) Our proof of conjecture (Sur) is based on two general statements. The first one is well known, cf. [St,1.5].

**Theorem 1.** Let \( \phi : X \to Y \) be a morphism of affine algebraic varieties. Then the corormorphism \( \phi^* : k[Y] \to k[X] \) is surjective if and only if \( \phi \) is a closed embedding.

The second statement gives a criterion of closedness of certain orbits (as a matter of fact for the proof of conjecture (Sur) we need only the implication (ii) \( \Rightarrow \) (i)).

**Theorem 2.** Let \( V \) be a direct sum of finitely dimensional \( G \)-modules \( V_1, \ldots , V_s \) and \( v_i \in V_i \) a nonzero weight vector of a weight \( \lambda_i \in \mathfrak{X} \). Then the following properties of the vector \( v = v_1 + \ldots + v_s \in V \) are equivalent:

(i) \( G \cdot v \) is closed in \( V \);
(ii) 0 is an interior point of the convex envelope \( \text{conv} \{ \lambda_1 , \ldots , \lambda_s \} \) of the weights \( \lambda_1 , \ldots , \lambda_s \) in \( \mathfrak{X} \otimes \mathbb{Z} \mathbb{Q} \);
(iii) \( T \cdot v \) is closed in \( V \).

**Proof.** (ii) \( \Leftrightarrow \) (iii). This is well known, cf. [PV, 6.11] or [P1].

(iii) \( \Rightarrow \) (i). We can assume, according to the Lefschetz principle, that \( k = \mathbb{C} \), cf. for instance [Si,VI,§6]. Assume that \( T \cdot v \) is closed in \( V \). Let \( K \) be a maximal connected compact subgroup of \( G \). Fix in \( V \) a \( K \)-invariant Hermitian inner product \( \langle , \rangle \), such that \( V_i \) and \( V_j \) are orthogonal if \( i \neq j \). Since \( T \cdot v \) is closed, the function \( v \mapsto \langle v, v \rangle \) attains its minimum on \( T \cdot v \). Since \( t \cdot v_i \), like \( v_i \), is also a nonzero weight vector of the weight \( \lambda_i \), changing the notation we can assume that this minimum is attained at \( v \). Therefore,

\[
(t \cdot v, v) = 0.
\] (5.1)

Since \( g_\alpha (V_i)_{\lambda_i} \subseteq (V_i)_{\alpha + \lambda_i} \) for any root \( \alpha \), it follows from orthogonality of the weight spaces of different weights and pairwise orthogonality of the subspaces \( V_i \) that

\[
(g_\alpha \cdot v, v) = 0.
\] (5.2)
Since \( g = t \oplus (\oplus_{\alpha} g_{\alpha}) \), it follows from (5.1) and (5.2) that

\[
(g \cdot v, v) = 0. \tag{5.3}
\]

Now closedness of \( G \cdot v \) in \( V \) follows from (5.3) and the Kempf–Ness criterion, [KN] (cf. also [PV, 6.12]).

(i) \( \Rightarrow \) (ii). Assume that \( G \cdot v \) is closed in \( V \). If (ii) is not satisfied, there is a linear function \( l \in (\mathfrak{X} \otimes_{\mathbb{Z}} \mathbb{Q})^* \) such that all the numbers \( n_i := l(\lambda_i) \) are nonnegative and at least one of them, say \( n_1 \), is positive. Multiplying \( l \) by an appropriate positive integer one can assume that \( l(\chi) \in \mathbb{Z} \) for any \( \chi \in \mathfrak{X} \). Hence there is a homomorphism \( \mu : k^* \to T \) such that \( \mu(t) \cdot v = t^{n_1}v_1 + \ldots + t^{n_s}v_s \). Therefore the closure of \( T \cdot v \) in \( V \) contains the vector \( u := u_1 + \ldots + u_s \), where \( u_i = v_i \) if \( n_i = 0 \) and \( u_i = 0 \) if \( n_i > 0 \). Since \( G \cdot v \) is closed, this closure is contained in \( G \cdot v \). On the other hand, \( u \notin G \cdot v \), since \( u_1 = 0 \), a contradiction. \( \Box \)

**Remark.** The proof given here heavily depends on the fact that the ground field has characteristic zero. In fact Theorem 2 is valid over ground field of arbitrary characteristic. The characteristic free proof of it is given in our article [P2]. We will use this fact in a forthcoming publication whether the main results of this preprint will be proved over ground field of arbitrary characteristic.

