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Supported by Federal Ministry of Science and Transport, Austria
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(Dated: November 20, 2000)

By encoding a qubit in a harmonic oscillator and investigating the \(d \to \infty\) limit, we give an entirely new realization of continuous-variable quantum computation. The generalized Pauli group is generated by number and phase operators for harmonic oscillators. We describe a physical realization in terms of microwave cavities, coupled via a standard Kerr optical nonlinearity.

PACS numbers: 03.67.Lx, 02.20.-a, 42.50.-p

The use of continuous-variable (CV) quantum computing allows information to be encoded and processed much more compactly and efficiently than with discrete-variable (qubit) computing. With CV realizations, one can perform quantum information processes using fewer coupled quantum systems: a considerable advantage for the experimental realization of quantum computing. The rapidly developing field of CV quantum information theory has applications to quantum error correction [1], quantum cryptography [2] and quantum teleportation [3], including an experimental realization of CV quantum teleportation [4].

At present, the proposed realization of CV quantum computation employs position eigenstates as a computational basis [1]; these states are approximated experimentally using highly squeezed states [3]. Here, we introduce new CV realizations, where the generalized Pauli group is generated by the number operator \(\hat{N}\) and a phase operator \(\hat{\theta}\), and the computational basis is given either by harmonic oscillator number eigenstates or phase eigenstates. These realizations are obtained formally by taking the \(d \to \infty\) limit of the qudit, the \(d\)-dimensional generalization of the qubit, and are important for five key reasons:

1. these CV realizations are entirely distinct from the position eigenstate computational basis realization, both in terms of the computational basis and in terms of the SUM gate;
2. the SUM gate employs a standard Kerr optical non-linearity to couple two modes;
3. these realizations give natural extensions of the qubit-based (discrete-variable) Pauli group, with a well-defined limiting procedure;
4. this CV quantum computation presents an appealing realization in terms of coupled microwave cavities using the powerful methods of state preparation and endoscopy in such cavities [5]; and
5. these realizations give a new implementation of the well-studied phase operator [6].

In this letter, we review the generalized Pauli group for qudits, and we construct the generators of the generalized Pauli group in \(d\) dimensions using operators that will be shown to be expressible in terms of the SU(2) angular momentum and phase operators. This construction allows us to conveniently view the \(d\)-dimensional space of qudits as the Hilbert space of a \(d\)-dimensional irrep of SU(2). We also express the qudits in terms of harmonic oscillator states and investigate the \(d \to \infty\) limit, and show that it is not the common generalization of the Pauli group for CV quantum information (i.e., the Heisenberg–Weyl group) with position eigenstates as the computational basis. Instead, we obtain a new CV realization, where the generalized Pauli group is generated by the number operator \(\hat{N}\) and a phase operator \(\hat{\theta}\). We also construct a second realization of qudits in terms of phase states and show that this realization is “dual” to the first realization given here. Finally, we discuss a physical realization of CV quantum computation in coupled harmonic oscillators with a coupled system of nonlinear microwave cavities introduced as a specific example. We establish a SUM gate, which serves as the CV analogue of the CNOT gate; this SUM gate employs a \(\chi^{(3)}\) optical nonlinearity and is distinct in operation from the SUM gate suggested [1] for the position-eigenstate computational basis.

We begin by reviewing the Pauli group of a qubit, and its generalization to the qudit. A qubit is realized as a state in a two-dimensional Hilbert space \(\mathcal{H}_2\). It is customary to choose two normalized orthogonal states, \(|0\rangle\) and \(|1\rangle\), to serve as a computational basis for \(\mathcal{H}_2\). The unitary operators \(\{X_2 \equiv \sigma_x, Z_2 \equiv \sigma_z\}\), where the \(\sigma_i\) are the Pauli spin matrices, generate the Pauli group using matrix multiplication. The elements of this group are known as Pauli operators and provide a basis of unitary operators on \(\mathcal{H}_2\).
Just as a qubit is realized as a state in a Hilbert space of dimension two, a qudit is realized as a state in a $d$-dimensional Hilbert space $\mathcal{H}_d$. It is useful to choose a computational basis $\{|s\}; s = 0, 1, \ldots, d - 1\}$ for $\mathcal{H}_d$, which serves as the generalization of the binary basis $\{|0\}, \{|1\}\}$ of the qubit.

