Some Results on the Structure of Quantum Family Algebras

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SOME RESULTS ON THE STRUCTURE
OF QUANTUM FAMILY ALGEBRAS

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Abstract. We continue the study of family algebras introduced by the author. In this paper we describe completely the structure of quantum family algebras for two cases of representations with a simple spectrum.

1. Generalities about family algebras

1.1. Basic definitions.

A new class of associative algebras related to simple complex Lie algebras (or root systems) was introduced and studied in [K]. They were named classical and quantum family algebras.

The aim of the this paper is to expose some results about the structure of quantum family algebras. In particular, we give a partial answer to the last of several open questions formulated in [K]:

In general, it would be very interesting to find out which quantum family algebras are commutative and which classical algebras are spanned over $I(\mathfrak{g})$ by powers of $M$ or analogous elements related to other generators of $I(\mathfrak{g})$.

We assume that the reader is acquainted with the general background of the theory of semi-simple Lie algebras (see e.g. [OV]).

Let $\mathfrak{g}$ be a simple complex Lie algebra with the canonical decomposition

$$\mathfrak{g} = n_- \oplus \mathfrak{h} \oplus n_+.$$ 

We denote by $P$ (respectively by $Q$) the weight (resp. root) lattice in $\mathfrak{h}^*$ and by $P_+$ (resp. $Q_+$) the semigroup generated by fundamental weights $\omega_1, \omega_2, \ldots, \omega_l$ (resp. by simple roots $\alpha_1, \alpha_2, \ldots, \alpha_l$).

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For every $\lambda \in P_+$ let $(\pi_\lambda, V_\lambda)$ be an irreducible representation of $\mathfrak{g}$ with highest weight $\lambda$. We denote by $d(\lambda)$ the dimension of $V_\lambda$.

Let $\lambda^*$ denote the highest weight of the dual (or contragredient) representation which acts in $V_\lambda^*$ by $\pi_{\lambda^*}(X) = -(\pi_\lambda(X))^*$. It is clear that $d(\lambda) = d(\lambda^*)$.

The space $\text{End} V_\lambda \simeq V_\lambda \otimes V_\lambda^*$ is isomorphic to the matrix space $\text{Mat}_{d(\lambda)}(\mathbb{C})$ and has a $\mathfrak{g}$-module structure defined by

$$X \cdot A = [\pi_\lambda(X), A].$$

Recall that the symmetric algebra $S(\mathfrak{g})$ and the enveloping algebra $U(\mathfrak{g})$ also have (isomorphic) $\mathfrak{g}$-module structures.

Let $G$ be a connected and simply connected Lie group with $\text{Lie}(G) = \mathfrak{g}$. The action of $\mathfrak{g}$ on $\text{End} V_\lambda$, $S(\mathfrak{g})$ and $U(\mathfrak{g})$ gives rise to the corresponding action of $G$ by automorphisms of these algebras.

We define two kinds of family algebras: the classical family algebra $C_\lambda(\mathfrak{g})$ and the quantum family algebra $Q_\lambda(\mathfrak{g})$ as

$$(1) \quad C_\lambda(\mathfrak{g}) := (\text{End} V_\lambda \otimes S(\mathfrak{g}))^G, \quad Q_\lambda(\mathfrak{g}) := (\text{End} V_\lambda \otimes U(\mathfrak{g}))^G.$$

It turns out that several important questions in the theory of semi-simple Lie algebras and their representations can be formulated, studied and sometimes solved in terms of these family algebras.

### 1.2. Family algebras related to a representation with a simple spectrum.

The following result was obtained in [K].

**Theorem 1.** Assume that $\pi_\lambda$ has a simple spectrum (i.e. all weights have multiplicity 1). Then the algebra $C_\lambda(\mathfrak{g})$ is commutative.

For the convenience of the reader we give here the proof.

