Bordered Complex Hessians

John P. D’Angelo


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John P. D’Angelo
Dept. of Mathematics
University of Illinois
Urbana, IL 61801 USA

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Abstract: We record some basic facts about bordered complex Hessians and logarithmically plurisubharmonic functions. These enable us to prove that a nonnegative bihomogeneous polynomial is plurisubharmonic if and only if it is log-plurisubharmonic; we give a more general version for twice differentiable homogeneous functions. The proof relies on the vanishing of the determinant of the bordered complex Hessian; we go on to find general classes of solutions to the nonlinear PDE given by setting the determinant of a bordered complex Hessian equal to zero.

Introduction

Suppose that \( r \) is a function of class \( C^2 \) defined on an open set in complex Euclidean space \( \mathbb{C}^n \). The complex Hessian \( H(r) \) of \( r \) is the Hermitian \( n \times n \) matrix whose entries are the second order partial derivatives \( r_{ij,\bar{k}} \). It is a standard fact that a \( C^2 \) function \( r \) is plurisubharmonic if and only if \( H(r) \) is nonnegative definite at all points. If \( r \) is also nonnegative, then the plurisubharmonicity of \( \log(r) \) can be characterized (see Proposition 3) by the nonnegative definiteness of the bordered complex Hessian \( B(r) \). This is the \( n + 1 \) by \( n + 1 \) Hermitian matrix

\[
B(r) = \begin{pmatrix}
    r_{z_1\bar{z}_1} & r_{z_1\bar{z}_2} & \cdots & r_{z_1\bar{z}_n} & r_{\bar{z}_1}
    \\
    r_{z_2\bar{z}_1} & r_{z_2\bar{z}_2} & \cdots & r_{z_2\bar{z}_n} & r_{\bar{z}_2}
    \\
    \cdots & \cdots & \cdots & \cdots & \cdots
    \\
    r_{z_n\bar{z}_1} & \cdots & \cdots & r_{z_n\bar{z}_n} & r_{\bar{z}_n}
    \\
    r_{\bar{z}_1} & \cdots & \cdots & r_{\bar{z}_n} & r
\end{pmatrix}.
\]

(1)

The bordered complex Hessian appeared as early as 1906 in work of Levi on pseudo-convexity. Let \( \delta \) denote the distance to the boundary of a domain \( \Omega \). It is a standard fact \( [H] \) that \( \Omega \) is pseudoconvex if and only if \( -\log(\delta) \) is continuous and plurisubharmonic on \( \Omega \). When \( \delta \) is smooth, we see that pseudoconvexity is characterized by the definiteness of \( B(\delta) \). When boundary \( \Omega \) is smooth, one can also characterize pseudoconvexity via the Levi form. The Levi form \( \lambda(r) \) of a defining function \( r \) is the restriction of \( H(r) \) to the subspace of \( (1,0) \) vectors tangent to the zero set of \( r \), and is thus an \( n - 1 \) by \( n - 1 \) Hermitian matrix. The Levi form of a defining function \( r \) for a domain in two complex variables can be identified with a function; this function can also be expressed in terms of the determinant of \( B(r) \). We have

\[
det(B(r)) = r \det(H(r)) - \lambda(r).
\]

(2)
An analogous formula holds in higher dimensions.

The bordered complex Hessian has appeared also in work of Fefferman [F] on the theory of the Bergman kernel function, and in work of Kohn [K] on subelliptic multipliers. Given a strongly pseudoconvex domain $\Omega$, Fefferman discussed a boundary value problem for an unknown function $r$. This function must satisfy $r = 0$ on the boundary of $\Omega$ and

$$\det(B(r)) = 1$$

on $\Omega$. Solutions to this nonlinear problem help provide approximations to the Bergman kernel function. In fact, for the unit ball, the Bergman kernel is a power of an obvious solution to this equation.

Kohn’s work on subelliptic multipliers for smoothly bounded pseudoconvex domains relies heavily on properties of the determinant of the Levi form. Because of the close relationship between the Levi form and the bordered Hessian of a defining function, some formulas there have a simpler appearance when expressed in terms of the bordered Hessian.