(6) Now we can give a proof of conjecture (Sur).

**Theorem 3.** Homomorphism (4.1) is surjective.

**Proof.** Consider the action of \( G \) on itself by the left translations. Since \( k \)-algebra \( S \) is finitely generated, there is an irreducible affine algebraic variety \( X \) endowed with a regular action of \( G \) and a dominant equivariant morphism \( \alpha : G \to X \), such that \( \alpha^*(k[X]) = S \). Let \( x = \alpha(e) \). The orbit \( G \cdot x \) is open and dense in \( X \). Let \( \pi : G \to G/U \) be the canonical morphism. Then \( \pi^*(k[G/U]) = S \), cf. (3.1) and [PV, 3.1]. Hence, there is an equivariant morphism \( \iota : G/U \to X \) such that

\[
\alpha = \iota \circ \pi. \tag{6.1}
\]

Since \( U \) has no nontrivial characters, \( G/U \) is quasiprojective, cf. [PV, 3.7], and therefore \( k(G/U) \) is the field of fractions of \( k[G/U] \). Since \( k(X) \) is the field of fractions of \( k[X] \), and, by (6.1), \( \iota^* : k[X] \to k[G/U] \) is an isomorphism, it follows from here that \( \iota \) is a birational isomorphism. Since \( \iota \) is equivariant, this means that \( \iota \) is an open embedding and its image is \( G \cdot x \).

It follows from (6.1) that \( G_x = U \). Therefore, if \( y = w_0 \cdot x \) and \( \beta : G \to X, g \mapsto g \cdot y \), then \( G_y = U^- \) and \( \beta^*(k[X]) = S^- \). Consider the morphism

\[
\gamma = \alpha \times \beta : G \to X \times X.
\]

We canonically identify the functions on each of the factors of \( X \times X \) with the functions on \( X \times X \). Then

\[
\tau : k[X] \otimes_k k[X] \to k[X \times X], \ f \otimes h \mapsto fh,
\]
is an equivariant isomorphism. Now it follows directly from the definitions of \(\alpha, \beta, \gamma, \tau\) and \(\mu\) that the composition of the homomorphisms

\[
S \otimes_k S \xrightarrow{(\alpha^*)^{-1} \otimes (\beta^*)^{-1}} k[X] \otimes_k k[X] \xrightarrow{\tau} k[X \times X] \xrightarrow{\gamma^*} k[G].
\] (6.2)

coincides with \(\mu\). Therefore surjectivity of \(\mu\) is equivalent to surjectivity of \(\gamma^*\). By Theorem 1, this means that the problem is reduced to proving that \(\gamma\) is a closed embedding.

Let \(z = (x, y) \in X \times X\). Then \(G_z = G_x \cap G_y = U \cap U^{-} = \{e\}\). Therefore \(\gamma\) is an equivariant isomorphism of \(G\) and \(G \cdot z\). Hence, it remains to prove that \(G \cdot z\) is closed in \(X \times X\).

Let \(\{\lambda_1, \ldots, \lambda_s\}\) be a set of generators of the monoid \(P_{++}\) and \(v_i\) a highest vector of \(R(\lambda_i)\) with respect to \(B\). Let

\[
V = R(\lambda_1) \oplus \ldots \oplus R(\lambda_s), \quad v = (v_1, \ldots, v_s),
\] (6.3)

and \(\overline{G \cdot v}\) be the closure of \(G \cdot v\) in \(V\). Restricting the functions to \(G \cdot v\), we identify \(k[\overline{G \cdot v}]\) with a subalgebra of \(k[\overline{G \cdot v}]\).

Since \(U \subseteq G_{v_i}\) for each \(i\), we have \(U \subseteq G_{v_i}\). Therefore, if \(\delta : G \longrightarrow G \cdot v, \quad g \mapsto g \cdot v\), then \(\delta^*(k[\overline{G \cdot v}]) \subseteq S\). Since the submodule \(R(\lambda_i)^* = R(\lambda_i^*)\) of \(V^* = R(\lambda_1)^* \oplus \ldots \oplus R(\lambda_s)^*\) is simple and the projection of \(G \cdot v\) to \(R(\lambda_i)\) is nonzero, the restriction of functions to \(\overline{G \cdot v}\) defines an embedding of \(R(\lambda_i)^*\) into \(k[\overline{G \cdot v}]\). Hence, \(\delta^*(k[\overline{G \cdot v}])\) contains \(S_{\lambda_i}\) for all \(i\). It follows from here and (3.3) that \(\delta^*(k[\overline{G \cdot v}]) = S\). Therefore one can take \(X = \overline{G \cdot v}\) and \(x = v\).