Let us consider an element $A \in C_\lambda(\mathfrak{g})$ as a polynomial map $A : \mathfrak{g}^* \to \text{End} V_\lambda$ which is equivariant with respect to the action of $G$. It is enough to check the commutativity of $A(F)$ and $B(F)$ for generic $F \in \mathfrak{g}^*$. Since generic element of $\mathfrak{g}^* \cong \mathfrak{h}$ is conjugate to an element of the Cartan subalgebra, we can assume that $F \in \mathfrak{h}$. We identify $\text{End} V_\lambda$ with $\text{Mat}_{d(\lambda)}(\mathbb{C})$ using the weight basis in $V_\lambda$. The simplicity of the spectrum of $\pi_\lambda$ implies that values of $A(F)$ and $B(F)$ commute with some diagonal matrices with distinct eigenvalues. Therefore they themselves are diagonal matrices, hence commute.

\[ \square \]

Recall now that the special element $M$ was introduced in [K] both in quantum and classical family algebra. Namely

$$(2) \quad M := \pi_\lambda(X_i) \otimes X^i$$

where $\{X^i\}$ and $\{X_i\}$ are dual bases in $\mathfrak{g}$ with respect to some $\text{Ad}$-invariant bilinear form.

This element belongs to $C_\lambda(\mathfrak{g})$ (resp. to $Q_\lambda(\mathfrak{g})$) if we interpret $X^i$ as an element of $S(\mathfrak{g})$ (resp. of $U(\mathfrak{g})$).
The most investigated examples of representations with a simple spectrum are the symmetric powers \( (\pi_{k\omega_1}, V_{k\omega_1}) \) of the standard representation of \( \mathfrak{sl}(N) \) or \( \mathfrak{gl}(N) \).
(Of course, the dual representations have the same property).

We consider here two simplest examples:
1) the standard representation of \( \mathfrak{sl}(N) \) or \( \mathfrak{gl}(N) \).

2) the \( N \)-dimensional irreducible representation of \( \mathfrak{sl}(2) \).

In all these examples the classical family algebra is generated over the subalgebra of scalar matrices by the element \( M \). Thus, it is an algebraic extension of the polynomial algebra \( \mathbb{C}[\mathfrak{g}]^G \) defined by the Hamilton – Cayley identity:

\[
M^N = \sum_{k=1}^{N} c_k \cdot M^{N-k}
\]

where \( c_k = (-1)^{k-1} \text{tr} \wedge^k M \) are \( G \)-invariant polynomials on \( \mathfrak{g} \).

The quantum family algebra is a deformation of the classical one, since \( U(\mathfrak{g}) \) is a deformation of \( S(\mathfrak{g}) \).

It follows that the quantum family algebra \( \mathcal{Q}_\lambda(\mathfrak{g}) \) is an algebraic extension of the center \( Z(\mathfrak{g}) \) of \( U(\mathfrak{g}) \) defined by the quantum analogue of the Hamilton – Cayley identity. This analogue has the same form (3) but with different coefficients \( c_k \). Below we shall describe explicitly these coefficients for the two series of examples mentioned above.

2. \textbf{QUANTUM FAMILY ALGEBRA AND LAPLACE-CASIMIR OPERATORS FOR }\textit{GL}(\textit{N}).

2.1. Problem setting.

Let \( \mathcal{A} \) be the quantum family algebra for the Lie algebra \( \mathfrak{g} = \mathfrak{gl}(N, \mathbb{C}) \) and the irreducible representation \( \pi_{\omega^*_1} \), dual to the standard one. Recall that \( \mathcal{A} \) consists of \( N \times N \) matrices \( A \) with entries from \( U(\mathfrak{gl}(N, \mathbb{C})) \) satisfying the relation

\[
(gA g^{-1})_{i,j} = \text{Ad}_g A_{i,j} \quad \text{for any } g \in GL(N, \mathbb{C}).
\]

Let \( M \) denote the element of \( \mathcal{A} \) given by

\[
M = -\sum_{i,j} E_{i,j} \otimes \pi_{\omega^*_1}(E_{i,j}) = \begin{pmatrix} E_{11} & E_{12} & \ldots & E_{1N} \\
E_{21} & E_{22} & \ldots & E_{2N} \\
\ldots & \ldots & \ldots & \ldots \\
E_{N1} & E_{N2} & \ldots & E_{NN} \end{pmatrix}
\]