Nevertheless there seem to be many simple facts about the bordered Hessian that have not been recorded. The purposes of this paper are to record these facts and then to derive from them some surprising and useful results. For example we use the bordered Hessian to prove (See Theorem 1) that a nonnegative bihomogeneous polynomial is plurisubharmonic if and only if its logarithm is plurisubharmonic. We obtain the same conclusion for twice differentiable nonnegative functions satisfying $r(tz) = |t|^{2m}r(z)$ for all complex numbers $t$. Here $m$ is an arbitrary positive number. One step in the proof is to show that $\det(B(r))$ vanishes identically for such homogeneous functions. This suggests trying to find other large classes of solutions to the nonlinear partial differential equation $\det(B(r)) = 0$. We do so in Theorem 2. Along the way we provide some insights into logarithmic plurisubharmonicity and related concepts.

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I. Plurisubharmonicity and Log Plurisubharmonicity

In this paper we consider only functions of class $C^2$, although many of our results have appropriate analogues for continuous functions, obtained via regularization. We recall that a $C^2$ function $r$ is plurisubharmonic on an open set if its complex Hessian is nonnegative definite at every point. The complex Hessian $H(r)$ is the $n$ by $n$ matrix

$$
\begin{pmatrix}
    r_{z_1 \overline{z}_1} & r_{z_1 \overline{z}_2} & \cdots & r_{z_1 \overline{z}_n} \\
    r_{z_2 \overline{z}_1} & r_{z_2 \overline{z}_2} & \cdots & r_{z_2 \overline{z}_n} \\
    \vdots & \vdots & \ddots & \vdots \\
    r_{z_n \overline{z}_1} & \cdots & \cdots & r_{z_n \overline{z}_n}
\end{pmatrix}
$$

Thus $r$ is plurisubharmonic on $\mathbb{C}^n$ if, for all $z, w \in \mathbb{C}^n$, we have

$$\sum_{j,k=1}^n r_{z_j \overline{z}_k}(z) w_j \overline{w}_k \geq 0$$
A nonnegative smooth function $r$ is called \textit{logarithmically plurisubharmonic} if the matrix $M(r) = (r r_{z_j \overline{z}_k} - r_{z_j \overline{z}_j} r_{z_k \overline{z}_k})$ is nonnegative definite at each point. Thus $r$ is logarithmically plurisubharmonic on $\mathbb{C}^n$ if, for all $z, w \in \mathbb{C}^n$, we have

$$r(z) \sum_{j,k=1}^{n} r_{z_j \overline{z}_k}(z) w_j \overline{w}_k \geq \left| \sum_{j=1}^{n} r_{z_j}(z) w_j \right|^2 \quad (6)$$

This definition is equivalent to saying that the function $\log(r)$ is plurisubharmonic; calculation shows that $r^2 H(\log(r)) = M(r)$. The advantage of the formulation (6) is that we need not worry about the zeroes of $r$, where $\log(r)$ is not smooth. Furthermore a version of (6) plays a crucial role in a vanishing theorem for harmonic forms proved by McNeal [M]. In McNeal’s inequality the factor of $r(z)$ on the left side of (6) is replaced by $A + Br(z)$, for positive numbers $A$ and $B$.

It is easy to show that the composition of a plurisubharmonic function with a convex increasing function is itself plurisubharmonic. Thus, if $\log(r)$ is plurisubharmonic then $r$ also is. We present a simple example of a homogeneous polynomial where $r$ is plurisubharmonic and nonnegative, but $\log(r)$ is not plurisubharmonic. Later we will show that the two conditions are identical for bihomogeneous polynomials.

\textbf{Example 1.} Suppose $n = 1$. Put $r(z) = |z|^2 + c(z^2 + \overline{z}^2)$. It is evident that $r$ is plurisubharmonic for all $c$ and that its values are nonnegative for $|c| \leq \frac{1}{2}$. A simple computation shows that $\log(r)$ is plurisubharmonic only when $c = 0$. This example does not contradict Theorem 1 because here $r(tz) = |t|^2 r(z)$ only for $t$ real.