Since \(\lambda_1^*, \ldots, \lambda_s^*\) is a set of generators of \(P_{++}\) as well, the similar arguments show that if \(u_i\) is a highest vector of \(R(\lambda_i^*)\) with respect to \(B\) and \(u = (u_1, \ldots, u_s) \in V^* = R(\lambda_1^*) \oplus \ldots \oplus R(\lambda_s^*)\), then one can take \(X\) to be the closure \(G \cdot u\) of \(G \cdot u\) in \(V^*\), \(x = u\) and \(y = w_0 \cdot u = (w_0 \cdot u_1, \ldots, w_0 \cdot u_s)\).

Hence, under these identifications, \(X \times X = \overline{G \cdot v} \times \overline{G \cdot u} \subset V \oplus V^* = R(\lambda_1) \oplus \ldots \oplus R(\lambda_s) \oplus R(\lambda_1^*) \oplus \ldots \oplus R(\lambda_s^*)\) and \(z = (v_1, \ldots, v_s, w_0 \cdot u_1, \ldots, w_0 \cdot u_s)\).

Since the weight of \(v_i\) is \(\lambda_i\) and the weight of \(w_0 \cdot u_i\) is \(w_0 \cdot \lambda_i^* = -\lambda_i\), the convex envelope of the weights of the projections of \(z\) to the irreducible direct summands of \(V \oplus V^*\) is \(C := \text{conv}\{\lambda_1, \ldots, \lambda_s, -\lambda_1, \ldots, -\lambda_s\}\).

Clearly \(0 \in C\). Since \(-C = C\), \(0\) is an interior point of \(C\). Whence, by Theorem 2, the orbit \(G \cdot v\) is closed in \(V \oplus V^*\), and therefore closed in \(X \times X\) as well. \(\square\)

(7) Now we pass to a proof of conjecture (\textbf{Ker}).

\textbf{Theorem 4.} The kernel of homomorphism (4.1) is generated by \(G\)-invariants.

\textbf{Proof.} We use the results obtained in the course of proof of Theorem 3 and keep the notation introduced therein.

Since the composition of homomorphisms in (6.2) coincides with \(\mu\), the statement of Theorem 4 is equivalent to the following one:

\textbf{The kernel of homomorphism} \(k[X \times X] \longrightarrow k[G \cdot z], \quad f \mapsto f|_{Gz}\), \textbf{is generated by its intersection} with \(k[X \times X]^G\).

In turn, this statement follows from a general Theorem 5 proved below. \(\square\)
Theorem 5. Let \( Y \) be an affine algebraic variety endowed with a regular action of a reductive algebraic group \( H \). Let \( y \in Y \) be a point such that

(i) \( H \cdot y \) is closed in \( Y \);
(ii) \( H_y \) is trivial.

Then the kernel of homomorphism \( k[Y] \rightarrow k[H \cdot y] \), \( f \mapsto f|_{H \cdot y} \), is generated by its intersection with \( k[Y]^H \).

Proof. There is a closed equivariant embedding of \( Y \) in a linear action, cf. [PV, 1.2], therefore one can assume that \( Y \) is an \( H \)-module.

Since \( Y \) is smooth, one can consider the Luna stratifications of \( Y \), [L], (cf. also [PV, 6.9]). It follows from (i) and (ii) that \( y \) lies in the principal stratum of this stratification. Therefore the canonical morphism \( \pi_{Y/H} : Y \rightarrow Y/H =: \text{Spec} \, k[Y]^H \) is smooth at any point of \( H \cdot y \), [L, Cor. 6] (cf. also [PV, Th. 6.11]).

