Results in Gevrey and Analytic Hypoellipticity

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In the field of Several Complex Variables, there arise noncoercive boundary value problems for the Laplace operator, which in turn lead, under suitable convexity conditions of the boundary, to subelliptic partial differential equations, whose prototype is

\[-\sum X_j^2 + X_0 + c\]

Here is a brief chronology of classic results in this area.

1963-64: J.J. Kohn strict pseudoconvexity \(\Rightarrow\) \(C^\infty\) hypo- via sub-ellipticity:

\[ \|u\|_{1/2}^2 \leq C(Q(u, u) + \|u\|_{L^2}^2), \quad u \in D(\bar{\partial}^\ast), \]

where

\[ Q(u, v) = (\bar{\partial} u, \bar{\partial} v)_{L^2} + (\bar{\partial} u, \bar{\partial}^\ast v)_{L^2} \]

and

\[ \|v\|_{1/2}^2 \leq C(Q_b(v, v) + \|v\|_{L^2}^2), \]

where

\[ Q_b(u, v) = (\bar{\partial}_b u, \bar{\partial}_b v)_{L^2} + (\bar{\partial}_b u, \bar{\partial}_b^\ast v)_{L^2}. \]

1967: J.J. Kohn and L. Nirenberg subellipticity \(\Rightarrow\) hypoellipticity quite generally:

\[ Lu = -\sum a^{ij}(x) u_{x_i x_j} + \sum a^i u_{x_i} + au \]

with smooth coefficients and \(a^{ij}(x) \geq 0\), then \(L\) is hypoelliptic wherever \(L\) is subelliptic.

1972: M. Derridj & C. Zuily, M. Derridj, D.S. Tartakoff subellipticity implies Gevrey hypoellipticity - for

\[ L_{SS} = -\sum X_j^2 + X_0 + c, \]

(the sum of squares case) the estimate

\[ \|v\|_{1/m}^2 \leq C(\Re(Lv, v)_0 + \|v\|_0^2) \]
implies hypoellipticity in the Gevrey class $G^s, s \geq m$, where near $x_0,$

$$u \in G^s \equiv \exists C : \forall \alpha \left| D^\alpha u \right| \leq C |\alpha|^{s+1} e^{s}$$

However, hopes that $G^1$ hypoellipticity might obtain were dashed by the celebrated counterexample of

M. S. Baouendi & C. Goulaouic (1971): the operator

$$L_{BG} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + x^2 \frac{\partial^2}{\partial t^2}$$

is not analytic hypoelliptic - $\exists u \in G^2 \setminus G^1$ with $L_{BG} = 0$.

1976: G. Komatsu, D.S. Tartakoff global $G^1$ for the $\bar{\partial}$-Neumann problem in strictly pseudo-convex domains.

1976: M. Derridj & D.S. Tartakoff global $G^1$ for $\Box_b$ on similar domains.

1977: D.S. Tartakoff, M. Derridj & D.S. Tartakoff local $G^s$ hypoellipticity, $s > 1,$ and in $C^l = k \log k$ (QA), strictly pseudo-convex.


1981-82: G. Métivier, Séminaire Goulaouic-Meyer-Schwartz no. 12, same, multiple characteristics

ca. 1981: A. Melin - a different proof.

1982, 1983: J. Sjöstrand, another proof, including $\Box_b$ on $\Im w = |z|^4$ in $\mathbb{C}^2$.

1984, 1985: A. Grigis, J. Sjöstrand, local $G^1$ for $\Im w = |z|^4$ and a somewhat more general class of the form $\Sigma^2 X^2$ where

$$X_1 = \frac{\partial}{\partial x_1} - b(x_1, x_2) g(x_1, x_2) \frac{\partial}{\partial t}$$

$$X_2 = \frac{\partial}{\partial x_2} + a(x_1, x_2) g(x_1, x_2) \frac{\partial}{\partial t}$$

where the functions $a, b, c,$ and $g$ are analytic and satisfy

$$(a \frac{\partial}{\partial x_1} + b \frac{\partial}{\partial x_2}) g = cg,$$

$$c + \frac{\partial a}{\partial x_1} + \frac{\partial b}{\partial x_2} \neq 0,$$

and

$$g = 0 \quad \text{iff} \quad (x_1, x_2) = (0, 0).$$
**Corollary:** If \( q(x_1, x_2) \) is quasihomogeneous with weight \((\ell_1, \ell_2)\), and semi-elliptic, then

\[
P = \left( \frac{\partial}{\partial x_1} - \ell_2 x_2 q \frac{\partial}{\partial t} \right)^2 + \left( \frac{\partial}{\partial x_2} + \ell_1 x_1 q \frac{\partial}{\partial t} \right)^2
\]

is locally \( G^1 \) analytic hypoelliptic in \( \mathbb{R}^3 \).