We continue to discuss connections between plurisubharmonicity and logarithmic plurisubharmonicity. Proposition 1 is well known; its generalization in Proposition 2 was obtained at ESI.

\textbf{Proposition 1.} Suppose that $r$ is a smooth positive function. Then $\log(r)$ is plurisubharmonic if and only if $|e^{a(z)}|^2 r(z)$ defines a plurisubharmonic function for each linear function $a$.

Proof. Suppose first that $\log(r)$ is plurisubharmonic. Since $z \to 2 \text{Re}(a(z))$ is harmonic, we see that

$$\log(r) + 2 \text{Re}(a) \quad (7)$$

is also plurisubharmonic. Since exponentiation is a convex increasing function on $\mathbb{R}$, the exponential of a plurisubharmonic function is also plurisubharmonic. Exponentiating (7) shows that $|e^{a}|^2 r$ defines a plurisubharmonic function.

To show the converse assertion we first define $u$ by

$$u(z) = |e^{a(z)}|^2 r(z). \quad (8)$$

We will compute the Hessian of $u$ at $z$ and then choose $a$ intelligently. By assumption $u$ is plurisubharmonic, so its Hessian is nonnegative:

$$0 \leq (u_{z_k \overline{z}_l}) = |e^{a(z)}|^2 (r_{z_k \overline{z}_l} + r_{z_k} \overline{a}_l + a_k r_{z_l} + r_{a_k} \overline{a}_l) \quad (9)$$
Recall that we are assuming $r$ does not vanish. In (9) set $a_k = \frac{-r_{zk}(z)}{r}$ and substitute. After cancelling the positive factor $|e^a(z)|^2$ and multiplying by the positive number $r(z, \overline{z})$ we discover that

$$0 \leq M(r) = r_{zk} \overline{r} - r_{zk} r_{z_k}$$

(10)

Since $z$ is an arbitrary point, $M(r)$ is nonnegative definite at each point. Thus $r$ is logarithmically plurisubharmonic. ♣

The ideas in Proposition 1 easily extend to convex increasing functions other than the exponential. Let $\phi$ be a smooth function on the real line, and suppose that both $\phi'$ and $\phi''$ are positive. We next characterize those functions $r$ for which $\phi^{-1}(r)$ are plurisubharmonic. In Proposition 2, we write $g = \phi^{-1}(r)$ so the notation is nice. The notation becomes even more attractive when we introduce an equivalence class: we say that $h \in E(g)$ if $H(g) = H(h)$.

**Proposition 2.** Let $\phi$ be a smooth strictly convex increasing function on $\mathbb{R}$. Suppose that $g$ is a smooth function defined on some open set in $\mathbb{C}^n$. Then $g$ is plurisubharmonic if and only if $\phi(h)$ is plurisubharmonic for every $h$ in $E(g)$.

**Proof.** First we compute the Hessian of a composition:

$$H(\phi(h)) = \phi''(h) |\partial h|^2 + \phi'(h)H(h).$$

(11)

Suppose that $g$ is plurisubharmonic and that $h \in E(g)$. Since $H(g) = H(h)$, and $\phi'$ is positive, (11) implies that

$$H(\phi(h)) \geq \phi'(h)H(h) = \phi'(h)H(g) \geq 0.$$

(12)

But (12) implies that $H(\phi(h))$ is nonnegative definite, so $\phi(h)$ is plurisubharmonic.

To show the converse assertion we know that $H(\phi(h))$ is nonnegative definite for each $h \in E(g)$. Given a point $p$, we select $h \in E(g)$ such that $\partial h$ vanishes at $p$. This is always possible; we choose $h$ so that $g - h$ is twice the real part of the linear function $z \to \partial g(p)(z)$. This choice simplifies (11); at $p$ we see from (11) that $H(\phi(h)) = \phi'(h)H(h)$, which is a positive multiple of $H(h)$ at $p$. Thus $H(h)$ is nonnegative definite at $p$. Since $g$ and $h$ have the same Hessian, $H(g)$ is also nonnegative there. Since $p$ is an arbitrary point, $H(g)$ is nonnegative definite everywhere, and $g$ is plurisubharmonic. ♣

We now return to the special case where $\phi$ is the exponential function.