Let \( \xi = \pi_{Y/H}(y) \). Since each fibre of \( \pi_{Y/H} \) contains a unique closed orbit which lies in the closure of any orbit contained in this fiber, cf. [PV, 4.4], it follows from (i) and (ii) that

\[ \pi_{Y/H}^{-1}(\xi) = H \cdot y. \] (7.1)

Therefore \( \pi_{Y/H} \) is a smooth morphism at any point of \( \pi_{Y/H}^{-1}(\xi) \). Hence, the fiber \( \pi_{Y/H}^{-1}(\xi) \) is reduced, cf. [R, Lemma 2.3]. This means that the ideal of functions in \( k[Y] \) vanishing on \( \pi_{Y/H}^{-1}(\xi) \) is generated by \( \pi_{Y/H}^*(m_\xi) \), where \( m_\xi \) is the ideal of \( \xi \) in \( k[Y/H] \). Since \( \pi_{Y/H}^*(k[Y/H]) = k[Y]^H \), this completes the proof because of (7.1).

(8) Now we want to give an explicit description of the generators of \( \ker \mu \). By Theorem 5, this is reduced to giving an explicit description of the algebra \( (\ker \mu)^G \).

By (3.3) and (3.6),

\[ (S \otimes_k S^-)^G = \bigoplus_{\lambda, \gamma \in \mathbb{P}^+} (S_\lambda \otimes_k S^-_\gamma)^G. \] (8.1)

Since \( S_\lambda \otimes_k S^-_\gamma \simeq (S_\lambda^*)^* \otimes_k S^-_\gamma \simeq \text{Hom}(S_\lambda^*, S^-_\gamma) \), it follows from the Schur Lemma that

\[ \dim (S_\lambda \otimes_k S^-_\gamma)^G = \begin{cases} 0, & \text{if } \gamma \neq \lambda^*, \\ 1, & \text{if } \gamma = \lambda^*, \end{cases} \] (8.2)

and therefore, by (8.1) and (8.2),

\[ (S \otimes_k S^-)^G = \bigoplus_{\lambda \in \mathbb{P}^+} (S_\lambda \otimes_k S^-_\lambda)^G. \] (8.3)

By (3.3) and (3.6), the decomposition (8.3) is a structure of \( \mathbb{P}^+ \)-graded algebra on \( (S \otimes_k S^-)^G \), i.e.

\[ (S_\lambda \otimes_k S^-_\lambda)^G \cdot (S_\gamma \otimes_k S^-_\gamma)^G = (S_{\lambda + \gamma} \otimes_k S^-_{(\lambda + \gamma)^*})^G. \] (8.4)

(9) First we describe the structure of \( (\ker \mu)^G \) as a vector space.
Proposition 1. (a) Any finite sum
\[ f = \sum_{\lambda} (f_{\lambda} - f_{\lambda}(e,e)), \quad f_{\lambda} \in (S_{\lambda} \otimes_k S_{\lambda}^*)^G, \tag{9.1} \]
lies in \((\ker \mu)^G\).

(b) Conversely, any element \(f \in (\ker \mu)^G\) can be written in a unique way in the form (9.1).

Proof. (a) We consider \((S \otimes_k S^-)^G\) in a natural way as a subalgebra of \(k[G \times G]\). Since \(\mu : (S \otimes_k S^-)^G \rightarrow k[G]^G = k\), we have \(\mu(f) = f(e,e)\) for any \(f \in (S \otimes_k S^-)^G\), and the claim follows.

(b) By (8.3), there is a unique decomposition
\[ f = \sum_{\lambda} f_{\lambda}, \quad f_{\lambda} \in (S_{\lambda} \otimes_k S_{\lambda}^*)^G, \tag{9.2} \]
whence (9.1), since \(0 = \mu(f) = f(e,e) = \sum_{\lambda} f_{\lambda}(e,e)\). Uniqueness of (9.1) follows from uniqueness of (9.2). \(\square\)

(10) To describe the algebra structure of \((\ker \mu)^G\) we need a few additional facts.

Lemma 1. Let \(V \) and \(W\) be two simple \(G\)-modules and \(P(V,W)\) the vector space of all invariant bilinear pairings \(\langle , \rangle \) : \(V \times W \rightarrow k\), \((v,w) \mapsto \langle v,w \rangle\). Then:

(i) \(\dim P(V,W) = 0 \) or \(1\), and the latter case happens if and only if \(W \cong V^*\).