**1988, 1991:** S.-C. Chen global \( G^1 \) (with symmetry).


**Counterexamples**

**1973, 1974:** Oleinik (& Radkeivich): counterexamples to analytic hypoellipticity when certain ODE’s have solutions.

**1991, 1992:** Christ, Hanges & Himonas, Himonas, Christ & Geller: local non-analytic hypoellipticity for Szegö projector, \( \Box_b \), some weakly pseudo convex domains of finite type.

\[
\frac{\partial^2}{\partial x^2} + \left( \frac{\partial}{\partial y} + x^{k-1} \frac{\partial}{\partial t} \right)^2
\]

**1996:** Francsics & Hanges: explicit study of ‘Treves curves’ and the analytic and Gevrey wave front relations for the Szegö kernel for the domains in \( \mathbb{C}^2 \):

\[
\exists w > (\mathbb{R}^2)^m
\]

for even integers \( m \). They show that for \( s \geq m > 2 \) the \( G^s \) Gevrey wave front set of the Szegö projection of a function \( u \) is equal to that of \( u \) together with its translates along the Treves curves.

**1996:** Christ: global analytic counterexample for Szegö.

**1996:** Cordaro/Himonas; Tartakoff: positive global results where local results fail, and some global/local results.

**Non-isotropic, sharp \( G^s \) hypoellipticity**

Consider the Baouendi-Goulaouic operator

\[
P_{BG} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + x^2 \frac{\partial^2}{\partial t^2}.
\]

From the result of Derridj & Zuily, this operator is \( G^2 \) hypoelliptic since we have

\[
||v||_{1/2} \leq C(P_{BG} v) + ||v||_0^2
\]

**Theorem** (1996 Tartakoff, Bove & Tartakoff) \( P_{BG} \) is \( G^{3/2, 1, 2} \) hypoelliptic - that is, defining, near a point,

\[
v \in G^{d_1, d_2, d_3} \iff \exists C_v:
\]

\[
|D_x^\alpha D_y^\beta D_\gamma v| \leq C^{||\alpha|| + ||\beta|| + ||\gamma|| + 1} \alpha_1 d_1 \beta_1 d_2 \gamma_1 d_3
\]
and this result is sharp - it is not hypoelliptic in any non-isotropic Gevrey space $C^{\tilde{d}_1, \ldots, \tilde{d}_s}$ with any $\tilde{d}_j \leq d_j$, $\sum \tilde{d}_j < \sum d_j$.

**Other cases:** Consider the Grušin-type operators

$$P_{12} = \frac{\partial^2}{\partial x^2} + \left( x \frac{\partial}{\partial y} \right)^2 + \left( x^2 \frac{\partial}{\partial t} \right)^2$$

and, with $p \leq q$,

$$P_{pq} = \frac{\partial^2}{\partial x^2} + \left( x^{p-1} \frac{\partial}{\partial y} \right)^2 + \left( x^{q-1} \frac{\partial}{\partial t} \right)^2$$

**Theorem** (Christ 1996) The operator $P_{12}$ is sharply $G^{3/2}$ hypoelliptic.

**Theorem** (Bove & Tartakoff, 1996) The operator $P_{12}$ is sharply $G^{5/4,1,3/2}$ hypoelliptic.

**Theorem** (Hanges & Himonas, 1996(8)) The operator $P_{m,2m}$ has a homogeneous solution which is in no $G^s$ with $s < 2$.