Let \( (\pi_\lambda, V_\lambda) \) be a finite dimensional irreducible representation of \( \mathfrak{g} \) with the highest weight \( \lambda \) and highest weight vector \( v_\lambda \). We denote by \( (\pi_\lambda^*, V_\lambda^*) \) the dual representation and by \( v_\lambda^* \) the lowest weight vector in \( V_\lambda^* = \bar{V}_\lambda^* \) normalized by the condition

\[
\langle v_\lambda^*, v_\lambda \rangle = 1.
\]

The well known Harish-Chandra map from \( U(\mathfrak{g}) \) to \( U(\mathfrak{h}) \simeq S(\mathfrak{h}) \simeq \mathbb{C}[\lambda_1, \ldots, \lambda_N] \) can be defined as

\[
\beta(A) = \langle v_\lambda^*, \pi_\lambda(A)v_\lambda \rangle.
\]

For technical reason it is convenient to introduce the order in \( \mathbb{R}^N \) which is opposite to the standard alphabetical order. Then dominant weights \( \lambda \) will satisfy \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \). The advantage is that the recurrent formula below will not contain \( N \) explicitly.
It is known that the image of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ under $\beta$ consists of all polynomials which are invariant under the **twisted action** of the Weyl group $W$. In our case $W \simeq S(N)$ and the action is

\[(8) \quad s \cdot \lambda = s(\lambda + \tilde{\rho}) - \tilde{\rho}, \quad \text{where} \quad \tilde{\rho} = (0, 1, \ldots, N - 2, N - 1).\]

In other words,

\[(8') \quad (s \cdot \lambda)_i = \lambda_{s^{-1}(i)} + i - s^{-1}(i).\]

**Remark 1.** Note that our vector $\tilde{\rho}$ differs from the commonly used half sum of positive roots $\rho = (\frac{1-N}{2}, \frac{3-N}{2}, \ldots, \frac{N-1}{2})$ by the vector $\frac{N-1}{2} \cdot (1, 1, \ldots, 1)$ which is $W$-invariant. Of course, the difference disappears when we pass to a simple subalgebra $\mathfrak{s}(N, \mathbb{C}) \subset \mathfrak{g}(n, \mathbb{C})$.

★

It is easy to derive from (4) that the elements $T_k = \text{tr} (M^k)$ belong to $Z(\mathfrak{g})$. We are interested in the explicit formula for the polynomials $\Delta_k(\lambda) = \beta(T_k)$ which are symmetric functions in the variables $\mu_i = \lambda_i + \tilde{\rho}_i = \lambda_i + i - 1$.

Actually, this expression can not be very simple. E.g., in [PP] it is expressed as a sum of all matrix elements of the $N$-th power of a certain $N \times N$ matrix. Similar results were obtained by [O], [S], and others. The approach via family algebras allows to give a transparent interpretation of the formula.

**Remark 2.** According to well-known Arnold’s Law no formula bears the name of its discoverer (including the Arnold’s Law itself). During my study of quantum family algebras I found several relatively manageable expressions for $\Delta_k$. Some of them I later found in the literature but not all.

Since my computations were finished on 24 August of 2000, the 26-th birthday of my best friend Pavel Ilyin, I suggest to call the result **Ilyin’s formula**.

★

**Theorem 2.** The polynomials $\Delta_k(\lambda) = \beta(T_k)$ are given by the following Ilyin formula:

\[(\text{Ilyin}) \quad \Delta_k(\mu) = \sum_{s \geq 0} (-1)^s \sum_{1 \leq i_1 < \cdots < i_{s+1} \leq N} h_{k-s}(\mu_{i_1}, \ldots, \mu_{i_{s+1}}).\]