(ii) If \(\{ v_i \}\) and \(\{ w_i \}\) are the dual bases of \(V \) and \(W\) with respect to a nonzero pairing \(\langle , \rangle \in P(V,W)\), then \(\sum_i (v_i \otimes w_i)\) is a nonzero element of \((V \otimes_k W)^G\) which does not depend on the choice of bases \(\{ v_i \}\) and \(\{ w_i \}\).

(iii) Let \(W \cong V^* \) and \(v^+ \in V, w_- \in W\) be respectively a highest and a lowest weight vectors with respect to \(B\). Then \(\langle v^+, w_- \rangle \neq 0\) for any nonzero pairing \(\langle , , \rangle \in P(V,W)\). For any nonzero constant \(\alpha \in k\), there is a unique pairing \(\langle , , \rangle \in P(V,W)\) such that \(\langle v^+, w_- \rangle = \alpha\).

Proof. (i) The \(G\)-module of all bilinear pairings \(V \times W \rightarrow k\) is isomorphic to the \(G\)-module \(\operatorname{Hom}(V,W^*)\), whence the claim by the Schur Lemma.

(ii) A nonzero pairing \(\langle , , \rangle\) defines an isomorphism of \(G\)-modules \(\varepsilon : V \otimes_k W \rightarrow \operatorname{Hom}(V,V), \sum_i (e_i \otimes f_i) \mapsto \{ v \mapsto \sum_i \langle v,f_i \rangle e_i \}\). It follows from here that \(\varepsilon(\sum_i (v_i \otimes w_i)) = \operatorname{Id}_V\), whence the claim.

(iii) Let \(\{ v_i \}, v_1 = v^+, \) be a basis of \(V\) consisting of the weight vectors, and \(\{ w_i \}\) the dual basis of \(W\) with respect to \(\langle , , \rangle\). The linear span \(V'\) of \(\{ v_i \mid i > 1 \}\) is \(U^*\)-invariant and \(u \cdot v_1 = v_1 + v', \quad v' = v'(u) \in V'\), for any \(u \in U^*\). Hence \((u \cdot w_1)(\sum_i \alpha_i v_i) = w_1(\sum_i \alpha_i (u^{-1} \cdot v_i)) = w_1(\alpha_1 v_1 + \text{an element in } V') = \alpha_1 = w_1(\sum_i \alpha_i v_i)\). Therefore, \(u \cdot w_1 = w_1\), hence \(w_- = \lambda w_1\) for a certain nonzero \(\lambda \in k\). Whence, \(\langle v^+, w_- \rangle = \lambda \neq 0\). The second statement follows from here and (i). \(\square\)
Theorem 6. \( \mu : (S_\lambda \otimes_k S_{\lambda^*})^G \to k[G]^G = k \) is an isomorphism.

Proof. By (8.2), one has to show that there is \( f \in (S_\lambda \otimes_k S_{\lambda^*})^G \) such that \( \mu(f) = f(e,e) \neq 0 \).

Let \( v^+ \) and \( v^- \) be a highest weight vector and a lowest weight vector of \( S_{\lambda^*} \), and \( S_\lambda \) and \( S_{\lambda^*} \) respectively. Consider the morphisms \( \alpha : G \to S_{\lambda^*}, \ g \mapsto g \cdot v^+ \), and \( \beta : G \to S_\lambda, \ g \mapsto g \cdot v^- \). Then \( \alpha^* : (S_{\lambda^*})^* \isom S_\lambda, \ \beta^* : (S_\lambda)^* \isom S_{\lambda^*} \), and therefore \( (\alpha \times \beta)^* : k[S_{\lambda^*} \times S_\lambda] = k[S_{\lambda^*}] \otimes_k k[S_\lambda] \to k[G \times G] = k[G] \otimes_k k[G] \) induces \( (\alpha \times \beta)^*((S_{\lambda^*})^* \otimes_k (S_\lambda)^*) = S_\lambda \otimes_k S_{\lambda^*} \).

Fix a nonzero invariant bilinear pairing \( \langle \ , \rangle : (S_{\lambda^*})^* \otimes_k (S_\lambda)^* \to k \). Let \( \{p_i\} \) and \( \{q_i\} \) be the dual bases of \( (S_{\lambda^*})^* \) and \( (S_\lambda)^* \) with respect to \( \langle \ , \rangle \). It follows from Lemma 1, (ii), that \( h := \sum_i (p_i \otimes q_i) \in ((S_{\lambda^*})^* \otimes_k (S_\lambda)^*)^G, h \neq 0 \). Therefore, \( f := (\alpha \times \beta)^*(h) \in (S_\lambda \otimes_k S_{\lambda^*})^G, f \neq 0 \).