**Theorem** (Christ 1996) The operator $P_{pq}$ is sharply $G^{q/p}$ hypoelliptic.

**Theorem** (Bove & Tartakoff, 1996) The operator $P_{pq}$ is sharply $G^{1-1/q+1/p,1,q/p}$ hypoelliptic.

**Question of Treves** In conjunction with a conjecture of Treves, formulated and refined below, there arose the question of analytic hypoellipticity of the operator

$$D_1^2 + (D_2 + x_1 x_2 x_3 D_3)^2 + x_1^4 D_3^2 + x_2^4 D_3^2.$$  

or, more generally,

$$D_1^2 + (D_2 + a(x_1, x_2, x_3) D_3)^2 + x_1^4 D_3^2 + x_2^4 D_3^2$$  

for any analytic

$$a(x_1, x_2, x_3) = O(x_1^2 + x_2^2).$$

(Treves’ conjecture has to do with the analytic hypoellipticity of operators in terms of the symplecticity of each of a sequence of subvarieties of the cotangent space; two possible definitions of these subvarieties seemed equally reasonable, and they already differ for this example, and hence whether or not this example is ahe would make one choice of the subvarieties possible and the other not.)

Bernardi, Bove & Tartakoff 1997, Christ 1997 (Grušin 1971:) the example is ahe.

For the case of an operator $P = \Sigma X_j^2$, with real vector fields $X_j$ with real analytic coefficients, the ideals whose vanishing defines the subvarieties in question are generated by:

$$I_1 = \{ \sigma(X_j)(x, \xi) \}$$

$$I_2 = \{ \sigma(X_j), \sigma(\{X_j, X_k\}) \}$$
\[ I_3 = \{\sigma(X_j), \sigma(\{X_j, X_k\}), \sigma(\{\{X_j, X_k\}, X_t\})\} \]

etc.

Assuming that each \( \Sigma_j \) is a stratified manifold and that the symplectic form has constant rank on each stratum, Treves’ conjecture may be stated:

**Conjecture** (Treves) The operator \( P \) is analytic hypoelliptic if all \( \Sigma_j \neq \emptyset \) are symplectic.

We have refined this conjecture to read as follows:

**Conjecture** (Bernardi - Bove - Tartakoff) The operator \( P = \sum X_j^2 \) is Gevrey hypoelliptic for all \( s \geq N(P)/\ell(P) \), where \( N(P) \) is the first \( j \) for which \( \Sigma_j = \emptyset \), and \( \ell(P) \) is the first \( j \) for which \( \Sigma_j \) is not symplectic.

Note that when \( \ell = 1 \), this is the result of Derridj & Zúily from 1972.

Christ has given examples which he asserted had better Gevrey regularity than that suggested by our conjecture, and we have published proofs for these examples which are as elementary as those for the simpler cases stated at the beginning. The examples in question are:

**Theorem** Let \( m, h, k \in \mathbb{N} \), \( h \leq m \), and denote by \( P = P(x, t, D_x, D_t) \) the operator
\[
D_x^2 + x^{2m} D_t^2 + x^{2(m-h)} t^{2k} D_t^2;
\]
then \( P \) is \( G^s \)-hypoelliptic at the origin for every \( s \geq s(h, k, m) \), where
\[
s(h, k, m) = \frac{k}{k - \frac{h}{m + 1}}.
\]

and

**Theorem** Let \( m - 1, k, p \in \mathbb{N} \) all be even, \( k \leq m - 1 \), and denote by \( L = L_{m,k,p}(x, t, D_x, D_t) \) the operator
\[
D_x^2 + (x^{m-1} + x^{m-k-1} t^p) D_t^2;
\]
then \( L \) is \( G^s \)-hypoelliptic at the origin for every \( s \geq s(p, k, m) \), where
\[
s(p, k, m) = \frac{pm}{pm - k} = \frac{p}{p - \frac{k}{m}}.
\]

**Gevrey thresholds**

For general operators with given Hörmander numbers, however, our results are unfortunately still sketchy - for example for the operators \( P_1 \) and \( P_2 \) given by
\[
P_1 = D_x^2 + x^{2(p-1)} (D_t + x^{q-p} D_s)^2 + x^{2(q-1)} D_s^2
\]
$P_1 = D_x^2 + x^{2(p-1)} (D_t + x^{q-p} D_z)^2 + x^{2(q-1)} D_t^2,$

we have

**Theorem** Let $q \geq p \geq 1$. Then

(a) $P_1$ is $G^{q/p}$-hypoelliptic.