Here $h_k$ denotes the standard full symmetric function of degree $k$: the sum of all different monomials of that degree.
In particular, we have:

\[ \Delta_0(\lambda) = N; \quad \Delta_1(\lambda) = \sum_i \mu_i - \binom{N}{2} = \sum_i \lambda_i; \]

\[ \Delta_2(\lambda) = \sum_i \mu_i^2 - \sum_{i<j} (\mu_i + \mu_j) + \binom{N}{3} = \sum_{i=1}^{N} (\lambda_i + \rho_i)^2 - \sum_{i=1}^{N} \rho_i^2; \]

\[ \Delta_3(\lambda) = \sum_i \mu_i^3 - \sum_{i<j} (\mu_i^2 + \mu_i \mu_j + \mu_j^2) + \sum_{i<j<k} (\mu_i + \mu_j + \mu_k) - \binom{N}{4} = \]

\[ \sum_{i=1}^{N} \left[ \mu_i^3 - \left( N - \frac{3}{2} \right) \mu_i^2 + \frac{(N-1)(N-2)}{2} \mu_i \right] - \frac{1}{2} \left( \sum_{i=1}^{N} \mu_i \right)^2 - \binom{N}{4} = \]

\[ \sum_i (\lambda_i + \rho_i)^3 - \frac{N}{2} \left( \sum_i (\lambda_i + \rho_i)^2 - \sum_i \rho_i^2 \right) - \frac{N^2 - 1}{4} \sum_i \lambda_i - \frac{1}{2} \left( \sum_i \lambda_i \right)^2. \]

Sometimes it is useful to rewrite this formula using another basis in the space of symmetric polynomials, that of monomial symmetric functions:

\[ m_{\delta}(\mu) = \text{sum of all different monomials of the form } \mu_{i_1}^{\delta_1} \mu_{i_2}^{\delta_2} \cdots \mu_{i_k}^{\delta_k}. \]

It looks as

\[ \Delta_k(\mu) = \sum_{0 \leq |\delta| \leq k} (-1)^{k-|\delta|} \binom{N - l(\delta)}{k - |\delta| - l(\delta) + 1} m_{\delta}(\mu). \]

Further, one can introduce the generating function

\[ \Delta(t; \mu) := \sum_{k=0}^{\infty} t^k \cdot \Delta_k(\mu). \]

In terms of this function the Ilyin formula takes the form

\[ \Delta(t; \mu) = t^{-1} - t^{-1} \prod_{i=1}^{N} \left( 1 - \frac{t}{1 - t \mu_i} \right) \]

as one easily derives using the identity

\[ \sum_{k=0}^{\infty} t^k \mu_k(\{\mu_i, i \in I\}) = \prod_{i \in I} (1 - t \mu_i)^{-1}. \]

(See details in the proof below).

The last form allows to represent the sequence \( \Delta_k(\mu) \) as a sum of geometric progressions with ratio \( \mu_i, 1 \leq i \leq N \). Namely,

\[ \Delta_k(\mu) = \sum_{i=1}^{N} a_{i}^{(N)} \cdot \mu_i^k \quad \text{where} \quad a_{i}^{(N)} = \prod_{i \neq j, 1 \leq j \leq N} \left( 1 + \frac{1}{\mu_j - \mu_i} \right). \]

This form of the result was actually known (see [O]).
2.2. The connection between different variants of the result.

To derive the equality (12) from Ilyin formula we introduce the rational function

\[ F(z) = - \prod_{j=1}^{N} \left( 1 - \frac{1}{z - \mu_j} \right), \]

where \( \mu_1, \ldots, \mu_N \) are complex parameters.

It is evident that \( \lim_{z \to \infty} F(z) = -1 \) and the direct computation shows that

\[ \text{Res}_{z=\mu_i} F = \prod_{j \neq i} \left( 1 + \frac{1}{\mu_j - \mu_i} \right) = a_i^{(N)}. \]

Therefore

\[ F(z) = -1 + \sum_{i=1}^{N} \frac{a_i^{(N)}}{z - \mu_i}. \]

We introduce one more parameter \( t \) and consider the integral

\[ I = \frac{1}{2\pi i} \oint_{|z|=R} \frac{F(z)}{1 - tz} \, dz \]

where \( R \) is supposed to be greater than \( \max |\mu_i| \) but less than \( t^{-1} \).