**Definition.** Let $r$ be a smooth function of $n$ complex variables. Its bordered Hessian $B(r)$ is the $n + 1$ by $n + 1$ Hermitian symmetric matrix

$$
\begin{pmatrix}
  r_{z_1 \overline{z}_1} & r_{z_1 \overline{z}_2} & \cdots & r_{z_1 \overline{z}_n} & r_{z_1} \\
  r_{z_2 \overline{z}_1} & r_{z_2 \overline{z}_2} & \cdots & r_{z_2 \overline{z}_n} & r_{z_2} \\
  \cdots & \cdots & \cdots & \cdots & \cdots \\
  r_{z_n \overline{z}_1} & \cdots & \cdots & r_{z_n \overline{z}_n} & r_{z_n} \\
  r_{\overline{z}_1} & \cdots & \cdots & r_{\overline{z}_n} & r
\end{pmatrix} = \begin{pmatrix} H(r) & \partial_r \\ \frac{\partial}{\partial r} & r \end{pmatrix}.
$$

(13)
Remark 1. One could also take the transpose of (13) as the definition. With the
convention implied by (13), the Hermitian form \( \mathcal{B}(r)(v, v) \) is computed using the matrix
multiplication \( v' \mathcal{B}(r) \bar{v} \). Here \( v \) is a column vector, \( \bar{v} \) is its conjugated column vector, and
\( v' \) is its (unconjugated) row vector.

Proposition 3. A smooth nonnegative function is logarithmically plurisubharmonic
if and only if its bordered Hessian is nonnegative definite.

Proof. The statement is equivalent to showing that \( \mathcal{B}(r) \) and \( M(r) \) are simultaneously
nonnegative definite. In case \( r \) vanishes somewhere, its gradient also vanishes there, and
its complex Hessian is nonnegative definite there. From (13) we see that \( \mathcal{B}(r) \) is also
nonnegative definite there. Also, the matrix \( M(r) = (r r_{ij} z_k - r_{ij} r_{zk}) \) vanishes there,
and hence it is also nonnegative definite. Thus when \( r \) vanishes both \( M(r) \) and \( \mathcal{B}(r) \)
are nonnegative definite. We may therefore assume that \( r > 0 \).

Suppose first that \( \mathcal{B}(r) \) is nonnegative definite. For all \( a \in \mathbb{C}^n \), and for all \( b \in \mathbb{C} \) we
therefore have

\[
\sum r_{ij} z_k a_j \bar{a}_k + 2 Re(\bar{b} \sum r_{ij} a_j) + r |b|^2 \geq 0
\]  

Since \( r > 0 \), we may choose \( b = -\frac{\sum a_j r_{ij}}{r} \) in (14). This yields

\[
\sum r_{ij} z_k a_j \bar{a}_k - \frac{|\sum r_{ij} a_j|^2}{r} \geq 0
\]  

Conversely, suppose that \( M(r) \geq 0 \). The inequality

\[
r 2 Re \left( \bar{b} \sum r_{ij} a_j \right) \geq - |b|^2 - \sum |r_{ij} a_j|^2
\]

implies, for all \( a \in \mathbb{C}^n \), and for all \( b \in \mathbb{C} \), that

\[
r \sum r_{ij} z_k a_j \bar{a}_k + 2 Re \left( \bar{b} r \sum r_{ij} a_j \right) + r^2 |b|^2 \geq r \sum r_{ij} z_k a_j \bar{a}_k - \sum |r_{ij} a_j|^2
\]

The expression on the right in (17) is \( M(r)(a, a) \), and hence is nonnegative. Therefore the
term on the left is nonnegative, and hence \( r \mathcal{B}(r) \) is nonnegative definite. Since \( r \) is positive
we see that \( \mathcal{B}(r) \) is nonnegative definite. \( \blacktriangleleft \).