(b) If $q \geq 2p$, then $P_2$ is $G^{q/p}$-hypoelliptic.

(c) If $p \leq q < 2p$, $P_2$ is $G^{3-2(q/p)}$-hypoelliptic.

**Homogeneity questions**

Christ, in a 1998 paper from the conference in honor of Pierre Lelong, has suggested that one might do well to study examples where the Levi form is given by non-polyhomogeneous functions such as

$x^k + y^k + x^2 y^2$

for $k > 4$ but even. In some polyhomogeneous cases this had been addressed by Grigis and Sjöstrand in 1985, as noted above, but not in general.

In 1988, 1991, and later, Derridj and Tartakoff (CPDE 1988, 1521-1600) proved local analyticity for $\Box_b$ on a variety of weakly pseudoconvex domains in $\mathbb{C}^n$, using elementary, $L^2$ methods. Included among these were some domains considered by Sjöstrand (using the FBI transform), namely domains whose boundary is given by

$\Im w > h(|z|^2),$

with $h$ real analytic and zero only at 0, and many domains where Sjöstrand’s hypotheses were not satisfied, namely

$\Im w > |z'|^2 + h(|z''|^2),$

These proofs makes no use of homogeneity per se, and obtain their flexibility by using *a priori* estimates.

As usual, the crucial step is that to localize to a neighborhood of $|z| = |w| = 0$, one need only consider localizing functions $\phi(\Re w)$ since any derivatives on a function of $z$ may be taken to be supported in a region where the Levi form is non-degenerate, and use the 1978 results of Treves and Tartakoff cited above. And when one of the vector fields $X_j$ lands on such a localizing function, it deposits the coefficient of $\partial / \partial t, t = \Re w$, which occurs in $X_j$. Finally, in order to ‘correct’ this derivative on a localizing function without a full gain in powers of $\partial / \partial \Re w$, this coefficient must be reasonably divisible by the Levi form. This is the content of the hypotheses of Grigis and Sjöstrand.

Work in progress with A.Bove is concerned with generalizing the $L^2$ methods of Derridj-Tartakoff from 1988, 1991, etc. to non-homogeneous cases.

**Sketches of proofs**
We start with the Baouendi-Goulaouic example

$$L_{BG} = \sum_{j=1}^{3} X_j^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + x^2 \frac{\partial^2}{\partial t^2}$$

For this case, $G^2$ behavior in $t$ is known and is optimal - let us accept it. In the variable $x$, we must estimate $\phi(x, y, t)D_x^r$ (in $L^2$ norm), and observe that derivatives in $x$ landing on $\phi$ are of no consequence since for $x \neq 0$, the operator is elliptic and hence analytic hypoelliptic. The brackets that occur in

$$\sum_{j=1}^{3} \|X_j v\|_2^2 + \|v\|_{1/2}^2 \leq C\{(Lv, v) + \|v\|_0^2\}$$

with $v = \phi(x, y, t)D_x^r u$ are essentially

$$\sum_{j=1}^{3} \|[X_j, \phi(x, y, t)D_x^r]u\|_0^2.$$

The most difficult of these is when $j = 3$, and this gives rise to two kinds of terms, one with $[X_3, \phi]u = x\phi_t u$, in which case we bring another $D_x = X_1$ to the left and iterate. This is a gain in power of $D_x$, one derivative on $\phi$, a good trade if for $\phi$ one uses the functions introduced by Ehrenpreis in 1960 and used heavily in our work and that of Hörmander: that is, for any pair of open sets $\omega$ compactly contained in $\Omega$, there exists a constant $C_0$ and, for any $N$, a $C_0^\infty(\Omega)$ function $\phi_N \equiv 1$ on $\omega$ and such that for $|\alpha| \leq N$,

$$|D^\alpha \phi_N| \leq C_0 C_0^{|\alpha|} N^{|\alpha|}.$$

Observe that when $|\alpha|$ is close to $N$, these bounds resemble the bounds one would have if $\phi$ were analytic, since

$$N^N \leq C N^{1/2} e^{N^2} N! \leq C^{N+1} N!$$

by Stirling's formula.