The mapping \( S_{\lambda^*} \times S_\lambda \to k, \ (a,b) \mapsto h(a,b) \), is a nonzero invariant bilinear pairing. Since \( f(e,e) = (\alpha \times \beta)^*(h)(e,e) = h(a(e), \beta(e)) = h(v^+, v^-) \), the claim follows from Lemma 1, (iii). \( \square \)

(11) It follows from Proposition 2 that for any \( \lambda \in P_+ \) there is a unique element \( h_\lambda \in (S_\lambda \otimes_k S_{\lambda^*})^G \) such that \( \mu(h_\lambda) = h_\lambda(e,e) = 1 \).

This element \( h_\lambda \) can be described in either of two ways:

(a) Fix a nonzero invariant pairing \( \langle \ , \rangle : S_\lambda \times S_{\lambda^*} \to k \). Let \( \{p_i\} \) and \( \{q_i\} \) be a pair of dual bases of \( S_\lambda \) and \( S_{\lambda^*} \), with respect to \( \langle \ , \rangle \). Then
\[
h_\lambda = \left( \sum_i p_i \otimes q_i \right) / \left( \sum_i p_i(e)q_i(e) \right).
\]

(b) Let \( \{X_i\} \) and \( \{X_i^*\} \) be a pair of dual bases of the vector space \( g \) with respect to the Killing form. The elements \( X_i \) and \( X_i^* \) act on \( S_\lambda \) and \( S_{\lambda^*} \) linearly. Consider the linear operator
\[
\Delta = \sum_i (X_i \otimes X_i^* + X_i^* \otimes X_i)
\]
on \( S_\lambda \otimes_k S_{\lambda^*} \). Let \( \langle \ , \rangle \) be the inner product on the lattice of weights of \( G \) induced by the Killing form, \( \rho \) half the sum of all positive roots and
\[
c(\lambda) = -\langle \lambda + 2\rho|\lambda \rangle - \langle \lambda^* + 2\rho|\lambda^* \rangle.
\]

Proposition 2. Let \( f \in S_\lambda \otimes_k S_{\lambda^*} \). Then the following properties are equivalent:

(i) \( f = h_\lambda \);
(ii) \( \Delta \cdot f = c(\lambda)f, \ f(e,e) = 1 \).

Proof. Let \( \Omega \in \mathcal{U}(g) \) be the Casimir operator given by the formula
\[
\Omega = \sum_i X_i X_i^*.
\]
If \( \lambda \in \mathbb{P}_{++} \), then \( \Omega \) acts on a simple \( \mathcal{U}(\mathfrak{g}) \)-module with the highest weight \( \lambda \) as scalar multiplication by \( \langle \lambda + 2\rho | \lambda \rangle \), cf. [Bo]. Therefore for any \( s \in S_\lambda, t \in S_{\lambda}^* \), one has the equality

\[
\Omega \cdot (s \otimes t) = \sum_i ((X_i X_i^* \cdot s) \otimes t + (X_i^* \cdot s) \otimes (X_i \cdot t) +
(X_i \cdot s) \otimes (X_i^* \cdot t) + s \otimes (X_i X_i^* \cdot t)) = -c(\lambda)(s \otimes t) + \bigtriangleup(s \otimes t). \tag{11.1}
\]

Since \( \langle \lambda + 2\rho | \lambda \rangle > 0 \) if \( \lambda \neq 0 \), the kernel of \( \Omega \) in any \( \mathcal{U}(\mathfrak{g}) \)-module \( V \) is \( V^G \). The claim follows from here, (8.2) and (11.1). \( \square \)

(12) Now we can describe the algebra structure of \( (\ker \mu)^G \) and indicate the generators of the ideal \( (\ker \mu) \).

**Theorem 7.** Let \( \lambda_1, \ldots, \lambda_s \) be a set of generators of the monoid \( \mathbb{P}_{++} \). Then \( (\ker \mu)^G \) coincides with \( k[\mathfrak{h}_{\lambda_1} - 1, \ldots, \mathfrak{h}_{\lambda_s} - 1]_+ \), the augmentation ideal of \( k[\mathfrak{h}_{\lambda_1} - 1, \ldots, \mathfrak{h}_{\lambda_s} - 1] \).