Next we investigate the kernel of the bordered Hessian. We write \( (a, b) \) for an \( n + 1 \)
tuple of \( \mathbb{C}^n \) functions, where \( a = (a_1, ..., a_n) \). We see that the conjugated column vector
(\( \bar{a}, \bar{b} \)) is in the kernel of \( \mathcal{B}(r) \) if and only if both (18) and (19) hold:

\[
\sum \bar{a}_j r_{ij} z_j + \bar{b} r = 0 \quad (18)
\]

For \( i = 1, ..., n \) we have

\[
\sum \bar{a}_j r_{zi} z_j + \bar{b} z_i = 0. \quad (19)
\]
We denote by \( L \) the \((1, 0)\) vector field defined by

\[
L = \sum_{j} a_j \frac{\partial}{\partial z_j}.
\tag{20}
\]

In this notation (18) becomes

\[
\mathcal{T}_r + \bar{b} r = 0
\tag{21}
\]

and the equations in (19) become

\[
\mathcal{T}_r(z_i) + \bar{b} r z_i = 0.
\tag{22}
\]

We obtain the following result.

**Proposition 4.** Let \( r \) be a \( C^2 \) function. Then \( \det(\mathcal{B}(r)) = 0 \) if and only if there is a \((1, 0)\) vector field \( L \) such that 1) and 2) hold:

1) \( L \) does not vanish where \( r \) does not vanish.

2) For \( i = 1, ..., n \) we have

\[
r \frac{\partial}{\partial z_i} = r z_i \frac{\partial}{\partial r}.
\tag{23}
\]

Furthermore \( \mathcal{T} \) satisfies (23) if and only if the \( n \) tuple of its coefficient functions \((\mathcal{T}_1, ..., \mathcal{T}_n)\) is in the kernel of the matrix \( M(r) = (r \ r z_i - r z_i \ r z_i) \).

Proof. Suppose first that \( \det(\mathcal{B}(r)) = 0 \). We can then find \((a, b)\) satisfying (18) and (19) such that \(|a|^2 \) and \( b \) do not simultaneously vanish. Define \( L \) by (20), and let \( \Omega \) denote the open set points where \( r \) does not vanish. On \( \Omega \) we may solve (21) for \( b \), substitute the result in (22), and therefore eliminate \( b \). After simplifying we obtain (23). Conversely, if there is a nonzero \((a, b)\) in the kernel of \( \mathcal{B}(r) \), the determinant must vanish. This holds on \( \Omega \), and hence everywhere by continuity.

The second statement is a reformulation of (23). \( \blacklozenge \)

We next apply Propositions 3 and 4 in the presence of homogeneity. Suppose first that \( r \) is a real-valued polynomial on \( \mathbb{C}^n \), and let \( m \) be a positive integer. Then \( r \) is called \textit{bihomogeneous} of degree \( 2m \) if it is homogeneous of degree \( m \) in both \( z \) and in \( \bar{z} \). Then \( r \) satisfies (24). More generally let \( m \) be a positive real number. We consider continuous functions that are \( C^2 \) away from their zero sets, and that satisfy (24) for all \( z \in \mathbb{C}^n \), and all \( t \in \mathbb{C} \).

\[
r(tz) = |t|^{2m} r(z)
\tag{24}
\]

One class of examples arises by raising a bihomogeneous polynomial to a fractional power. Our primary interest in the next two results remains nonetheless the case where \( r \) is a bihomogeneous polynomial.

**Proposition 5.** Suppose \( r \) is twice differentiable and satisfies (24) for some positive number \( m \). (In particular \( r \) could be a bihomogeneous polynomial of degree \( 2m \).) Then \( \det(\mathcal{B}(r)) \) vanishes identically.

Proof. Suppose that \( r \) is twice differentiable and (24) holds. By differentiating (24) with respect to \( t \) and then setting \( t = 1 \) we obtain
\[
\sum_{k=1}^{n} \tau_k(z)\bar{\tau}_k = mr(z)
\]  \hspace{1cm} (25)

Differentiating (25) with respect to \( z_i \) we further obtain, for \( i = 1, \ldots, n, \)
\[
\sum_{k=1}^{n} \tau_k \bar{z}_k (z) \bar{\tau}_k = mr_{zi}(z)
\]  \hspace{1cm} (26)

It follows immediately from (25) and (26) that the nonzero column vector \((\bar{\tau}_1, \ldots, \bar{\tau}_n, -m)\)
lies in the kernel of the matrix \( \mathcal{B}(r) \). Hence the determinant vanishes. In the language of
Proposition 4, the vector field given by \( L = \sum z_j \frac{\partial}{\partial z_j} \) satisfies (23). \( \blacklozenge \).