The other bracket that enters is when $[X_3 = x D_t, \phi(x, y, t)D_x^r]$ generates

$$\phi[x, D_x^r]D_t = x\phi D_x^{r-1} D_t.$$

Iterating this leads to $C^r r!! \phi D_x^{r/2} u$ (where $r!! = r(r - 2)(r - 4) \ldots$), and, together with the $G^2$ behavior in the variable $t$, leaves $G^{3/2}$ in $x$.

Finally in $y$, the above brackets yield only analytic-type growth: $[X_j, \phi D_y^r]$ contains only $x\phi_t D_y^r \sim x D_y \phi_t D_y^{r-1}$, which is fine even without the factor of $x$ upon iteration and $\phi_y D_y^r \sim D_y \phi_y D_y^{r-1}$, which is just as good, and leads to $||\phi^r u||_{L^2}$, which, as we’ve seen above, is as good as $C^{r+1} r!!$ when $\phi = \phi_N$ with $N \sim r$.

For the sharpness in each index, we consider the function

$$u_\epsilon = \int_0^\infty e^{\text{exp}[i \epsilon^2 s - t \rho - \rho^2 x^2 / 2 - \rho^t]} d\rho$$
for $\varepsilon > 1$ which solves $L_{BG} u_\varepsilon = 0$ yet brief calculations show that $u_\varepsilon$ satisfies

$$|\partial_t^k u_\varepsilon(0)| = \left| \int_0^\infty e^{-\rho^\varepsilon \rho^k d\rho} \right| \sim C^k k!^{1/\varepsilon},$$

$$|\partial_s^k u_\varepsilon(0)| = \left| \int_0^\infty e^{-\rho^\varepsilon \rho^{2k} d\rho} \right| \sim C^k k!^{2/\varepsilon},$$

and

$$|\partial_x^{2k} u_\varepsilon(0)| = \left| \int_0^\infty e^{-\rho^\varepsilon \rho^{2k} d\rho} \right| \sim C^k k!^{1+2/\varepsilon} \sim C^k (2k)!^{1/2+1/\varepsilon},$$

showing that for any $\varepsilon > 1$,

$$u_\varepsilon \in G^{1/2+1/\varepsilon,1/\varepsilon,2/\varepsilon}$$

and no better.

For the general case, the reasoning is similar but we need the solve a particular O.D.E. in a way that goes back to Oleinik-Radkeivich (1974) and has been used since that time by a number of authors:

In considering the operator of Grušin type

$$P_{pq} = \frac{\partial^2}{\partial x^2} + \left( x^{p-1} \frac{\partial}{\partial y} \right)^2 + \left( x^{q-1} \frac{\partial}{\partial t} \right)^2,$$

we know from Oleinik & Radkeivich that there exists a real number $\varepsilon$ such that the ordinary differential equation

$$u'' = x^{2(q-1)} u - x^{2(p-1)} u,$$

has a non trivial solution $u$ defined on the whole real line and rapidly decreasing at infinity.

Repeated differentiation of the O.D.E. shows that we have

$$|u^j(0)| \leq C^{1+j} j!^{1-\frac{1}{\varepsilon}},$$

for every $j \in \mathbb{N}$, where $C$ is a suitable positive constant.

Now we define the function $u_\varepsilon(x,t,s)$ by

$$\int_0^{+\infty} \exp(i\rho s + i\sqrt{\varepsilon} \rho^{1/2} t - \rho^\varepsilon) u(\rho^{1/2} x) d\rho$$

for any $\varepsilon > \frac{p}{q}$. This function is easily seen to be a solution of the differential equation. Yet brief computations show that $u_\varepsilon$ satisfies

$$|\partial_t^j u_\varepsilon(0)| \sim C \int_0^{+\infty} e^{-\rho^\varepsilon \rho^{2j} d\rho} \sim C^j j!^{1/\varepsilon},$$
\[ |\partial_z^j u_\varepsilon(0)| \sim C \int_0^{+\infty} e^{-\rho^\varepsilon} \rho^j d\rho \sim C^j j!^{1+\varepsilon}, \]

and

\[ |\partial_t^j u_\varepsilon(0)| = |v^{(j)}(0)| \int_0^{+\infty} e^{-\rho^\varepsilon} \rho^j d\rho \sim C^j \left| v^{(j)}(0) \right| j!^{1+\varepsilon} \sim C^j j!^{1-\frac{1}{p}+\frac{1}{q}+\varepsilon}, \]

showing that, for any \( \varepsilon > \frac{p}{q} \),

\[ u_\varepsilon \in G^{1-\frac{1}{q}+\frac{1}{p},\frac{1}{q},\frac{1}{p}} \]

and no better.