A basis for unitary operators on $\mathcal{H}_d$ is given by the \textit{generalized Pauli operators} [1, 7, 8]

$$(X_d)^a (Z_d)^b = X_{d}^{a} Z_{d}^{b} = \exp(2\pi i s / d) |s\rangle\langle s|, \quad a, b = 0, 1, \ldots d - 1,$$

where $X_d$ and $Z_d$ are defined by their action on the computational basis as follows:

$$X_d |s\rangle = |s + 1 \text{ (mod } d)\rangle,$$
$$Z_d |s\rangle = \exp(2\pi i s / d) |s\rangle.$$

The operators $X_d$ and $Z_d$ generate a group under matrix multiplication, known as the \textit{generalized Pauli group}. Note that $X_d$ and $Z_d$ are non-commutative and obey

$$Z_d X_d = \exp(2\pi i / d) X_d Z_d.$$

In the following, we give a representation of $X_d$ and $Z_d$ in the $d$-dimensional Hilbert space of a SU(2) irrep of highest weight (angular momentum) $j = (d - 1)/2$. The relevant generalized Pauli operators can be viewed in terms of SU(2) angular momentum and phase operators [9].

Consider the standard basis for the su(2) algebra \{\(J_x, J_y, J_z\)\}. Let \{\(|j, m\rangle; m = -j, \ldots, j\}\) denote the standard weight basis for the Hilbert space $\mathcal{H}_{d = 2j + 1}$ for an SU(2) irrep of highest weight (angular momentum) $j$. We use the simplifying notation of Vourdas [9] where we allow $m$ to take all the integer (or half-integer) values modulo $2j + 1$, thus defining $|j, j + 1\rangle = |j, -j\rangle$.

With the computational basis defined to be

$$|s\rangle \equiv |j, s - j\rangle, \quad s = 0, 1, \ldots, d - 1,$$

we now write the generators of the generalized Pauli group in terms of operators that act in a natural way on our SU(2) basis states. Because the basis states are eigenstates of $J_z$, we have

$$Z_d = \exp(2\pi i J_z / d),$$

which is unitary and satisfies Eq. (3). For the generalized Pauli operator $X_d$, we use

$$X_d = \sum_{m = -j}^{j} |j, m\rangle \langle j, m + 1|.$$  

One can easily check that $X_d$ satisfies Eq. (2) under the identification of Eq. (7) and that it is unitary. The operators $X_d$ and $Z_d$ satisfy Eq. (4) and together generate a representation of the generalized Pauli group for a qudit. It is convenient to view $X_d$ as the exponent of a Hermitian operator $X_d = \exp(2\pi i \theta_d / d)$, just as $Z$ is generated by the operator $J_z$. The operator $\theta_d$ is the SU(2) phase operator of Vourdas [9].

It is also possible to realize the operators $X_d$ and $Z_d$ as operators that act naturally on the space $\mathbb{H}_d$ of dimension $d$ spanned by harmonic oscillator states of no more than $d - 1$ bosons. We define the computational basis to be the set of harmonic oscillator energy eigenstates

$$|s\rangle \equiv |n = s\rangle_{\text{HO}}, \quad s = 0, 1, \ldots, d - 1,$$

where $\mathbb{H}|n\rangle_{\text{HO}} = n|n\rangle_{\text{HO}}$. Again, we apply the cyclic notation $d = \{0\}$. Encoding a qudit in an oscillator is important not only for investigating the $d \to \infty$ limit, but also for realizing a qudit experimentally and for creating error-correcting codes for qudit-based computation [1].

In this Hilbert space, the generators $X_d$ and $Z_d$ of the generalized Pauli group for a qudit become

$$X_d \mapsto \sum_{s = 0}^{d - 1} |s\rangle \langle s|, \quad Z_d \mapsto \exp(2\pi i \tilde{N} / d),$$

which are unitary on $\mathbb{H}_d$. It is convenient to view $X_d$ as the exponent of a Hermitian operator $\theta_d$, such that $X_d = \exp(2\pi i \theta_d / d)$; the operator $\theta_d$ is the Pegg-Barnett phase operator [6]. We will call this representation of the generalized Pauli group the \textit{number representation}.