We can compute the integral (14) via the residues formula. Since our function \( F \) has inside the disc \( |z| \leq R \) only simple poles at \( \mu_1, \ldots, \mu_N \), we get the result

\[ I = \sum_{i=1}^{N} \frac{a_i^{(N)}}{1 - t\mu_i} = \sum_{k \geq 0} t^k \sum_{i=1}^{N} a_i^{(N)} \mu_i^k. \]

But we can also use the fact that outside the disc \( F \) has a simple pole at \( t^{-1} \)
and a non-zero residue at \( \infty \). Then we come to the expression

\[ I = -\text{Res}_{z=\infty} F - \text{Res}_{z=t^{-1}} F = t^{-1} \prod_{j=1}^{N} \left( 1 - \frac{t}{1 - t\mu_i} \right). \]

This expression can be rewritten in the form

\[ t^{-1} \left( 1 - \sum_{I \subseteq [1, \ldots, N]} \prod_{i \in I} \frac{-t}{1 - t\mu_i} \right) = \sum_{s \geq 0} (-t)^s \cdot \sum_{|I| = s} h_{k-s}(\{\mu_i, i \in I\}). \]

Comparing the two expressions for the integral, we obtain the identity

\[ \sum_{i=1}^{N} a_i^{(N)} \mu_i^k = \sum_{s \geq 0} (-t)^s \cdot \sum_{|I| = s+1} h_{k-s}(\{\mu_i, i \in I\}). \]

Thus, we proved that the Ilyin formula is equivalent to (12).
2.3. Recurrence relation.

We shall use the notation similar to those introduced in [MNO]; the matrix elements of \( M^k \) are denoted by \( E_{i,j}^{(k)} \), \( k \geq 0 \).

The elements \( E_{i,j}^{(k)} \in U(\mathfrak{g}) \) have zero weight. Therefore they commute with elements of \( U(\mathfrak{h}) \) and for any representation \((\pi_\lambda, V_\lambda)\) the operator \( \pi_\lambda(E_{i,j}^{(k)}) \) preserves the weight decomposition of \( V_\lambda \). In particular, there is a constant \( c_{i,j}^{(k)}(\lambda) \) such that

\[
\pi_\lambda(E_{i,j}^{(k)})v_\lambda = c_{i,j}^{(k)}(\lambda)v_\lambda.
\]

We shall use the equality

\[
E_{i,i}^{(k+1)} = \sum_s E_{i,s}^{(k)}E_{s,i}
\]

to derive a recurrence relation for the quantities \( c_{i,j}^{(k)} \). For this end we note that \( E_{i,j}v_\lambda = 0 \) if \( i > j \). So, when we apply both sides of (16) to \( v_\lambda \), it is enough to take the summation in the right hand side only for \( s \leq i \).

Moreover, for \( s = i \) we get \( E_{i,i}^{(k)}E_{i,i}v_\lambda = c_{i,j}^{(k)}(\lambda) \cdot \lambda_i \cdot v_\lambda \) and for \( s < i \) we have \( E_{i,s}^{(k)}v_\lambda = 0 \), hence \( E_{i,s}^{(k)}E_{s,i}v_\lambda = [E_{i,s}^{(k)}, E_{s,i}]v_\lambda = (E_{i,i}^{(k)} - E_{s,s}^{(k)})v_\lambda \). It gives the desired recurrence relation

\[
c_{i}^{(k+1)}(\lambda) = (\lambda_i - 1 + i) \cdot c_{i}^{(k)}(\lambda) - \sum_{1 \leq s < i} c_{s}^{(k)}(\lambda)
\]

or

\[
c_{i}^{(k+1)}(\mu) = \mu_i \cdot c_{i}^{(k)}(\mu) - \sum_{1 \leq s < i} c_{s}^{(k)}(\mu).
\]

Note, that the quantity \( c_{i}^{(k)}(\mu) \) depends only on \( \mu_1, \ldots, \mu_i \) as well as the sum \( s_{i}^{(k)} = \sum_{i=1}^{n} c_{i}^{(k)} \).

We shall prove the Ilyin formula by induction on \( k \) and \( N \). We rewrite it in terms of quantities \( s_{i}^{(k)} \) using (12):

\[
s_{N}^{(k)} = \sum_{j=1}^{N} a_{i}^{(N)} \mu_i^k.
\]

For \( k = 0 \) it reduces (see (9)) to \( \sum_{i=1}^{N} a_{i}^{(N)} = N \). It follows from (13) and the relation \( \lim_{z \to \infty} z(F(z) + 1) = N \).