Finally, for sharpness in each index, we merely need note that, for example,

\[ u_\varepsilon(x, t, s) \in G^{1-\frac{1}{q}+\frac{1}{p},\frac{1}{q},\frac{1}{p}} \setminus G^{1-\frac{1}{q}+\frac{1}{p},\frac{1}{q},\frac{1}{p}-\delta,\frac{1}{q},\frac{1}{p}} \]

for any \( \delta > 0 \), so that this operator cannot be hypoelliptic in \( G^{d_1,\frac{1}{q},\frac{1}{p}} \) for any \( d_1 < 1-\frac{1}{q}+\frac{1}{p} \) and similarly in the other two places.

**The other cases**

Proofs in the other cases are very similar. Some added subtlety comes from balancing different terms that have different powers of \( x \). For example, for the operator \( P_{pq} = \sum_1^3 X_j^2 \):

\[ P_{pq} = \frac{\partial^2}{\partial x^2} + \left( x^{p-1} \frac{\partial}{\partial t} \right)^2 + \left( x^{q-1} \frac{\partial}{\partial s} \right)^2, \]

the a priori estimate we use is:

\[ \sum_1^3 \left| \langle X_j v \rangle \right|^2 + \left| \| v \| \right|^2 \leq \]

\[ \leq C \left( \| \operatorname{Re}(P_{pq} v, v) \|_0 + \| v \|^{2} \right) \]

and setting \( v = \phi D_x^r u \) as before, with \( \phi_x \) supported in the elliptic region, we find the principal errors are

\[ \| [X_j, \phi D_s^r] u \|_{L^2}^2. \]

Initially \( j = 1 \) causes no trouble (ellipticity for \( x \neq 0 \)) but \( j = 2 \) creates the term

\[ \| x^{p-1} \phi D_s^r u \|_{L^2}^2 \]

and \( j = 3 \) a similar one, with a better power of \( x \) and an equivalent derivative on \( \phi \). On the next round, using the subelliptic part of the a priori estimate, we may obtain as errors either

\[ \| x^{2p-2} \phi'' D_s^{-1/2} u \|_{L^2}^2 \]
(from $j = 2, 3$) or, from $j = 1$,

$$(p-1) \left\| x^{p-2} \phi_t D_s^{r-1/q} \right\|_{L^2}^2.$$

For the former of these, once the powers of $x$ reach $q-1$, we may pull out a $D_s$ since $x^{q-1}D_s = X_3$, on which we have maximal control.

The latter of the two errors above has decreasing powers of $x$, and the only salvation here is that this term no longer occurs once the power of $x$ has been reduced to zero; the next iteration is forced to raise the powers of $x$ significantly.

In fact one can already see here that the effect of differentiating the powers of $x$ until they are gone produces a decrease in power of $D_s$ of roughly $p/q$ for a single derivative on $\phi$, which translates readily into the Gevrey class $G^{q/p}$.

For the former type of error, the powers of $x$ will mount until they combine with a $D_s$ to give $X_3$ at the same rate, though this is not quite so clear, since there are more derivatives on $\phi$; if one is lucky and, say, $d(p-1) = q-1$ for some $d$, then this term will iterate to

$$\left\| x^{q-1}\phi^{(d)} D_s^{r-(d-1)/q} u \right\|_{L^2}^2 =$$

or a gain of $1 + (d-1)/q$ powers of $D_s$ at the expense of $d$ derivatives on the localizing function. Normalized, this reduces to a gain of $\frac{1+(d-1)/q}{d}$ per derivative on $\phi$, or, inversely, $\frac{d}{1+(d-1)/q}$ derivatives on $\phi$ for each gain of one $D_s$. When one unravels the fractions, bearing in mind that $d = \frac{q-1}{p-1}$, this (seemingly miraculously) reduces to $q/p$ derivatives on $\phi$ per gain of one in $D_s$ — and this rapidly leads to $G^{q/p}$ hypoellipticity.