The explicit realization of $X_d$ and $Z_d$ as unitary operators on the harmonic oscillator Hilbert space enables us to investigate the $d \to \infty$ limit; the limiting procedure for phase operators has been thoroughly investigated [6, 10]. In this limit, the computational basis remains the harmonic oscillator energy eigenstates (now including all states $s = 0, 1, \ldots, \infty$), following Eq. (8). It is natural to generalize the operator $X_d$ to a continuous transformation $X(x)$, generated by the phase operator $\theta_d$; i.e.,

$$X(x) \equiv \exp(iz \theta_d), \quad x \in \mathbb{R}.$$  

Similarly, the CV generalization of $Z_d$ is obtained by replacing the finite angle $2\pi / d$ in the expression for $Z_d$ in Eq. (9) by the continuous angle $z \in \mathbb{R}$, so that we now have the unitary transformation $Z(z)$ defined by

$$Z(z) \equiv \exp(iz \tilde{N}).$$

Note that, in extending the above representation of the generalized Pauli group from qudits to CV representations, we do not obtain the usual generalization as the Heisenberg-Weyl group, with position $z$ and momentum $\tilde{p}$ operators as generators. Instead, the generalized Pauli operators are generated by the number operator $\tilde{N}$ and the phase operator $\theta_d$; these operators are in a sense “conjugate” like momentum and position, but there exist challenging problems with defining the phase operator on the infinite-dimensional Hilbert space $\mathbb{H}_\infty$ of the harmonic oscillator [6, 10]. It is also interesting to note that
the states of the computational basis for the limiting case remain harmonic oscillator energy eigenstates, not position (or momentum) eigenstates or squeezed Gaussians as are commonly used for CV quantum computing. In what follows, we will refer to this representation as the number representation of the generalized CV Pauli group.

It is possible to construct another realization of $X_d$ and $Z_d$ in the Hilbert space $\mathcal{H}_d$ for an irrep of $SU(2)$ where the computational basis is given by $SU(2)$ phase states. This representation is “dual” to the number representation. Consider the relation $i X_2 = \exp(i(\pi/2)X_2)$ for qubits; i.e., that

$$|1\rangle = X_2 |0\rangle = (-i) \exp(i(\pi/2)X_2) |0\rangle.$$  

(12)

The Pauli operator $X_2$ has two interpretations, each of which can be generalized in a different way. In the number representation, we interpreted $X_2$ as a number state raising operator $|1\rangle = X_2 |0\rangle$ and generalized this operator as such. However, using the relation (12), we can also view $X_2$ as a rotation. (Using the $SU(2)$ representation $X_2 = 2 J_x$, this rotation is about the $x$-axis.) Thus, the state $|1\rangle$ is obtained (up to a phase) by rotating $|0\rangle$ by an angle $\pi$ about the $x$-axis. The computational basis states needed for this type of generalization to qudits are “$SU(2)$ phase states” and have been investigated by Vourdas [9] (although using rotations generated by $J_x$ rather than $J_\varphi$). These states form an orthonormal basis for the $SU(2)$ irrep and are “dual” to the weight basis. Let $\{ |j, m\rangle; m = -j, \ldots, j \}$ be the weight basis for an $SU(2)$ irrep of angular momentum $j = (d-1)/2$, where $J_x$ rather than $J_\varphi$ is diagonal; i.e., $J_x |j, m\rangle = m |j, m\rangle$. For this representation, we define the computational basis states to be

$$|\psi\rangle \equiv \begin{cases} 
\frac{1}{\sqrt{d}} \sum_{m=-j}^{j} \exp(2\pi i m s/d) |j, m\rangle, & \text{d odd,} \\
\sum_{m=-j}^{j} \exp(2\pi i (m + \frac{1}{2}) s/d) |j, m\rangle, & \text{d even.}
\end{cases}$$  

(13)

These states form an orthonormal basis for $\mathcal{H}_d$ [9]. They are referred to as $SU(2)$ phase states because they are eigenstates of the $SU(2)$ phase operator, defined below.

The generalized Pauli operator $X_d$ on this computational basis is given by

$$X_d \mapsto \begin{cases} 
\exp(2\pi i J_x/d), & \text{d odd,} \\
\exp(-i\pi/d) \exp(2\pi i J_x/d), & \text{d even,}
\end{cases}$$  

(14)

satisfying Eq. (2). Note that $(X_d)^d = 1$ for both $j$ integral and half-integral (i.e., spinor). The generalized Pauli operator $Z_d$ is given by

$$Z_d \mapsto \sum_{s=0}^{d-1} \exp(2\pi i s/d) |s\rangle \langle s|,$$  

(15)

which is unitary and satisfies Eq. (3). Note that we can express $Z_d$ as the exponent of a Hermitian operator,

$$Z_d = \exp(2\pi i \hat{\theta}_d/d), \quad \hat{\theta}_d \equiv \sum_{s=0}^{d-1} s |s\rangle \langle s|;$$  

(16)

the operator $\hat{\theta}_d$ is the $SU(2)$ phase operator.