Assume that it is true for \( k \leq n \) and all \( N \) and also for \( k = n + 1 \) and the number of variables \( < N \). Our goal is to check it for \( k = n + 1 \) and \( N \) variables. We shall use the relation

\[
c_{i}^{(k)}(\mu) = s_{i}^{(k)}(\mu_1, \mu_2, \ldots, \mu_i) - s_{i-1}^{(k)}(\mu_1, \mu_2, \ldots, \mu_{i-1})
\]

to rewrite (17) as

\[
s_{i}^{(k+1)}(\mu_1, \mu_2, \ldots, \mu_i) - s_{i-1}^{(k+1)}(\mu_1, \mu_2, \ldots, \mu_{i-1}) =
\mu_i \cdot s_{i}^{(k)}(\mu_1, \mu_2, \ldots, \mu_i) - (\mu_i + 1) \cdot s_{i-1}^{(k)}(\mu_1, \mu_2, \ldots, \mu_{i-1}).
\]
Then for \( s_N^{(n+1)} \) we obtain the expression

\[
s_N^{(n+1)} = s_N^{(n+1)} + \mu_N \cdot s_N^{(n)} - (\mu_N + 1) \cdot s_{N-1}^{(n)}.
\]

Since (18) is supposed to be true for all members of right hand side, we get

\[
s_N^{(n+1)} = \sum_{j=1}^{N} b_j^{(N)} \mu_j^{n+1}
\]

where

\[
b_j^{(N)} = \begin{cases} 
(1 - \frac{\mu_{j+1}}{\mu_j})a_j^{(N-1)} + \frac{\mu_N}{\mu_j}a_j^{(N)} & \text{for } i < N \\
a_N^{(N)} & \text{for } i = N.
\end{cases}
\]

Using the relation \( a_j^{(N)} = a_j^{(N-1)} \left( 1 + \frac{1}{\mu_j-\mu_N} \right) \), we see that in both cases we have \( b_j^{(N)} = a_j^{(N)} \).

\( \square \)

2.4. Discussion.

First of all we want to observe that the formula (12) looks as the trace of the \( k \)-th power of an operator with eigenvalues \( \mu_i \) and multiplicities \( a_i^{(N)} \), \( 1 \leq i \leq N \).

The sum of these multiplicities is equal to \( N \), the size of the matrix, but the multiplicities themselves are not integers!

Second, the matrix \( M \) satisfies the quantum analogue of the Cayley-Hamilton identity:

\[
M^N = \sum_{k=1}^{N} c_k M^{N-k}
\]

where \( c_k \) are some elements of \( Z(g) \).

For a general matrix \( A \) with entries from a non-commutative algebra \( A \) there is no reason to have something like this. There is however a case when the powers of \( A \) are linearly dependent over the center \( Z \) of \( A \). It happens, when \( A \) is similar to a matrix with elements from \( Z \). We show, that it is exactly the case when we consider the element \( M \) in our quantum family algebra.

Recall that a generic matrix over a commutative field can be reduced to the so-called second normal form:

\[
A = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 \\
a_n & a_{n-1} & a_{n-2} & \cdots & a_1
\end{pmatrix}.
\]

Indeed, if \( A \) possesses a cyclic vector \( v \), i.e. such that vectors \( v_k = A^{k-1}v \) are linearly independent for \( 1 \leq k \leq n \), then \( A \) takes the form (20) in the basis \( v_1, \ldots, v_n \).
It turns out, that over the skew-field $D(\mathfrak{g})$ generated by $U(\mathfrak{g})$ the matrix $M$ can be reduced to the second normal form. For $N = 2$ it looks as follows:

$$M = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ E_{21} & E_{22} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ a_2 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ E_{21} & E_{22} \end{pmatrix},$$

where

$$a_1 = E_{11} + 1 + E_{22} = \text{tr} M + 1,$$

$$a_2 = E_{21} E_{12} - (E_{11} + 1)E_{22} = \frac{\text{tr} M^2 - (\text{tr} M)^2 - \text{tr} M}{2}.$$ 

It implies the recurrence relation

$$M^n = (\text{tr} M + 1)M^{n-1} + \frac{(\text{tr} M)^2 + \text{tr} M - \text{tr} (M^2)}{2} M^{n-2},$$

which is the quantum analogue of the Hamilton–Cayley identity.