We are now prepared to prove a surprising result. The case when \( r \) is homogeneous
of degree 1 is stated in [JP].

**Theorem 1.** Suppose \( r \) is a nonnegative function, twice differentiable away from its
zero set, and assume \( r \) satisfies (24) for some positive \( m \). (In particular \( r \) could be an
arbitrary nonnegative bihomogeneous polynomial). Then \( r \) is logarithmically plurisubharmonic
if and only if \( r \) is plurisubharmonic.

Proof. We have noted that if \( \log(r) \) is plurisubharmonic, then so is \( r \). We are interested
in the converse assertion; we assume that \( r \) is plurisubharmonic and we will prove that
\( \log(r) \) also is.

For a positive number \( m \) we consider the function \( \phi \) given by \( \phi(z) = ||z||^{2m} \). A
standard computation, which uses the Cauchy-Schwarz inequality in case \( 0 < m < 1 \), shows
that \( \phi \) is strongly plurisubharmonic away from the origin. Given \( r \), and \( \epsilon > 0 \), we consider
the strongly plurisubharmonic function \( r + \epsilon\phi \). We claim that \( \mathcal{B}(r + \epsilon\phi) \) is nonnegative
definite. It suffices to show two things: first that the first \( n \) leading principal minor determinants are positive, and second that the full determinant (the \( n + 1 \)-st leading principal
minor determinant) is nonnegative. The positivity of the first \( n \) minor determinants is
equivalent to the strong plurisubharmonicity of \( r + \epsilon\phi \), so the first statement holds. Both \( r \) and \( \phi \) satisfy (24), so \( r + \epsilon\phi \) does also. Therefore Proposition 5 guarantees that the
determinant of its bordered Hessian vanishes identically. This proves the second statement,
and hence \( \mathcal{B}(r + \epsilon\phi) \) is nonnegative definite. Letting \( \epsilon \) tend to zero we see that \( \mathcal{B}(r) \) is also
nonnegative definite. By Proposition 3, \( r \) is logarithmically plurisubharmonic. \( \blacklozenge \).

Theorem 1 applies in particular to quotients of bihomogeneous polynomials. Suppose
that \( r \) and \( u \) are bihomogeneous polynomials of degrees \( 2(m + d) \) and \( 2d \), and \( u \) is positive
away from the origin. Then \( \frac{r}{u} \), defined to be 0 at the origin, satisfies (24). Therefore the
determinant of its bordered Hessian vanishes.

**II. Solutions to** \( \det(\mathcal{B}(r)) = 0 \)

Proposition 5 reveals that bihomogeneous polynomials are solutions to the nonlinear
equation \( \det(\mathcal{B}(r)) = 0 \). After introducing some standard concepts, we reexamine the
proof to obtain additional solutions.

We let \( T^{1,0} \) denote the bundle whose sections are smooth \((1,0)\) vector fields on \( \mathbb{C}^n \). In
coordinates, such vector fields can be written as in (20) where the \( a_j \) are smooth functions.
We let $O_{1,0}$ denote the sections of $T_{1,0}$ for which the coefficient functions are holomorphic. Suppose that $L$ is such a vector field. A simple computation shows that (23) can be replaced with the condition

$$\frac{\partial}{\partial z_i} \left( \frac{L(r)}{r} \right) = 0 \hspace{1cm} (27)$$

for all $i$. The truth of (27) for all $i$ is equivalent to the holomorphicity of the ratio $\frac{L(r)}{r}$.

This illuminates the bihomogeneous case. In Proposition 5 the vector field $L$ is given by

$$L = \sum z_j \frac{\partial}{\partial z_j}$$

and $L(r) = mr$. Therefore (27) holds because the ratio is the constant $m$.