**Proof.** Since \( \mathfrak{h}_{\lambda_i} - 1 \in (\ker \mu)^G \), one has \( (\ker \mu)^G \supseteq k[\mathfrak{h}_{\lambda_1} - 1, \ldots, \mathfrak{h}_{\lambda_s} - 1]_+ \).

Let \( \lambda \in \mathbb{P}_{++} \). Then \( \lambda = \sum_i n_i \lambda_i \) for some integers \( n_i \geq 0 \). It follows from (8.4) that \( \mathfrak{h}_\lambda = \prod_i \mathfrak{h}_{\lambda_i}^{n_i} = \prod_i ((\mathfrak{h}_{\lambda_i} - 1) + 1)^{n_i} \). Therefore, \( \mathfrak{h}_\lambda - 1 \in k[\mathfrak{h}_{\lambda_1} - 1, \ldots, \mathfrak{h}_{\lambda_s} - 1]_+ \). Hence, by (8.2) and Proposition 1, \( (\ker \mu)^G \subseteq k[\mathfrak{h}_{\lambda_1} - 1, \ldots, \mathfrak{h}_{\lambda_s} - 1]_+ \). \( \square \)

**Remark.** The proofs of Theorem 3, Theorem 4, Theorem 5, Theorem 6, Theorem 7 given here use the assumption that the ground field has characteristic zero. However these statements are true over the ground field of arbitrary characteristic. The proofs will be given in a forthcoming publication.

(13) Now we shall indicate the explicit presentation of the algebras \( S \) and \( S^- \) by the generators and relations. For the first time such presentation was obtained (but not published) by B. Kostant; his result and proof appears in [LT]. We just present it here in a slightly different form than in [LT]. We use the results obtained in the course of the proof of Theorem 3 and keep the notation introduced therein.

The decomposition (6.3) induced the natural \( \mathbb{N}^s \)-grading of \( k[V] \).

Let \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^s \). It follows from (6.3) that

\[
k[V]_{e_i + e_j} \simeq \begin{cases} 
R(\lambda_i)^* \otimes_k R(\lambda_j)^*, & \text{if } i \neq j; \\
S^2 R(\lambda_i)^*, & \text{if } i = j.
\end{cases} \tag{13.1}
\]

By (13.1), for all \( 1 \leq i \leq j \leq s \) the \( G \)-module \( k[V]_{e_i + e_j} \) contains a unique simple submodule with the highest weight \( \lambda_i^* + \lambda_j^* \), the Cartan component of \( k[V]_{e_i + e_j} \). Denote by \( Q_{i,j} \) the \( G \)-invariant direct complement in \( k[V]_{e_i + e_j} \) to this submodule. Let \( J \) be the ideal in \( k[V] \) generated by all \( Q_{i,j} \), \( 1 \leq i \leq j \leq s \).

Since \( V^U \) is \( B \)-stable, \( G \cdot V^U \) is closed in \( V \). It is shown in [Br, 4.1]\(^1\) that \( J \) is the ideal of \( G \cdot V^U \) in \( k[V] \).

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\(^1\)The part of Theorem in [Br, 4.1] concerning the ideal \( I \) contains a mistake but the part concerning the ideal \( J \) is correct.
Theorem 8. Assume that $\lambda_1, \ldots, \lambda_s$ is a free system of generators of $P_+$ (i.e. each element of $P_+$ uniquely expressed as a linear combination of these generators with the nonnegative integer coefficients). Then $S$ is $G$-isomorphic to $k[V]/J$.

Proof. The condition on $\lambda_1, \ldots, \lambda_s$ is equivalent to the equality $s = \dim T$. Since $G_v = U$, we have $T_v = \{e\}$. Therefore, $\dim T \cdot v = \dim T = s = \dim V^U$. Since $\dim T \cdot v \subset V^U$ and $G \cdot V^U$ is closed, it follows from here that $X = \overline{G \cdot v} = G \cdot V^U$. □

Remark. The presentation given in Section (13) heavily depends on the assumption that the ground field has characteristic zero. In a forthcoming publication we will give a characteristic free presentation. It is related to the theory of standard monomials.

References