Note also the following expression which is valid for any analytic function $f$:

$$f(M) = \frac{f(\mu_1) - f(\mu_2)}{\mu_1 - \mu_2} \cdot M + \frac{\mu_1 f(\mu_2) - \mu_2 f(\mu_1)}{\mu_1 - \mu_2} \cdot 1$$

where $\mu_{1,2}$ are roots of the quadratic polynomial $\mu^2 - a_1 \mu - a_2$.

3. Quantum Family Algebra for Irreducible Representations of $\mathfrak{sl}(2)$

The results and conjectures discussed in this section are based on the direct computations made by N. Rojkovskaya. It is a challenge to find a conceptual explanation of these results and extend them to the case $\mathfrak{g} = \mathfrak{sl}(N)$ or $\mathfrak{gl}(N)$.

3.1. Problem setting.

Let $\mathfrak{g} = \mathfrak{sl}(2)$ and $E, F, H$ be a standard basis in $\mathfrak{g}$. We prefer to replace it by the new basis $\tilde{E} = \hbar \cdot E$, $\tilde{F} = \hbar \cdot F$, $\tilde{H} = \hbar \cdot H$ so that the commutation relations take the form

$$[\tilde{H}, \tilde{E}] = 2\hbar \tilde{E}, \quad [\tilde{H}, \tilde{F}] = -2\hbar \tilde{F}, \quad [\tilde{E}, \tilde{F}] = \hbar \tilde{H}$$

and give the explicit deformation of the abelian Lie algebra.

Let $(\pi_n, V_n)$ be a $(n + 1)$-dimensional irreducible representation of $\mathfrak{g}$, the $n$-th symmetric power of the standard 2-dimensional representation $\pi_1$.

In an appropriate basis in $V_n$ the operators of the representation look like

$$\pi_n(aE + bH + cF) =$$

$$= \hbar \begin{pmatrix} b_2^n & a\sqrt{1 \cdot n} & 0 & \cdots & 0 & 0 \\ c\sqrt{n \cdot 1} & b_2(\frac{n}{2} - 1) & a\sqrt{2 \cdot (n - 1)} & \cdots & 0 & 0 \\ 0 & \sqrt{n - 1 \cdot 2} & b_2(\frac{n}{2} - 2) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b(1 - \frac{\alpha}{n}) & a\sqrt{1 \cdot n} \\ 0 & 0 & 0 & \cdots & c\sqrt{1 \cdot n} & -b_2^n \end{pmatrix}. $$

The problem is to find the explicit formulae for the quantum eigenvalues and quantum multiplicities of the element

$$M = \frac{1}{\hbar} \left( E \cdot \pi_n(F) + F \cdot \pi_n(E) + \frac{1}{2} H \cdot \pi_n(H) \right).$$
2.2. Results and conjectures.

The main statement is the following conjectural formula based on the direct computation of the second normal form of the matrix $M$ for small values of $N$. We write it in the form of

**Proposition.** Let $\Delta = H^2 + 2EF + 2FE$ be the standard generator of $Z(\mathfrak{g})$. Then

a) Quantum eigenvalues of $M$ are

\begin{equation}
\mu_k^{(n)} = \left[ k^2 - \left( k + \frac{1}{2} \right) n \right] \hbar + \left( \frac{n}{2} - k \right) \sqrt{\Delta + \hbar^2}, \quad 0 \leq k \leq n.
\end{equation}

b) The corresponding multiplicities are

\begin{equation}
d_k^{(n)} = 1 + \frac{(n-2k)\hbar}{\sqrt{\Delta + \hbar^2}}, \quad 0 \leq k \leq n.
\end{equation}

c) The trace of $M^p$ is given by

\begin{equation}
\text{tr } M^p = \sum_{k=0}^{n} d_k^{(n)} \cdot \left( \mu_k^{(n)} \right)^p, \quad p \geq 0.
\end{equation}

I have no doubts that this Proposition is correct. It is quite possible that it could be proved by the method of [PWZ]. But in present time it is only confirmed by direct computations made for $N \leq 5$.

**References**


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