We next give another class of solutions to $\det(\mathcal{B}(r)) = 0$. Proposition 5 suggests comparing $\det(\mathcal{B}(r^m))$ with $\det(\mathcal{B}(r))$. We therefore try to relate $\mathcal{B}(\phi(r))$ and $\mathcal{B}(r)$.

We recall from the discussion after (6) that $M(r) = r^2 H(\log(r))$ is the matrix that determines whether $r$ is logarithmically plurisubharmonic. We also noted in the proof of Proposition 4 that $(\bar{\sigma}, \bar{\theta})$ lies in the kernel of $\mathcal{B}(r)$ if and only if $\bar{\sigma}$ lies in the kernel of $M(r)$ and $\bar{b}$ satisfies (18). Therefore it is both sufficient and natural to study $M(r)$.

**Lemma 1.** Let $\phi$ be a smooth function on $\mathbb{R}$. The following formulas hold. (In (30) the expressions $K$ and $L$ are defined by the formula).

$$M(\phi(r)) = \phi \phi' H(r) + (\phi \phi'' - (\phi')^2) |\partial r|^2 \hspace{1cm} (29)$$

$$M(\phi(r)) = (\phi \phi' + r(\phi \phi'' - (\phi')^2)) H(r) - (\phi \phi'' - (\phi')^2) M(r) = K(\phi, r) H(r) + L(\phi) M(r) \hspace{1cm} (30)$$

Proof. (29) is a simple exercise in the chain rule. To prove (30) we use the formula $M(r) = r H(r) - |\partial r|^2$ to eliminate $|\partial r|^2$ from (29) and then simplify. ♣

**Corollary 1.** Suppose that $\phi$ satisfies the conditions $\dot{K}(\phi, r) = 0$ and $L(\phi) \neq 0$. Then the kernels of $M(r)$ and $M(\phi(r))$ coincide.

**Lemma 2.** Suppose that $\phi$ is a $C^2$ function on $\mathbb{R}$. Then $\dot{K}(\phi(r), r) = 0$ identically if and only if $\phi(r) = cr^{k}$ for real constants $c$ and $k$. In this case, $L(\phi)$ is nonzero except in case $\phi$ is constant.

Proof. For ease on the eyes we have omitted the variable $r$ in the formulas, involving $\phi$ and its derivatives. By setting $\dot{K}(\phi(r), r)$ equal to 0 we obtain a second order differential equation that can be solved by elementary methods; the constants $c$ and $k$ arise as constants of integration. We omit the details. The second statement follows from the first. ♣

The following corollary enables us to produce additional solutions to the nonlinear equation $\det(\mathcal{B}(r)) = 0$. When $r$ is a solution, so is every power of $r$. This parses well with the result on homogeneity; later we find many solutions that are not homogeneous.
Corollary 2. Suppose that \( r \) is \( C^2 \) and \( \det(\mathcal{B}(r)) = 0 \). Then \( \det(\mathcal{B}(cr^k)) = 0 \) for every choice of \( c \) and \( k \).

We now use Proposition 5 and methods from [D] to find many more solutions to the equation \( \det(\mathcal{B}(r)) = 0 \). Suppose that \( r \) is a real-analytic function, defined near the origin in \( \mathbb{C}^n \). We expand \( r \) in a power series about the origin:

\[
 r(z, \overline{z}) = \sum c_{\alpha \beta} z^{\alpha} \overline{z}^{\beta}.
\]

We call the Hermitian matrix \((c_{\alpha \beta})\) from (31) the coefficient matrix of \( r \).

By the methods in [D] we can write

\[
 r(z, \overline{z}) = ||f(z)||^2 - ||g(z)||^2 = \sum_k |f^k(z)|^2 - \sum_j |g^j(z)|^2,
\]

where the functions \( f^k \) and \( g^j \) are holomorphic and linearly independent. When the coefficient matrix has \( P \) positive and \( N \) negative eigenvalues, we obtain (32) with \( P \) functions \( f^k \) and \( N \) functions \( g^j \).