Note that this representation of the generalized Pauli group is “dual” to the number representation in the same sense that the position and momentum representations of the harmonic oscillator are dual. For the number representation, the computational basis states are eigenstates of $J_z$, and the phase operator $\hat{J}_x$ generates the “ladder” transformations. In the phase representation given here, the computational basis states are eigenstates of the phase operator $\hat{\theta}_d$, i.e., “phase eigenstates”, and it is $J_x$ which generates the ladder transformations via rotations about the $x$-axis. Both of these representations can be considered natural generalizations of the qubit case, because the standard computational basis $|0\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$ and $|1\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$ are both eigenstates of $J_z$ and phase eigenstates of $\hat{J}_x$.

As with the number representation, this phase representation of the generalized Pauli group can be expressed in a harmonic oscillator Hilbert space. Again, the $d \to \infty$ limit yields challenging problems: it is well known that phase eigenstates do not exist in the infinite-dimensional Hilbert space $\mathcal{H}_\infty$ of the harmonic oscillator [6].

In any experimental realization, the problems associated with taking the $d \to \infty$ limit would not arise. A physically realistic system would have a finite energy cutoff (and an associated resolution in time and thus phase), and so experimental CV computation would in actuality involve qudits with finite (although possibly very large) $d$. As a result of our well-defined limiting procedure for qudits, the above realization of CV quantum computation is applicable to such a physically realistic system.

One approach to constructing a physical realization would be as a mode in a microwave cavity, as depicted in Fig. 1. State preparation and endoscopic measurement in a microwave cavity [5] can be used to create initial quantum states for computing, as well as for measurements of the final states. To perform universal CV computation [11], it is necessary to be able to realize an arbitrary unitary transformation on a single qudit, and to have a controlled two-qudit interaction gate such as the SUM gate [1]. An arbitrary unitary transformation on a single qudit, to any desired precision, can be performed efficiently using a combination of linear optics, parametric down-conversion, and a nonlinear optical Kerr medium [11]. By this combination, one can approximate (to arbitrary accuracy) any polynomial Hamiltonian in $a^\dagger$ and $\hat{a}$. Of particular importance is to realize the Fourier transform operation on a single qudit, which takes number eigenstates to phase eigenstates and
vice versa. This operation is the generalization of the Hadamard transformation for qudits.

For quantum computation, we must also realize a gate that performs a two-qudit interaction. Consider two oscillators coupled by the four-wave mixing interaction Hamiltonian $\chi \hat{N}_1 \hat{N}_2 = \chi \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2$. This Hamiltonian for an optical system describes a four-wave mixing process in which $\chi$ is proportional to the third-order nonlinear susceptibility [12]. Let oscillator 1 be in a state $|s_1\rangle_1$ encoded in the number state basis, and let oscillator 2 be in a state $|s_2\rangle_2$ encoded in the phase state basis. This interaction Hamiltonian generates the transformation

$$e^{-i\chi \hat{N}_1 \hat{N}_2/\hbar} |s_1\rangle_1 \otimes |s_2\rangle_2 = |s_1\rangle_1 \otimes |\chi s_1 + s_2\rangle_2.$$  

Thus, with time $t = \chi^{-1}$, this Hamiltonian generates the SUM transformation $|s_1\rangle_1 \otimes |s_2\rangle_2 \rightarrow |s_1\rangle_1 \otimes |s_1 + s_2\rangle_2$.

Quantum computation with multiple qudits could be performed by coupling many microwave cavities as shown in Fig. 2. The oscillator along the horizontal axis serves as a “bus,” interacting with any of the other cavities 1 through $N$ via a SUM interaction of the type described above. Note that the control qudit for the sum operation must be encoded in the number state basis, and the target qudit must be in the phase state basis. Thus, the Fourier transform operation is a vital component to performing SUM operations using this bus.

In summary, we have presented a new form of continuous-variable computation in terms of number and phase operators, and describe a realization in terms of coupled microwave cavities using linear optical elements, optical squeezing, and nonlinear Kerr media. This new approach has the advantage over position-eigenstate CV computation in that the computational basis states, for large but finite $d$, are well-defined and obtainable, and do not require “infinite-squeezing” of Gaussian wavepackets.

This project has been supported by an Australian Research Council Large Grant and by a Macquarie University Research Grant. We acknowledge helpful discussions with Samuel Braunstein.