The mixed cases are naturally mixtures of these arguments, which can be simplified formally using some seemingly complicated norms in the style of Grušin.

Finally, for the last two theorems (pp. 13, 14), the arguments are similar - the parity of the exponents enters in the estimates in a simple way - it is easy in these cases to bound, for instance, either $|x^{m-1} D_t w|$ or $|x^{m-k-1} t^p D_t w|$ by the combination present in the operator:

$$|(x^{m-1} + x^{m-k-1} t^p) D_t w| = |X_2 w|.$$

Whether, as Christ claims, the parity restrictions on the exponents are indeed an artifact of his proof remains to be seen. That they enter so naturally and trivially in this approach suggests that they may be important.

**Sketch of methods of work in progress with Bove**

We recall how to ‘correct’ localizations of high derivatives of $T = \frac{\partial}{\partial t}$ in earlier work. For the left invariant vector fields on the Heisenberg group, $X, Y$, we wrote

$$T_\phi^p = \sum_{r = |\alpha + \beta| \leq p} \frac{(-1)^{|\alpha|}}{\alpha! \beta!} \phi X^\alpha Y^\beta X^\beta Y^\alpha T^{p-r}$$

10
where \( \phi_{X^\alpha Y^\beta} = X^\alpha Y^\beta(\phi) \). This expression had excellent brackets with both \( X \) and \( Y \):

\[
[X, T_p] = 0, \quad [Y, T_p^\phi] = T_p^{\phi - 1} \circ Y,
\]

modulo \( C_p \) terms of the form \( \phi^{(p+1)}Z^p/p! \) where each \( Z \) may be either \( X \) or \( Y \).

When one is in a degenerate situation and the brackets are not so simple, higher order brackets must be considered to ‘correct’ the errors. Thus even in the simplest degenerate case, with say the vector fields \( X, Y \) given by

\[
X = \frac{\partial}{\partial x} - g_y(x, y) \frac{\partial}{\partial t}, \\
y = \frac{\partial}{\partial x} + g_x(x, y) \frac{\partial}{\partial t},
\]

and \( \lambda = \Delta_{x,y} \lambda = g_{xx} + g_{yy}, \) while we need primarily consider how these vector fields act on \( \phi(t) \), the coefficients are more complicated than in the Heisenberg case (where \( g_x = x, g_y = y \))

To correct

\[
[X, \phi T_p] = -g_y \phi_t T_p,
\]

one uses, first of all,

\[
\phi T_p - \frac{g_y}{\lambda} \phi_t T_p^{p-1} Y,
\]

since

\[
[X, \phi T_p - \frac{g_y}{\lambda} \phi_t T_p^{p-1} Y] = -(\frac{g_y}{\lambda})_x \phi_t T_p^{p-1} Y +
\]

\[
\frac{g_y}{\lambda} g_y \phi_{tt} T_p^{p-1} Y
\]

and to correct (the worst part of) this one adds

\[
\frac{1}{2!} \left( \frac{g_y}{\lambda} \right)^2 \phi_{tt} T_p^{p-2} Y^2,
\]

but here the trouble begins, since in the correcting factor \( [X, Y^2] \) there will occur non-trivial double brackets; and in general, instead of

\[
\frac{1}{k!} \left( \frac{g_y}{\lambda} \right)^k \phi^{(k)} T_p^{p-k} Y^k,
\]

one needs a whole polynomial in \( Y \):

Defining

\[
W_k = \sum_{\ell} A_{\ell}^k(x, y) Y^\ell,
\]

the conditions on the coefficients \( A_{\ell}^k(x, y) \) become

\[
[X, W_k] = W_{k-1} T
\]
exactly.

This can be achieved, but as in the 1988 paper with Derridj, the coefficients are closely related to the Bernoulli numbers and recurrence relations for them with shifting indices, as I well recall speaking about at the Stoll Fest at Notre Dame some 10 years ago. The actual coefficients at that time were essentially

\[ A^k_t = \left( \frac{t}{e^t - 1} \right)^{k+1} \frac{(k - \ell)}{(k - \ell)!} \]

And while this approach may seem calculational, it has the advantage of not being tied to homogeneous or even polyhomogeneous functions.