In order to make the ideas more transparent, it is natural to introduce a borded Jacobian. Suppose that \( \{h^k\} \) is a finite or countable collection of holomorphic functions of \( n \) variables. We let \( \mathcal{B}_J(h) \) denote the matrix obtained by adding an extra column as follows:

\[
\mathcal{B}_J(h) = \begin{pmatrix}
 h^1_{z_1} & h^1_{z_2} & \cdots & h^1_{z_n} & h^1_N \\
 h^2_{z_1} & h^2_{z_2} & \cdots & h^2_{z_n} & h^2_N \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 h^N_{z_1} & h^N_{z_2} & \cdots & h^N_{z_n} & h^N_N \\
 \end{pmatrix} = \begin{pmatrix}
 \partial h^1 & h^1 \\
 \partial h^2 & h^2 \\
 \vdots & \vdots \\
 \partial h^N & h^N \\
 \end{pmatrix}.
\]

The connection to bordered Hessians arises when considering real-analytic functions defined by \( \sum |h^k(z)|^2 = r(z, \overline{z}) \). In this case the transpose of the bordered Jacobian is a holomorphic square root of the bordered Hessian. In particular it follows that elements in the kernel of \( \mathcal{B}_J(h) \) are also in the kernel of \( \mathcal{B}(r) \). These ideas lead to the following result.

Theorem 2. Suppose that \( r \) is real-analytic on \( \mathbb{C}^n \), and that its coefficient matrix has rank at most \( n \). Then \( \det(\mathcal{B}(r)) = 0 \). More generally suppose that \( \{h^k\} \) is a finite or countable collection of holomorphic functions such that (with an arbitrary choice of plus-minus signs)

1) The sum \( \sum \pm |h^k(z)|^2 = r(z, \overline{z}) \) converges uniformly on compact subsets of some open set, and therefore \( r \) is real-analytic there.

2) rank(\( \mathcal{B}_J(h) \)) \( \leq n \).

Then \( \det(\mathcal{B}(r)) = 0 \).

Proof. We prove the more general statement first and derive the other from it. Given the collection of holomorphic functions \( \{h^k\} \), we define \( r \) by 1). Since the rank of \( \mathcal{B}_J(h) \) is smaller than \( n + 1 \), we can find holomorphic functions \( a_1, \ldots, a_n \) and \( b \), not simultaneously vanishing on any open set, such that

\[
 \langle \partial h^k, a \rangle + bh^k = 0
\]

(34)
for all $k$. Putting $L = \sum a_j \frac{\partial}{\partial z_j}$ we can rewrite this as $L(h^k) = -b h^k$ for each $k$.

We claim that the ratio $\frac{L(r)}{r}$ is the holomorphic function $-b$. This follows from applying $L$ to $r$:

$$L(r) = \sum \pm L(h^k) \overline{h^k} = -b \sum \pm |h^k|^2 = -br$$

Since $L \in \mathcal{O}^{1,0}$ this condition is equivalent to (27), and by Proposition 5 therefore implies that $\det(\mathcal{B}(r)) = 0$.

In case the matrix of coefficients has rank $d$, we may by linear algebra always write $r(z) = \sum \pm |h^k(z)|^2$ where the $h^k$ are holomorphic and there are at most $d$ terms in the sum. It follows immediately that $\mathcal{B}_J(h)$ has rank at most $d$. By the part already proved we see that $d \leq n$ implies the desired conclusion $\det(\mathcal{B}(r)) = 0$. ♣.

We close this paper with an example.

**Example 2.** Suppose that $g$ is holomorphic on some domain in $\mathbb{C}^2$, and that $|g| < 1$ there. Let $f$ be another holomorphic function on this domain, and define $r$ by

$$r(z) = \frac{|f(z)|^2}{1 - |g(z)|^2} = \sum_{k=0}^{\infty} |(fg^k)(z)|^2$$

Then $\det(\mathcal{B}(r)) = 0$. This follows from Theorem 2. Let $h_k$ be the function $fg^{k-1}$. Then $\text{rank}(\mathcal{B}(h_k))$ is easily seen to be at most 2.

**Bibliography**


